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Reflections on the hysteretic damping model

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Abstract. This paper presents a reflection on a recently proposed solution to the problem of the free vibration response with the constant hysteretic damping model, that has been presented in some conferences in recent years, by the author himself and some of his colleagues. On the one hand, as expected, the subject has been received with natural criticism, mainly due to the well-known non-causal behaviour of the model in free vibration. On the other hand, it was not easy to understand what could be wrong in that proposal, as apparently everything was perfect from a mathematical point of view. The author decided that this subject deserved a more careful and detailed analysis and – in this kind of tutorial paper – the issue seems to have been clarified. It is concluded that the proposed solution involving the constant hysteretic damping corresponds in fact to an equivalent viscously damped model; it is therefore concluded that the application of the constant hysteretic damping to model the free vibration of practical engineering problems should be considered only in the perspective of an equivalent viscously damped model.

Keywords: Hysteretic damping, free vibration

1. Introduction

The most popular model for damping is the viscous one, where the force developed by the damping element is directly proportional to the velocity of the response, i.e., $f = c\dot{x}$ (c is the damping coefficient). But models are models, i.e., mathematical abstractions that try to match as closely as possible the reality without being completely perfect. For instance, although the viscously damped model is quite accurate in describing the free vibration of a system, one should not forget that it tells us that the system never comes to a halt! The response only goes to zero at infinity, or if preferred, the system vibrates forever... So, the model is not perfect in the description of free vibration, but it is good enough from an engineering point of view and quite "friendly" from a mathematical perspective. It is also often used in the frequency domain, although in that case one must be careful, as when the frequency range of interest is considerably large some deviations between the response of the model and the true response may become apparent. Strictly speaking about damping, such a deviation may be due to the fact that the viscous model implies energy dissipation per cycle that is linearly proportional to the frequency ($W_{diss.} = \int_0^{2\pi/\omega} f\,\dot{x}\,dt = \pi\,c\omega X^2$). It is a well known result that the energy dissipated per cycle on most metallic structures is a consequence of the internal friction of the material itself, known as *material hysteresis*, and the experience shows that it is practically independent on the frequency of excitation [1]. This observation led to the introduction of the hysteretic model. Thinking in terms of the single-degree-of-freedom system (SDOF), while the force due to the viscous damping is proportional to the velocity, the force due to the hysteretic damping is proportional to the displacement, although still in phase with the velocity, as it is a dissipative force. With such a definition, the dissipated energy becomes independent of the frequency, as desired: $W_{diss.} = \pi\,d X^2$, where d is the hysteretic damping coefficient (note that its units are N/m, as a stiffness). For an SDOF system subjected to a harmonic force, the equilibrium equation is now given by:

$$m\ddot{x}(t) + k\,(1 + i\eta)\,x(t) = F\,e^{i\omega t} \tag{1}$$

where $\eta = d/k$ is the hysteretic damping factor and $k\,(1 + i\eta)$ is a complex stiffness. The steady-state solution (corresponding to the particular solution of (1)) presents no problem:

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$$x(t) = X e^{i\omega t} = \frac{F}{k - \omega^2 m + i k \eta} e^{i\omega t} \tag{2}$$

and the frequency response function is given by:

$$H(\omega) = \frac{1/m}{\omega_n^2 - \omega^2 + i\,\omega_n^2 \eta} \tag{3}$$

where ω_n is the undamped natural frequency of the system. This model, valid only for $\omega > 0$, is quite satisfactory in many engineering applications, namely in structural dynamics. The problem arises when one wishes to evaluate the free vibration response, i.e, when one tries to solve the homogeneous equation related to Eq. (1):

$$m\ddot{x}(t) + k\,(1 + i\eta)\,x(t) = 0 \tag{4}$$

How to obtain the free vibration response for a system with hysteretic damping has been the motive of all kind of discussions and it is amazing the interest that it has raised along so many decades (e.g. [2–9]). Eq. (4) seems absurd, as the solution has to be necessarily complex. From a physical point of view, it does not make any sense indeed. For instance, Inaudi and Kelly [10] state that the force $k\,(1 + i\eta)\,x(t)$ in Eq. (4) is clearly incorrect, as it would mean that a real response implies a complex force. Both the real part of the complex stiffness (the storage modulus) and the imaginary part (the loss modulus) are independent of the frequency and, as already mentioned, the dissipated energy becomes independent of the frequency. It is also very well known that the constant hysteretic damping model is non-causal (see, for instance [8,9,11,12]). In fact, as reminded by Inaudi and Kelly [10] and discussed in [13], it is not possible to formulate a causal model that has simultaneously a storage modulus and a loss modulus independent of the frequency. Back in 1958, Biot [14] proposed a viscoelastic damping model whose variation with frequency is very weak with respect to the loss modulus, and that can therefore come close – in practice – to the constant hysteretic model, although it leads to a storage modulus that increases with frequency.

As stated above, Eq. (4) does not make sense. However, in recent papers, Ribeiro et al. [15,16] have proposed to look at Eq. (4) from a different perspective: Eq. (4) has a mathematical complex solution $x(t)$ that is not the physical response of the system; the physical response of the system should be the real part of $x(t)$. As shown in [15], the solution is given by (after discarding the unstable part of the solution):

$$x(t) = Ce^{-\omega_n at} e^{i\omega_n bt} \tag{5}$$

where C is a complex constant and a and b are given by:

$$a = \sqrt{\frac{-1 + \sqrt{1 + \eta^2}}{2}} \quad b = \sqrt{\frac{1 + \sqrt{1 + \eta^2}}{2}} \tag{6}$$

Note that in order to solve Eq. (5) it is necessary to give two initial conditions, which must be complex quantities and – once more – the rationale is that the physical (measurable) quantities that one is used to (initial displacement and velocity) are the real parts of those complex initial conditions. In fact, such an approach is not really new. Similar results had already been reported by Sorokin [17,18]. Denoting the initial displacement and velocity as x_0 and v_0, it was shown [15] that:

$$x(t) = \left(x_0 - i\frac{v_0 + \omega_n a x_0}{\omega_n b} \right) e^{-\omega_n at} e^{i\omega_n bt} \tag{7}$$

And from this mathematical solution, the physical one – the response of the system – would be its real part:

$$x(t) = e^{-\omega_n at} \left(x_0 \cos\omega_n bt + \frac{v_0 + \omega_n a x_0}{\omega_n b} \sin\omega_n bt \right) \tag{8}$$

In this paper a discussion on the validity of this kind of approach will be carried on.

2. Impulse response function *versus* frequency response function

It is a known fact that the free response of a system modeled with the constant hysteretic damping has been reported over the years by various authors as having a non-causal behaviour (e.g. [8,9,11,12]), i.e., the system, initially at rest, has a response other than zero before any perturbation occurs, which clearly is not physically realizable. The effect cannot precede the cause. For a system with a real-valued time response $x(t)$, the impulse response function (IRF) $h\,(t)$ (response to a Dirac impulse) and the frequency response function $H(\omega)$ constitute a Fourier pair, i.e., $H(\omega) = \mathcal{F}(h(t))$ and $h\,(t) = \mathcal{F}^{-1}(H(\omega))$, where $\mathcal{F}$ means Fourier transform and $\mathcal{F}^{-1}$ its inverse. But for the response to be real-valued (the only one physically admissible), $H(\omega)$ must be Hermitian, or conjugate-even (real part even, imaginary part odd), i.e., $H(-\omega) = H^*(\omega)$, where * means complex conjugate. And the inverse is also true, if $H(\omega)$ is Hermitian, the impulse response is real. This can be easily proven:

$$\begin{aligned}
h\,(t) = \mathcal{F}^{-1}(H(\omega)) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} H(\omega)\,e^{i\omega t}d\omega = \frac{1}{2\pi}\left(\int_{-\infty}^{0} H(\omega)\,e^{i\omega t}d\omega + \int_{0}^{+\infty} H(\omega)\,e^{i\omega t}d\omega\right)\\
&= \frac{1}{2\pi}\left(-\int_{0}^{-\infty} H(\omega)\,e^{i\omega t}d\omega + \int_{0}^{+\infty} H(\omega)\,e^{i\omega t}d\omega\right)\\
&= \frac{1}{2\pi}\left(-\int_{0}^{+\infty} H(-\omega)\,e^{-i\omega t}d(-\omega) + \int_{0}^{+\infty} H(\omega)\,e^{i\omega t}d\omega\right)\\
&= \frac{1}{2\pi}\int_{0}^{+\infty}\left(H^*(\omega)\,e^{-i\omega t} + H(\omega)\,e^{i\omega t}\right)d\omega = \frac{1}{2\pi}\cdot 2Re\int_{0}^{+\infty} H(\omega)\,e^{i\omega t}\,d\omega
\end{aligned} \tag{9}$$

2.1. Viscous damping

For a system with viscous damping and harmonic excitation, the frequency response function is

$$H(\omega) = \frac{1/m}{\omega_n^2 - \omega^2 + i\,2\xi\omega_n\omega} = \frac{1}{m}\cdot\left(\frac{\omega_n^2 - \omega^2}{(\omega_n^2 - \omega^2)^2 + (2\xi\omega_n\omega)^2} + \frac{-i\,2\xi\omega_n\omega}{(\omega_n^2 - \omega^2)^2 + (2\xi\omega_n\omega)^2}\right), \tag{10}$$

whereξ is the viscous damping factor. It is clear that $H(\omega)$ is a Hermitian function of ω. The graphical representation of the real and imaginary parts of $H(\omega)$ against frequency (in a dimensionless way) is illustrated in Fig. 1a, b. Therefore, from Eq. (10), its impulse response function will be real. Applying Eq. (9), namely by contour integration, it can be shown that the impulse response function is given by:

$$h\,(t) = \frac{1}{m\omega_d}e^{-\xi\omega_n t}\sin\omega_d t \tag{11}$$

where $\omega_d = \omega_n\sqrt{1-\xi^2}$ is the damped natural frequency. It is clear that the system is at rest for $t = 0$, as $h\,(0) = 0$. The system has a causal behaviour.

2.2. Hysteretic damping

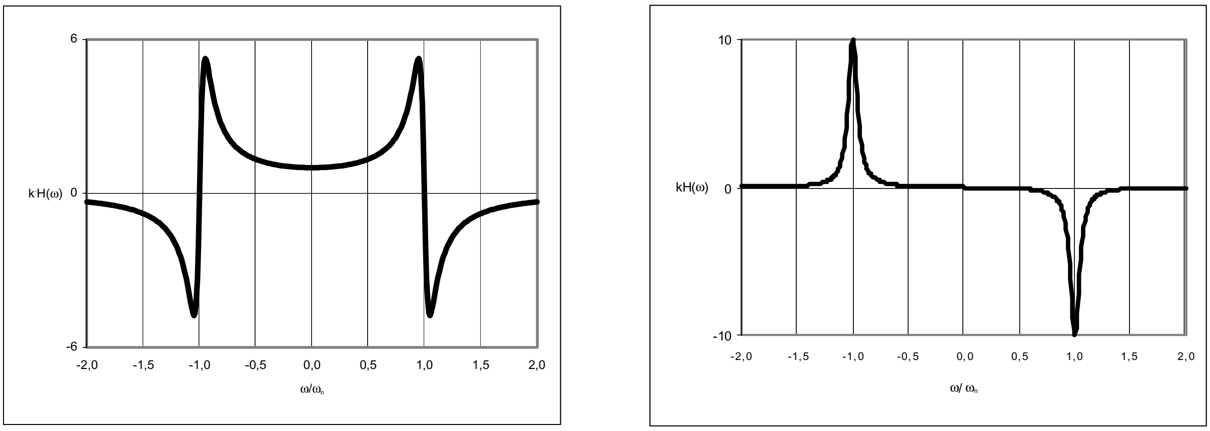
Let us now take the hysteretic damping case. Writing Eq. (3) as:

$$H(\omega) = \frac{1}{m}\cdot\left(\frac{\omega_n^2 - \omega^2}{(\omega_n^2 - \omega^2)^2 + (\eta\omega_n^2)^2} + \frac{-i\,\eta\omega_n^2}{(\omega_n^2 - \omega^2)^2 + (\eta\omega_n^2)^2}\right), \tag{12}$$

one notices that both the real and complex parts are even (Fig. 2a, b) and therefore, the impulse response function will not be real.

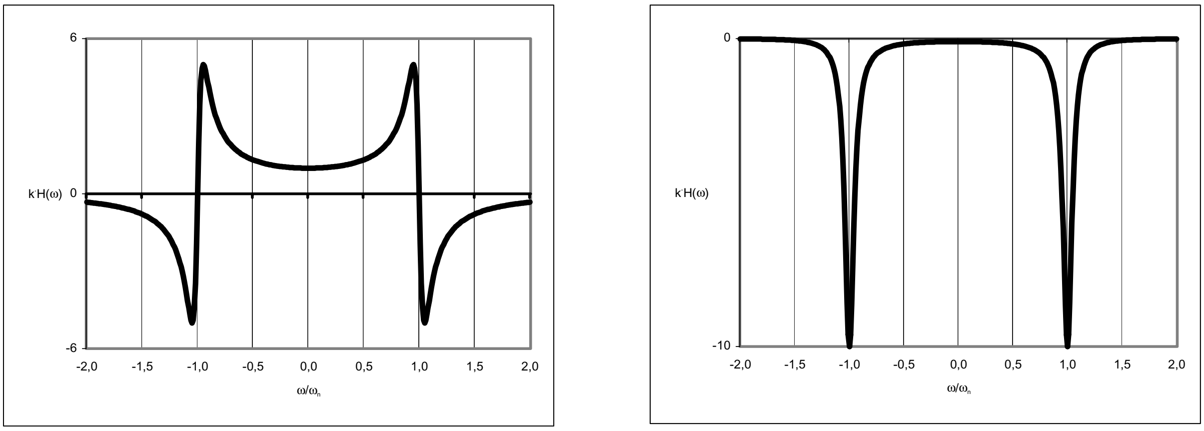
In order to force $h\,(t)$ to be real, it is usual to modify the frequency response function to

$$H(\omega) = \frac{1/m}{\omega_n^2 - \omega^2 + i\,\eta\omega_n^2 sgn\,(\omega)} \tag{13}$$



a) Real part of $H(\omega)$ (viscous damping case). b) Imaginary part of $H(\omega)$ (viscous damping case).

Fig. 1.



a) Real part of $H(\omega)$ (hysteretic damping case). b) Imaginary part of $H(\omega)$ (hysteretic damping case).

Fig. 2.

where $sgn\,(\omega)$ is the *signum* function, defined as:

$$sgn\,(\omega) = \begin{cases} 1 & for\; \omega > 0 \\ 0 & for\; \omega = 0 \\ -1 & for\; \omega < 0 \end{cases} \tag{14}$$

Such a modification has no effect on the steady-state response of the system and has the virtue of allowing the impulse response function to become real valued. Writing Eq. (13) in its real and imaginary parts,

$$H(\omega) = \frac{1}{m} \cdot \left(\frac{\omega_n^2 - \omega^2}{\left(\omega_n^2 - \omega^2\right)^2 + \left(\eta\omega_n^2\right)^2} + \frac{-i\,\eta\omega_n^2 sgn\,(\omega)}{\left(\omega_n^2 - \omega^2\right)^2 + \left(\eta\omega_n^2\right)^2} \right) \tag{15}$$

One can clearly observe that $H(\omega)$ is now Hermitian. In order to obtain the expression for the frequency response function with the *signum* function as in Eq. (13), Inaudi and Kelly [10] state that the correct formulation of Eq. (1) in the time domain should be:

$$m\ddot{x}(t) + kx(t) + k\eta\,\mathcal{H}\left(x(t)\right) = f(t) \tag{16}$$

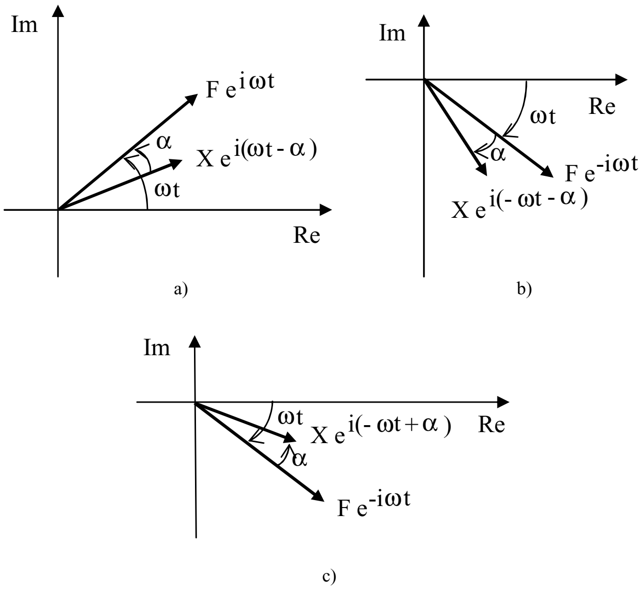


Fig. 3. a) Force and response in the complex plane for $\omega > 0$; b) Force and response in the complex plane for $\omega < 0$ with an incorrect phase angle; c) Force and response in the complex plane for $\omega < 0$ with the correct phase angle.

where $\mathcal{H}\left(x(t)\right)$ represents the Hilbert transform of $x(t)$, defined in the time domain as:

$$\mathcal{H}\left(x(t)\right) = -\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{x(\tau)}{t-\tau}\,d\tau \tag{17}$$

or

$$\mathcal{H}\left(x(t)\right) = x(t)\otimes\left(-\frac{1}{\pi\,t}\right), \tag{18}$$

where $\otimes$ is the convolution operator. In fact, applying Fourier transforms to both sides of Eq. (18), it turns out that

$$\mathcal{F}\left(\mathcal{H}\left(x(t)\right)\right) = \mathcal{F}\left(x(t)\right)\ \mathcal{F}\left(-\frac{1}{\pi\,t}\right) \quad \Rightarrow \quad \mathcal{H}\left(X(\omega)\right) = X(\omega)\ isgn(\omega) \tag{19}$$

Therefore, in the frequency domain, Eq. (16) becomes:

$$-\,m\,\omega^2 X(\omega) + k\,X(\omega) + \ i\,k\eta sgn(\omega)X(\omega) = F(\omega) \tag{20}$$

and the frequency response function is given by

$$H(\omega) = \frac{X(\omega)}{F(\omega)} = \frac{1}{k - m\,\omega^2\ +\ i\,k\eta sgn(\omega)} = \frac{1/m}{\omega_n^2\ -\ \omega^2\ +\ i\omega_n^2\,\eta sgn(\omega)} \tag{21}$$

as in Eq. (13). Moreover, the use of the *signum* function to change the sign for negative frequencies is also a necessary condition to have a force preceding the response. In fact, the phase angle between the response and the force (see Fig. 3a) is given by:

$$\alpha = tg^{-1}\left(\frac{\eta\omega_n^2 sgn\left(\omega\right)}{\omega_n^2 - \omega^2}\right) \tag{22}$$

Without the *signum* function, the angle would remain positive for negative frequencies and the response would lead the force (see Fig. 3b), something that cannot happen. Figure 3c illustrates the correct position of the response with respect to the applied force.

As the impulse response function is the inverse Fourier transform of the frequency response function, one must solve the following problem:

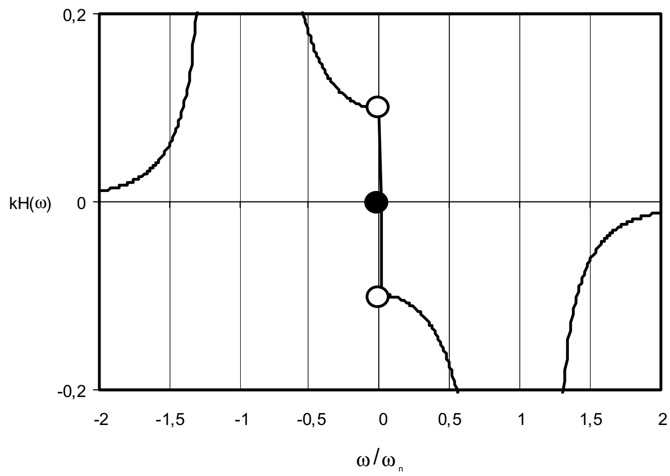


Fig. 4. Imaginary part of the modified $H(\omega)$.

$$h\left(t\right) = \mathcal{F}^{-1}\left(H(\omega)\right) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1/m}{\omega_n^2-\omega^2+i\omega_n^2\eta sgn\,\omega}\cdot e^{i\omega t}d\omega \tag{23}$$

Things become a bit complicated, as the *signum* function introduces a discontinuity at $\omega = 0$ in the imaginary part of $H(\omega)$ (see Fig. 4), which precludes the calculation of the impulse response function (for instance by contour integration) in a closed form.

Gaul et al. [11] succeeded in calculating an approximate solution for the impulse response function from Eq. (23) by contour integration, overcoming the singularity at $\omega = 0$, as will be explained in Section 3.1. They obtained, of course, a real valued solution for $h\left(t\right)$, but confirmed the non-causality of the model, as they found out a negative value (though small) for $h\left(0\right)$ (a precursor). Similar findings have been reported by other authors (e.g. [12,19]). Using the result expressed in Eq. (9), Eq. (23) can alternatively be written as

$$h\left(t\right) = \frac{1}{\pi m}\int_{0}^{+\infty}\frac{\left(\omega_n^2-\omega^2\right)\cos\omega t+\omega_n^2\eta\sin\omega t}{\left(\omega_n^2-\omega^2\right)^2+\left(\omega_n^2\eta\right)^2}\,d\omega \tag{24}$$

which can be evaluated numerically. The non-causal behaviour is again verified, but by using an iterative technique, Inaudi and Kelly [10] managed to obtain a solution that converged to an approximate response, where the non-causality has little expression.

Makris [20] states that the term $k\left(1+i\eta sgn(\omega)\right)$ leads to a non-causal behaviour, because the frequency response function does not verify the Kramers-Kroning conditions, which establish that for a linear, causal and stable system the real and imaginary parts of the frequency response function should constitute an Hilbert pair, where the imaginary part can be calculated from the real part and vice-versa:

$$Re\left(H(\omega)\right) = -\frac{2}{\pi}\int_{0}^{+\infty}\frac{Im\left(H(\Omega)\right)\Omega}{\Omega^2-\omega^2}\,d\Omega; \quad Im\left(H(\omega)\right) = \frac{2\omega}{\pi}\int_{0}^{+\infty}\frac{Re\left(H(\Omega)\right)}{\Omega^2-\omega^2}\,d\Omega \tag{25}$$

With this in mind, assuming a constant imaginary part ($= \eta k sgn\left(\omega/\varepsilon\right)$, where ε is an arbitrary constant just to account for the right dimensions), the real part must be related to the imaginary one through the Hilbert transform. In this way, Makris [20] obtains a real part that is necessarily a function of the frequency ω and that ensures the causal behaviour, i.e., the impulse response function results real and it is null for $t < 0$. With such a model (which he named "causal hysteretic element"), where the real part (the storage modulus) varies with frequency, instead of $k\left(1+i\eta sgn(\omega)\right)$, Makris obtained the expression $k\left(1+\frac{2}{\pi}\eta\ln\left|\frac{\omega}{\varepsilon}\right|+i\eta sgn\left(\frac{\omega}{\varepsilon}\right)\right)$, which is practically the same as Biot's model [14]. In fact, this model is the limiting case of Biot's linear viscoelastic model, with nearly frequency independent dissipation. However, the model is not valid for $\omega = 0$.

The question arises as "What about the solution expressed in Eq. (7)"? Does it correspond to a non-causal behaviour as well? It seems that it does not make much sense to talk about causality referring to Eq. (7), as it is a pure mathematical result. Equation (8) is (supposedly) the one representing the response. Introducing in Eq. (7) the initial conditions of a unitary impulse at $t = 0$, $x_0 = 0, v_0 = 1/m$, the solution becomes:

$$h(t) = -i\frac{1}{m\omega_n b}e^{-\omega_n at}e^{i\omega_n bt} = \frac{1}{m\omega_n b}e^{-\omega_n at}\left(-i\cos\omega_n bt + \sin\omega_n bt\right) \tag{26}$$

As the physical solution should be its real part, one has:

$$Re\left(h(t)\right) = \frac{1}{m\omega_n b}e^{-\omega_n at}\sin\omega_n bt \tag{27}$$

which is a very similar result to the viscously damped model. For the complex solution, (Eq. (26)), one has $Im\left(h(0)\right) = -\frac{1}{m\omega_n b} \neq 0$, but it is not clear the meaning of this result. However, the real part, expressed in Eq. (27) gives $Re\left(h(0)\right) = 0$. This means that apparently the solution proposed in [15] shows a causal behaviour! However, this solution comes from the free vibration equation (4) and the corresponding frequency response function still is Eq. (3), the one that is not Hermitian and therefore has a complex solution. So, in principle and ignoring any physical argument, it could seem logical to try and deduce the (complex) impulse response function from Eq. (3) and compare it to Eq. (26). However, as discussed before and illustrated in figure 3, from a physical point of view it does not make sense to integrate along negative frequencies without changing the sign of the imaginary part of the frequency-response-function, through the use of the *signum* function (or any other with a similar effect). Therefore, one concludes that the solution given in Eq. (26) does not make sense, as it cannot be obtained from any of the frequency response functions of either Eq. (3) or Eq. (13) through an inverse Fourier transform. In other words, one may say that the physics cannot be ignored all the way until the very end as in Ribeiro's *et. al.* approach, where one simply takes the real part of a complex result. The physical meaning must come into play earlier. If one does not attempt to reproduce those results by doing an inverse Fourier transform, then it may be difficult to understand where the mistake occurred.

In the next section, one will see how to compute, or come close to a practical solution for the IRF, using contour integration, for the hysteretic and mixed models. Taking a tutorial perspective, considerable detail is put in the developments that follow.

3. Calculating the IRF using contour integration

3.1. The hysteretic damping case

As discussed in Section 2.2, the impulse response function is given by Eq. (23), from which

$$2\pi m\, h(t) = \int_{-\infty}^{+\infty} \frac{e^{i\omega t}}{\omega_n^2 - \omega^2 + i\omega_n^2\eta sgn\omega}\, d\omega \tag{28}$$

One will follow the explanation given in [11], using Cauchy's Residue Theorem. One extends ω to the complex plane, defining a new variable $z = \omega + i\sigma$. As $\int_{-\infty}^{+\infty}$ should be understood as its *principal value*, i.e., $P.\ V.\ \int_{-\infty}^{+\infty} = \lim\limits_{R\to\infty}\int_{-R}^{+R}$, one needs to evaluate:

$$\lim_{R\to\infty}\int_{-R}^{+R} \frac{e^{izt}}{\omega_n^2 - z^2 + i\omega_n^2\eta sgn\left(Re(z)\right)}\, dz = \lim_{R\to\infty}\int_C f(z)dz = 2\pi i\sum_j Res.\left(f, z_j\right) \tag{29}$$

Because one has simple poles and $f(z)$ can be written as $f(z) = \frac{\phi(z)}{\psi(z)}$, the residues can be given by:

$$Res.\left(f, z_j\right) = \frac{\phi\left(z_j\right)}{\psi'\left(z_j\right)} \tag{30}$$

One has two possibilities for the roots of the denominator:

$$z^2 - \omega_n^2\left(1 + i\eta\right) = 0 \quad \text{for} \quad \omega > 0 \tag{31a}$$

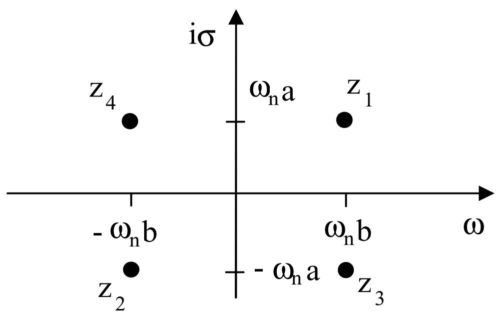


Fig. 5. Poles z_1, z_2, z_3 and z_4 for the hysteretic damping case.

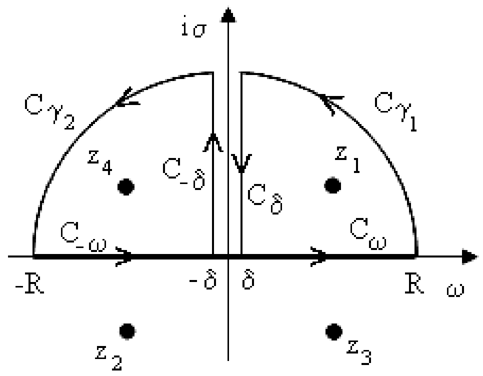


Fig. 6. Upper half-plane integration contour.

$$z^2 - \omega_n^2\,(1 - i\,\eta) = 0 \quad \text{for} \quad \omega < 0 \tag{31b}$$

which are:

$$\begin{array}{l} z_1 = \omega_n b + i\,\omega_n a \\ z_2 = -\omega_n b - i\,\omega_n a \end{array} \quad \text{for} \quad \omega > 0 \quad \text{and} \quad \begin{array}{l} z_3 = \omega_n b - i\omega_n a \\ z_4 = -\omega_n b + i\omega_n a \end{array} \quad \text{for} \quad \omega < 0 \tag{32}$$

with a and b given by Eq. (6). The four poles are represented in Fig. 5.

The contour in the upper half-plane has to be split in two, to avoid the singularity at $\omega = 0$ (as discussed in Section 2.2), which in the complex plane corresponds to avoid the imaginary axis by a small quantity δ (Fig. 6). These contours only encompass poles z_1 and z_4.

Therefore, one has:

$$\begin{aligned} \lim_{\substack{R\to\infty \\ \delta\to 0}} \int_C f(z)dz &= \lim_{\substack{R\to\infty \\ \delta\to 0}} \int_{C_\omega} f(\omega)d\omega + \lim_{R\to\infty} \int_{C_{\gamma_1}} f(z)dz + \lim_{R\to\infty} \int_{C_\delta} f(i\sigma)d(i\sigma) \\ &\quad + \lim_{\substack{R\to\infty \\ \delta\to 0}} \int_{C_{-\omega}} f(\omega)d\omega + \lim_{R\to\infty} \int_{C_{\gamma_2}} f(z)dz + \lim_{R\to\infty} \int_{C_{-\delta}} f(i\sigma)d(i\sigma) \\ &= 2\pi i\,(Res.\,(f, z_1) + Res.\,(f, z_4)) \end{aligned} \tag{33}$$

where

$$\lim_{R\to\infty} \int_{C_{\gamma_1}} f(z)dz = \lim_{R\to\infty} \int_0^{\pi/2} \frac{e^{i\,R\,e^{i\varphi}\,t}}{\omega_n^2 - R^2\,i\,e^{i2\varphi} + i\omega_n^2\,\eta}\; R\,i\,e^{i\varphi}d\varphi$$

$$= \lim_{R\to\infty}\int_0^{\pi/2} \frac{e^{iRe^{i\varphi}t}}{z_1^2 - R^2\, i\, e^{i2\varphi}}\, R\, i\, e^{i\varphi} d\varphi \tag{34}$$

$$\lim_{R\to\infty}\int_{C_{\gamma_2}} f(z)dz = \lim_{R\to\infty}\int_{\pi/2}^{\pi} \frac{e^{iRe^{i\varphi}t}}{z_4^2 - R^2\, i\, e^{i2\varphi}}\, R\, i\, e^{i\varphi} d\varphi$$

$$\lim_{R\to\infty}\int_{C_\delta} f(i\sigma)d(i\sigma) = \lim_{R\to\infty}\int_R^0 \frac{e^{i(i\sigma)t}}{\omega_n^2 - (i\sigma)^2 + i\omega_n^2\eta}\, i\, d\sigma = \lim_{R\to\infty}\int_R^0 \frac{e^{-\sigma t}}{z_1^2+\sigma^2}\, i\, d\sigma$$

$$\lim_{R\to\infty}\int_{C_{-\delta}} f(i\sigma)d(i\sigma) = \lim_{R\to\infty}\int_0^R \frac{e^{-\sigma t}}{z_4^2+\sigma^2}\, i\, d\sigma$$

As

$$2\pi m\, h\,(t) = \int_{-\infty}^{+\infty} \frac{e^{i\omega t}}{\omega_n^2 - \omega^2 + i\omega_n^2\eta sgn\omega}\, d\omega = \lim_{\substack{R\to\infty\\ \delta\to 0}}\int_{C_\omega} f(\omega)d\omega + \lim_{\substack{R\to\infty\\ \delta\to 0}}\int_{C_{-\omega}} f(\omega)d\omega, \tag{35}$$

one has

$$2\pi m\, h\,(t) = -\lim_{R\to\infty}\int_{C_{\gamma_1}} f(z)dz - \lim_{R\to\infty}\int_{C_{\gamma_2}} f(z)dz - \lim_{R\to\infty}\int_{C_\delta} f(i\sigma)d(i\sigma) \tag{36}$$
$$-\lim_{R\to\infty}\int_{C_{-\delta}} f(i\sigma)d(i\sigma) + 2\pi i\,(Res.\,(f,z_1) + Res.\,(f,z_4))$$

As $f(z) = \frac{e^{izt}}{\omega_n^2 - z^2 + i\omega_n^2\,\eta sgn\,(Re(z))}$, the residues are given by (from Eq. (30)):

$$Res.\,(f,z_1) = -\frac{e^{iz_1t}}{2z_1} \qquad Res.\,(f,z_4) = -\frac{e^{iz_4t}}{2z_4} \tag{37}$$

As the first two integrands on the r.h.s. of Eq. (36) are of order $R\big/(1+R^2)$, they tend to zero as R tends to infinity; having this into account and substituting Eqs (34) and (37) in Eq. (36), it follows that

$$2\pi m\, h\,(t) = -\lim_{R\to\infty} i\left(\int_R^0 \frac{e^{-\sigma t}}{\sigma^2+z_1^2}\, d\sigma + \int_0^R \frac{e^{-\sigma t}}{\sigma^2+z_4^2}\, d\sigma\right) - \pi i\left(\frac{e^{iz_1t}}{z_1} + \frac{e^{iz_4t}}{z_4}\right) \tag{38}$$

Substituting the values of z_1 and z_4, and after some development, it is not difficult to obtain:

$$m\,\omega_n h\,(t) = \frac{e^{-\omega_n at}}{a^2+b^2}\,(b\sin\omega_n bt - a\cos\omega_n bt) + \frac{\omega_n^3\eta}{\pi}\lim_{R\to\infty}\int_0^R \frac{e^{-\sigma t}}{\left(\sigma^2+\omega_n^2\right)^2 + \omega_n^4\eta^2}\, d\sigma \tag{39}$$

The residual integral in Eq. (39) cannot be evaluated analytically. However, it converges by an order of $1\big/R^3$ as $R\to\infty$ and decays exponentially in time. Therefore, it is possible to have an upper-bound for this integral at $t=0$ [11]. Let us call it $I(0)$:

$$I(0) = \frac{\omega_n^3\eta}{\pi}\lim_{R\to\infty}\int_0^R \frac{1}{\left(\sigma^2+\omega_n^2\right)^2+\omega_n^4\eta^2}\, d\sigma \tag{40}$$

In terms of z_1 and expanding in partial fractions, $I(0)$ can be expressed as:

$$I(0) = -\frac{\omega_n}{2\pi}\, Im\left(\frac{1}{i\,z_1}\lim_{R\to\infty}\left(\int_0^R \frac{1}{\sigma - i\,z_1}\, d\sigma - \int_0^R \frac{1}{\sigma + i\,z_1}\, d\sigma\right)\right) \tag{41}$$

and in this form, it can be directly calculated. Let $u = \sigma - iz_1 = \sigma + \omega_n a - i\omega_n b$ and $v = \sigma + iz_1 = \sigma - \omega_n a + i\omega_n b$:

$$\begin{aligned} I(0) &= -\tfrac{\omega_n}{2\pi}\, Im\left(\tfrac{1}{i\,z_1}\lim_{R\to\infty}\left[\ln|u| + i\, tg^{-1}\tfrac{Im(u)}{Re(u)} - \left(\ln|v| + i\, tg^{-1}\tfrac{Im(v)}{Re(v)}\right)\right]_0^R\right) \\ &= -\tfrac{\omega_n}{2\pi}\, Im\left(\tfrac{1}{i\,z_1}\lim_{R\to\infty}\left[\ln\tfrac{|u|}{|v|} + i\, tg^{-1}\tfrac{-\omega_n b}{\sigma+\omega_n a} - i\, tg^{-1}\tfrac{\omega_n b}{\sigma-\omega_n a}\right]_0^R\right) \\ &= -\tfrac{\omega_n}{2\pi}\, Im\left(\tfrac{1}{i\,z_1}\left(0 - i\left(tg^{-1}\tfrac{-b}{a} - tg^{-1}\tfrac{b}{-a}\right)\right)\right) = \tfrac{\omega_n}{2\pi}\, Im\left(\tfrac{1}{z_1}\left(-\theta - (-\theta+\pi)\right)\right) = \tfrac{a}{2(a^2+b^2)} \end{aligned} \tag{42}$$

Thus, Eq. (39), at $t = 0$, leads to:

$$m\omega_n \, h\,(0) = -\frac{a}{2\,(a^2 + b^2)} \tag{43}$$

which demonstrates the non-causal behaviour of the hysteretic model. However, this negative value is very small for small damping factors, approaching $-\eta/4$ for $\eta^2 \ll 1$. Moreover, as $I(t)$ dies out, the response approaches:

$$h\,(t) = \frac{e^{-\omega_n a\, t}}{m\,\omega_n\,(a^2 + b^2)}\,(b\sin\omega_n bt - a\cos\omega_n bt) \tag{44}$$

3.2. The mixed damping case

For a single degree of freedom with both viscous and hysteretic damping subjected to a harmonic force, the frequency response function, for $\omega > 0$, is

$$H(\omega) = \frac{1/m}{\omega_n^2 - \omega^2 + i\,\left(2\xi\omega_n\omega + \omega_n^2\,\eta\right)} \tag{45}$$

Once again, in order to obtain the impulse response function via an inverse Fourier transform, one must have a Hermitian function in order to have a real response and to contemplate the correct phase angle between response and force. Introducing again the *signum* function for the hysteretic damping factor,

$$2\pi m\,h\,(t) = \int_{-\infty}^{+\infty} \frac{e^{i\,\omega\,t}}{\omega_n^2 - \omega^2 + i\,\left(2\xi\omega_n\omega + \omega_n^2\,\eta sgn\omega\right)}d\omega \tag{46}$$

Note that for the mixed damping case, Ribeiro et al. [16] obtained the following solution:

$$h\,(t) = \left(-i\frac{1}{m\,\omega_n b}\right)e^{-\omega_n(\xi+a)\,t}e^{i\omega_n b\,t} \tag{47}$$

from which the real part would be the physical response:

$$Re\,(h\,(t)) = \frac{1}{m\,\omega_n b}e^{-\omega_n(\xi+a)\,t}\sin\omega_n b\,t \tag{48}$$

where a and $b$are now given by:

$$a = \sqrt{\frac{-\left(1-\xi^2\right) + \sqrt{\left(1-\xi^2\right)^2 + \eta^2}}{2}} \quad b = \sqrt{\frac{1-\xi^2 + \sqrt{\left(1-\xi^2\right)^2 + \eta^2}}{2}} \tag{49}$$

Eqs (47) and (48) are similar to Eqs (26) and (27). Going back to Eq. (46), one must again extend ω to the complex plane, making $z = \omega + i\sigma$. As before, one has two possibilities for the roots of the denominator:

$$z^2 - i2\xi\omega_n\,z - \omega_n^2\,(1 + i\eta) = 0 \quad \text{for} \quad \omega > 0 \tag{50a}$$

$$z^2 - i2\xi\omega_n\,z - \omega_n^2\,(1 - i\eta) = 0 \quad \text{for} \quad \omega < 0 \tag{50b}$$

which are:

$$\begin{array}{l} z_1 = \omega_n b + i\,\omega_n(\xi + a) \\ z_2 = -\omega_n b + i\,\omega_n(\xi - a) \end{array} \quad \text{for} \quad \omega > 0 \quad \text{and} \quad \begin{array}{l} z_3 = \omega_n b + i\omega_n\,(\xi - a) \\ z_4 = -\omega_n b + i\omega_n\,(\xi + a) \end{array} \quad \text{for} \quad \omega < 0 \tag{51}$$

where a and b are given by Eq. (49). One has again four poles (Fig. 7). Note that one must have $a > \xi$, which means $\eta > 2\xi$. Otherwise, there would be 4 poles in the upper-half plane and that would not replicate the viscously damped case when $\eta = 0$.

This case is very similar to the hysteretic one, as there is also a singularity at $\omega = 0$, precluding a closed form solution. Following the same steps as before, one ends up with a similar expression to Eq. (38):

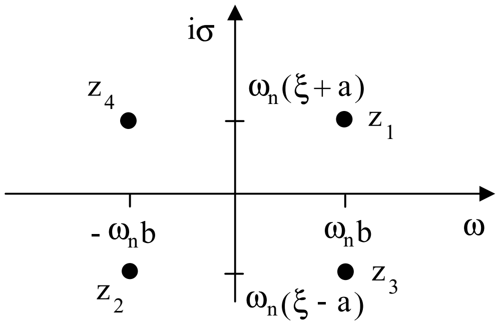


Fig. 7. Poles z_1, z_2, z_3 and z_4 for the mixed damping case.

$$2\pi m\, h\,(t) = -\lim_{R\to\infty} i \left(\int_R^0 \frac{e^{-\sigma t}}{\sigma^2 - 2\xi\omega_n\sigma + \omega_n^2(1+i\eta)}\, d\sigma + \int_0^R \frac{e^{-\sigma t}}{\sigma^2 - 2\xi\omega_n\sigma + \omega_n^2(1-i\eta)}\, d\sigma \right) \\ -\pi i \left(\frac{e^{i\,z_1\,t}}{z_1 - i\xi\omega_n} + \frac{e^{i\,z_4\,t}}{z_4 - \xi\omega_n} \right) \tag{52}$$

And thus, after some manipulation,

$$m\,\omega_n h\,(t) = \frac{e^{-\omega_n(\xi+a)\,t}}{a^2+b^2}\,(b\sin\omega_n bt - a\cos\omega_n bt) + \frac{\omega_n^3\eta}{\pi} \lim_{R\to\infty} \int_0^R \frac{e^{-\sigma t}}{\left(\sigma^2 - 2\xi\omega_n\sigma + \omega_n^2\right)^2 + \omega_n^4\eta^2}\, d\sigma \tag{53}$$

The upper-bound of the integral in Eq. (53) at $t = 0$ is now:

$$I(0) = \frac{\omega_n^3\eta}{\pi} \lim_{R\to\infty} \int_0^R \frac{1}{\left(\sigma^2 - 2\xi\omega_n\sigma + \omega_n^2\right)^2 + \omega_n^4\eta^2}\, d\sigma \\ = \frac{1}{2\pi} Im\left\{ \frac{1}{2\,(a-ib)} \lim_{R\to\infty} \int_0^R \left(\frac{1}{\sigma - \omega_n\,(\xi-a) - i\omega_n b} - \frac{1}{\sigma - \omega_n\,(\xi+a) + i\omega_n b} \right) d\sigma \right\} \\ = \frac{1}{2\pi} Im\left\{ \frac{1}{2\,(a-ib)} \left[\lim_{R\to\infty} \left(\ln\frac{|R - \omega_n\,(\xi-a) - i\omega_n b|}{|R - \omega_n\,(\xi+a) + i\omega_n b|} + i\left(tg^{-1}\frac{-\omega_n b}{R - \omega_n\,(\xi-a)} \right.\right.\right.\right. \\ \left.\left.\left.\left. -tg^{-1}\frac{\omega_n b}{R - \omega_n\,(\xi+a)} \right)\right) + \ln\frac{|-(\xi+a) + ib|}{|-(\xi-a) - ib|} + i\left(-tg^{-1}\frac{-b}{-(\xi-a)} + tg^{-1}\frac{b}{-(\xi+a)} \right) \right] \right\} \\ = \frac{1}{4\pi\,(a^2+b^2)} \left(\frac{1}{2} b\, \ln \frac{(\xi+a)^2 + b^2}{(\xi-a)^2 + b^2} + a\,\Delta \right) \tag{54}$$

and therefore, at $t = 0$ one has:

$$m\,\omega_n h\,(0) = \frac{-a}{a^2+b^2} + \frac{1}{4\pi\,(a^2+b^2)} \left(\frac{1}{2} b\, \ln \frac{(\xi+a)^2 + b^2}{(\xi-a)^2 + b^2} + a\,\Delta \right) \tag{55}$$

For small values of the damping factors $\Delta \approx \pi$, and Eq. (55) becomes:

$$m\omega_n\, h\,(0) \approx -\frac{3a}{4\,(a^2+b^2)} \tag{56}$$

As $I(t)$ dies out, the response approaches:

$$h\,(t) = \frac{e^{-\omega_n(\xi+a)\,t}}{m\,\omega_n\,(a^2+b^2)}\,(b\sin\omega_n bt - a\cos\omega_n bt) \tag{57}$$

It is obvious that expression Eq. (57) encompasses the particular cases of the viscous and hysteretic models.

4. Alternative: To find an equivalent viscous model

One possibility to circumvent the problem of obtaining a free response with constant hysteretic damping is to find an equivalent viscous damping model. One way to do this is to state that the roots of the denominator of the frequency response function containing the hysteretic damping term are the same as for an equivalent viscous damped case [11]. Let us take again the mixed damping case in the complex plane, where

$$H(z) = \frac{1/m}{\omega_n^2 - z^2 + i\,(2\xi\omega_n z + \omega_n^2\,\eta sgn\,(Re(z)))} \tag{58}$$

The roots are given by Eqs (51). For the equivalent system,

$$H_{eq}(z) = \frac{1/m}{\omega_{n_{eq}}^2 - z^2 + i\,2\xi_{eq}\omega_{n_{eq}}\,z} \tag{59}$$

with the roots:

$$z_{1_{eq}} = \omega_{n_{eq}}\sqrt{1-\xi_{eq}^2} + i\,\xi_{eq}\,\omega_{n_{eq}} \qquad z_{2_{eq}} = -\omega_{n_{eq}}\sqrt{1-\xi_{eq}^2} + i\,\xi_{eq}\,\omega_{n_{eq}} \tag{60}$$

The roots of the equivalent system stay in the upper half-plane, and so they must be equaled to z_1 and z_4, respectively, which corresponds to assume $\xi < a$, i.e., $\eta > 2\xi$. The equivalent parameters are

$$\omega_{n_{eq}} = \omega_n\sqrt{(\xi+a)^2 + b^2} \qquad \xi_{eq} = \frac{\xi + a}{\sqrt{(\xi+a)^2 + b^2}} \tag{61}$$

One will see in the next sub-section the implications of this result in the impulse response function.

4.1. The IRF for an equivalent viscous damping model

From Eq. (11) it is clear that the impulse response function is given by

$$h(t) = \frac{1}{m\,\omega_{n_{eq}}\sqrt{1-\xi_{eq}^2}}e^{-\xi_{eq}\omega_{n_{eq}}t}\sin\left(\omega_{n_{eq}}\sqrt{1-\xi_{eq}^2}\right)t \tag{62}$$

But from Eqs (51) and (60),

$$\omega_{n_{eq}}\sqrt{1-\xi_{eq}^2} = \omega_n b\xi_{eq} \qquad \xi_{eq}\omega_{n_{eq}} = \omega_n\,(\xi + a) \tag{63}$$

Therefore,

$$h(t) = \frac{1}{m\,\omega_n b}e^{-\omega_n(\xi+a)\,t}\sin\omega_n b\,t \tag{64}$$

But this is precisely the result of Eq. (48), obtained by Ribeiro et al. [16] (and of course equals Eq. (27) when there is only hysteretic damping). So, one concludes that the results proposed by those authors correspond in fact to an equivalent viscously damped model. That justifies the causal behaviour of the solutions. The differential equation corresponding to the free vibration of such a system is:

$$m\,\ddot{x} + 2\,m\,\omega_n\,(\xi + a)\,\dot{x} + k\left((\xi+a)^2 + b^2\right)\,x = 0 \tag{65}$$

which, for the hysteretic damping case becomes (after applying Eq. (6)):

$$m\,\ddot{x} + 2\,m\,\omega_n\sqrt{\frac{-1+\sqrt{1+\eta^2}}{2}}\,\dot{x} + k\sqrt{1+\eta^2}\,x = 0 \tag{66}$$

4.2. The logarithmic decrement

Note that the logarithmic decrement is the natural logarithm of the damping rate (rate between two consecutive amplitudes):

$$\delta = \ln \frac{x_i}{x_{i+1}} = 2\pi \frac{\xi_{eq}}{\sqrt{1-\xi_{eq}^2}} = 2\pi \frac{\xi + a}{b} \tag{67}$$

In the case of hysteretic damping, $\delta = 2\pi a/b$ can be used to evaluate the damping factor η.

5. Conclusions

From the study developed in this paper, one can draw the following conclusions:

– only a real solution of the differential equation describing the free vibration of a system makes sense from the physical point of view;
– if the homogeneous differential equation has a complex solution, as assumed in [15], the application of complex initial conditions makes sense. The mistake in [15] and [16] is that neither the real part of those conditions represents the physical measured quantities, nor the physical response is the real part of the complex solution;
– from points 1 and 2, it can be concluded that for a physically real system the homogeneous differential equation cannot have a complex solution and therefore the genuine impulse response function can only come from a Hermitian frequency response function;
– the reason why the real solutions proposed in [15] and [16] correspond to a causal behaviour and make sense is because they are – in fact – the solutions of an equivalent viscously damped system, and not the solutions of the initial problem.

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References

[1] B.J. Lazan, *Damping of Materials and Members in Structural Mechanics*, Oxford: Pergamon Press, 1968.
[2] W.W. Soroka, Notes on the relations between Viscous and Structural Damping Coefficients, *Journal of the Aeronautical Sciences* **16** (1949), 409–410, 448.
[3] N.O. Miklestad, The Concept of Complex Damping, *Journal of Applied Mechanics* **19** (1952), 284–286.
[4] R.E.D. Bishop, The Treatment of Damping Forces in Vibration Theory, *Journal of the Aeronautical Society* **59**(539) (1955), 738–742.
[5] T.J. Reid, Free Vibration and Hysteretic Damping, *Journal of the Royal Aeronautical Society* **60** (1956), 283.
[6] P. Lancaster, Free Vibration and Hysteretic Damping, *Journal of the Royal Aeronautical Society* **64** (1960), 229.
[7] T.K. Caughey, Vibration of Dynamic Systems with Linear Hysteretic Damping, *Proceedings of the 4th U.S. National Congress on Applied Mechanics, ASME* New York, (1962), 87–97.
[8] S.H. Crandall, Dynamic Response of Systems with Structural Damping, in: *Air, Space and Instruments*, S. Lees, ed., MaGraw-Hill, New York, 1963, pp. 183–193.
[9] S.H. Crandall, The Role of Damping in Vibratioin Theory, *Journal of Sound and Vibration* **11** (1970), 3–18.
[10] J.A. Inaudi and J.M. Kelly, Linear Hysteretic Damping and the Hilbert Transform, *Journal of Engineering Mechanics* **121**(5) (1995), 626–632.
[11] L. Gaul, S. Bohlen and S. Kelly, Linear Hysteretic Damping and the Hillbert Transform, *Journal ofEngineering Mechanics* **121**(5) (1995), 626–632.
[12] H.K. Milne, The Impulse Response Function of a Single Degree of Freedom System with Hysteretic Damping, *Journal of Sound and Vibration* **100**(4) (1985), 590–593.
[13] N. Makris and J. Zhang, Time-domain viscoelastic analysis of earth structures, *Earthquake Engineering and Structural Dynamics* **29**(6) (2000), 745–768.
[14] M.A. Biot, Linear thermodynamics and the mechanics of solids, *Proceedings of the 3rd US National Congress of Applied Mechanics* (1958), 1–18.

[15] A.M.R. Ribeiro, N.M.M. Maia, J.M.M. Silva, M. Freitas, L. Reis, Free Vibration Response Using the Constant Hysteretic Damping Model, *Proceedings of the XIth Int Conf on Vibration Engineering* Timisoara, Roménia, (2005), 65–70.
[16] A.M.R. Ribeiro, N.M.M. Maia and J.M.M. Silva, On the Modelling of Damping in Structural Vibrations, *Proceedings of ISMA 2006*, Noise and Vibration Engineering, Leuven, Belgium, 2006.
[17] E.C. Sorokin, Method for account of non-elastic strength of material in construction vibration, in: *Structural Dynamics Researches*, Gosstroiizdat, 1951, (in Russian).
[18] E.C. Sorokin, *On Theory of Internal Damping in Elastic System Vibration*, Gosstroiizdat, 1960, (in Russian).
[19] D.I.G. Jones, The Impulse Response Function of a Damped Single Degree of Freedom System, *Journal of Sound and Vibration* **106**(2), (1986), 353–356.
[20] N. Makris, Causal hysteretic element, *Journal of Engineering Mechanics* **123**(11) (1997), 1209–1214.