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Synchronization of two non-identical coupled exciters in a non-resonant vibrating system of linear motion. Part I: Theoretical analysis

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Abstract. In this paper an analytical approach is proposed to study the feature of frequency capture of two non-identical coupled exciters in a non-resonant vibrating system. The electromagnetic torque of an induction motor in the quasi-steady-state operation is derived. With the introduction of two perturbation small parameters to average angular velocity of two exciters and their phase difference, we deduce the Equation of Frequency Capture by averaging two motion equations of two exciters over their average period. It converts the synchronization problem of two exciters into that of existence and stability of zero solution for the Equation of Frequency Capture. The conditions of implementing frequency capture and that of stabilizing synchronous operation of two motors have been derived. The concept of torque of frequency capture is proposed to physically explain the peculiarity of self-synchronization of the two exciters. An interesting conclusion is reached that the moments of inertia of the two exciters in the Equation of Frequency Capture reduce and there is a coupling moment of inertia between the two exciters. The reduction of moments of inertia and the coupling moment of inertia have an effect on the stability of synchronous operation.

Keywords: Self-synchronization, vibrating system, frequency capture, stability

1. Introduction

Synchronization of two self-excited oscillatory system is a classical problem in the theory of synchronization [8]. The earliest detailed accounts on synchronized motion was made by Huygens [6], who observed that two clock pendulums suspended from stiff wooden beams could run in a steady-state and move in opposition to each other at the same angular velocity. Subsequently, Van der Pol [12] observed the synchronization of certain electrical-mechanical system and Rayleigh [9] found that two organ tubes could produce a synchronized sound when the outlets are close to each other. Belhman [2,3] proposed the theory of self-synchronization of vibrating machinery with double exciters. Recently, Teufel and Torger [11] studied the synchronization of two aerodynamically excited pendulums and Czolczynski et al. [5] investigated the synchronization of two non-identical self-excited oscillators suspended on the elastic structure by computer simulation.

Vibrating machines are categorized into non-resonant system and near-resonant system [16]. The operation frequency of a non-resonant system is far over from its natural frequency and that of a near-resonant system is close to its natural one. In a vibrating system with two-motor drives, the operation frequencies of two motors will keep a particular ratio due to the motion of the vibrating system. This ratio can be an integer or a fraction. The process that

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the system implements this particular ratio of operation frequency of two motors is called the frequency capture, and the result of frequency capture is called synchronization. According to the value of frequency ratio, the frequency capture can be divided into three groups, i.e., the foundational frequency capture (ratio = 1), the times frequency capture (ratio is an integer and > 1) and the fractional frequency capture (ratio is a fraction, p/q, p and q are coprime integrals). Foundational frequency capture can occur in a linear vibrating system and times frequency capture or fractional frequency capture can occur in a nonlinear vibrating system [16].

Although small parameter and averaging method developed by Blekhman [2,3] and Wen [13–16] can allow for a better understanding and theoretical explanation of the mechanism of self-synchronization, there are some deficiencies in current method. First, this method ignores the feature of frequency capture, i.e., the average angular velocity of two motors is assumed to be a constant and only phase difference between two exciters is considered to a variable small parameter. Two differential equations of motion of the exciters were merged into one differential equation of phase difference, and it is only suitable to analyze the synchronization of a vibrating system with two identical coupled exciters driven by two induction motors [2,3,13–16]. When there are significant differences in the parameters of two induction motors, the synchronization of the system can not even be implemented [16]. Next, the dynamic characteristics of induction motor are less considered. Actually, self-synchronization of a vibrating system stems from the effect of electric-mechanic motors coupling and the operation frequency of the system is dependent on the dynamic parameters of two induction motors.

In this paper, an analytical approach is employed to investigate the frequency capture of two non-identical coupled exciters in a non-resonant vibrating system. By introducing two variable perturbation parameters to average angular velocity of two exciters and their phase difference, the problem of synchronization is converted into that of existence and stability of zero solution for the Equation of Frequency Capture. The rest of this paper is organized as follows: Section 2 describes the dynamic model of the system. In Section 3, we derive the electromagnetic torque of an induction motor operating at the quasi-steady-state. In Section 4, we derive the Equation of Frequency Capture by averaging the equations of motion of two exciters over their average period, and deduce the condition of implementing frequency capture and that of stabilizing synchronous operation of the two exciters. The results of theoretical analysis are discussed in Section 5 and Section 6 shows our conclusions.

2. Equations of motion of a vibrating system

The dynamic model of a vibrating system is illustrated in Fig. 1, in which the springs are symmetrically connected to the body of the machine and two induction motors are symmetrically installed in the system. The two motors, which drive one eccentric lump to excite the system, respectively, are supplied with same electric source and rotate in opposite directions. When the phase difference between two exciters is zero and the masses of two lumps are equal, the motion of the vibrating system is a straight line in y-direction. This system is called a vibrating system of linear motion. $oxy$is the fixed frame and its origin $o$is the balance point of centroid of the machine body. Using the Langrangian Equations and choosing the variables $x, y, \psi, \varphi_1$and $\varphi_2$as the generalized coordinates, the differential equations of motion of the system can be derived as [3,15,16]:

$$\begin{aligned}
M\ddot{x} + f_x\dot{x} + k_x x &= m_1 r_1(\dot{\varphi}_1^2 \sin\varphi_1 + \ddot{\varphi}_1 \sin\varphi_1) - m_2 r(\dot{\varphi}_2^2 \cos\varphi_2 + \ddot{\varphi}_2 \sin\varphi_2) \\
M\ddot{y} + f_y\dot{y} + k_y y &= m_1 r(\dot{\varphi}_1^2 \sin\varphi_1 - \ddot{\varphi}_1 \cos\varphi_1) + m_2 r(\dot{\varphi}_2^2 \sin\varphi_2 - \ddot{\varphi}_2 \cos\varphi_2) \\
J\ddot{\psi} + f_\psi\dot{\psi} + k_\psi \psi &= (T_{e2} - T_{e1}) - m_1 r l_0[\dot{\varphi}_1^2 \sin(\varphi_1 + \beta) - \ddot{\varphi}_1 \cos(\varphi_1 + \beta)] \\
&\quad + m_2 r l_0[\dot{\varphi}_2^2 \sin(\varphi_2 + \beta) - \ddot{\varphi}_2 \cos(\varphi_2 + \beta)] \\
J_{01}\ddot{\varphi}_1 + f_{d1}\dot{\varphi}_1 &= T_{e1} - m_1 r[\ddot{y}\cos\varphi_1 - \ddot{x}\sin\varphi_1 - l_0\ddot{\psi}\cos(\varphi_1 + \beta)] \\
J_{02}\ddot{\varphi}_2 + f_{d2}\dot{\varphi}_2 &= T_{e2} - m_2 r[\ddot{y}\cos\varphi_2 + \ddot{x}\sin\varphi_2 + l_0\ddot{\psi}\cos(\varphi_2 + \beta)] \qquad (1)
\end{aligned}$$

where M is the mass of the vibrating system (including that of two eccentric lamps, m_1 and m_2); J the moment of inertia of the machine body; k_x, k_y and k_ψ the spring constants and f_x, f_y and f_ψ the damping constants in x-, y- and ψ-directions, respectively; J_{01} and J_{02} the moments of inertia of two eccentric rotors; T_{e1} and T_{e2} the electromagnetic torques of two motors; f_{d1} and f_{d2} the damping coefficients of motor axes; $(\dot{\ })$ and $(\ddot{\ })$ denote d/dt and d^2/dt^2.

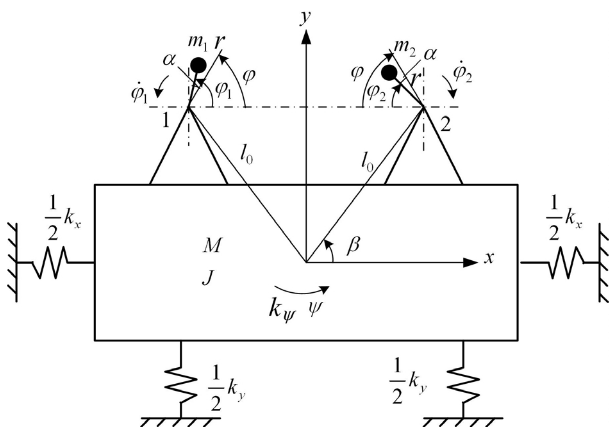

Fig. 1. Dynamic model of a vibrating system with two motors rotating in opposite directions.

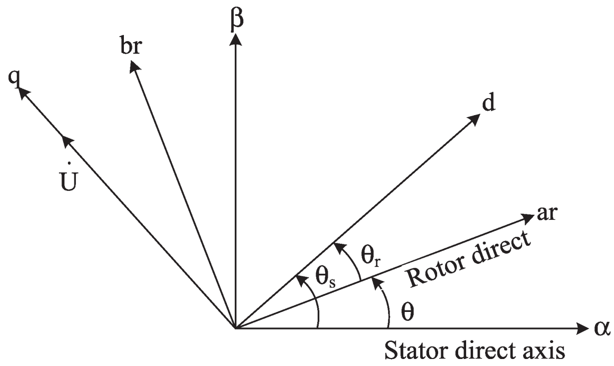

Fig. 2. Reference frames and space vector representation.

3. Electromagnetic torque of an induction motor in the quasi-steady-state operation

Figure 2 shows three different reference frames for a three-phase squirrel cage induction motor: stator reference frame (α, β); rotor reference frame (ar, br) and arbitrary reference frame (d, q). If the direction of $q-$axis is assumed to be that of the stator voltage vector, the frame is called the synchronous frame of stator voltage [4]. By referring to the synchronous frame of stator voltage, the dynamic model of a three-phase, balanced, single excited induction motor can be expressed as [4]:

$$\begin{aligned} 0 &= R_\mathrm{s} i_\mathrm{ds} + \frac{d\phi_\mathrm{ds}}{\mathrm{d}t} - \omega_\mathrm{s}\phi_\mathrm{qs} \\ U_\mathrm{s0} &= R_\mathrm{s} i_\mathrm{qs} + \frac{\mathrm{d}\phi_\mathrm{qs}}{\mathrm{d}t} + \omega_\mathrm{s}\phi_\mathrm{ds} \\ 0 &= R_\mathrm{r} i_\mathrm{dr} + \frac{\mathrm{d}\phi_\mathrm{dr}}{\mathrm{d}t} - \phi_\mathrm{qr}\frac{\mathrm{d}\theta_\mathrm{r}}{\mathrm{d}t} \\ 0 &= R_\mathrm{r} i_\mathrm{qr} + \frac{\mathrm{d}\phi_\mathrm{qr}}{\mathrm{d}t} + \phi_\mathrm{dr}\frac{\mathrm{d}\theta_\mathrm{r}}{\mathrm{d}t} \end{aligned} \tag{2}$$

and

$$\begin{aligned} \phi_\mathrm{ds} &= L_\mathrm{s} i_\mathrm{ds} + L_\mathrm{m} i_\mathrm{dr} \\ \phi_\mathrm{qs} &= L_\mathrm{s} i_\mathrm{qs} + L_\mathrm{m} i_\mathrm{qr} \end{aligned}$$

$$\phi_{\mathrm{dr}} = L_{\mathrm{r}} i_{\mathrm{dr}} + L_{\mathrm{m}} i_{\mathrm{ds}}$$
$$\phi_{\mathrm{qr}} = L_{\mathrm{r}} i_{\mathrm{qr}} + L_{\mathrm{m}} i_{\mathrm{qs}} \tag{3}$$

where U_{s0} is the amplitude of the stator voltage vector; i_{ds} and i_{qs} the d- and q-axis stator currents; i_{dr} and i_{qr} the d- and q-axis rotor currents; R_{s} the stator resistance and R_{r} the rotor resistance; ϕ_{ds} and ϕ_{qs} the d- and q-axis stator flux linkages; ϕ_{dr} and ϕ_{qr} the d- and q-axis rotor flux linkages; L_{s} the stator inductance; L_{r} the rotor inductance; L_{m} the mutual inductance; ω_{s} the synchronous electric angular velocity.

The electromagnetic torque of an induction motor can be expressed as [4]:

$$T_{\mathrm{e}} = n_{\mathrm{p}} \frac{L_{\mathrm{m}}}{L_s} (i_{\mathrm{dr}} \phi_{\mathrm{qs}} - i_{\mathrm{qr}} \phi_{\mathrm{ds}}) \tag{4}$$

where n_{p} is the number of pole pairs.

When the motor operates at the steady-state, the voltage, current or field variables are constant [4]. Based on a negligible R_{s}, $\psi_{\mathrm{qs}} \approx 0$ can be deduced from Eq. (2) [4]. Substituting Eq. (3) into Eq. (2) and eliminating i_{ds}, i_{qs}, ψ_{dr} and ψ_{qr}, the equations of an induction motor operating at the steady-state can be simplified as follows

$$\begin{aligned} &\phi_{\mathrm{ds0}} = \tfrac{U_{\mathrm{s0}}}{\omega_{\mathrm{s}}} \\ &i_{\mathrm{dr0}} - \omega_s s_0 \sigma \tau_{\mathrm{r}} i_{\mathrm{qr0}} = 0 \\ &\omega_{\mathrm{s}} s_0 \sigma \tau_{\mathrm{r}} i_{\mathrm{dr0}} + i_{\mathrm{qr0}} = -\tfrac{L_m}{L_{\mathrm{s}}} \tfrac{1}{R_{\mathrm{r}}} \omega_{\mathrm{s}} s_0 \phi_{\mathrm{ds0}} \end{aligned} \tag{5}$$

where $s_0 = 1 - n_{\mathrm{p}} \omega_{\mathrm{m0}} / \omega_{\mathrm{s}}$ denotes the motor slip at the steady-state and ω_{m0} is its mechanical angular velocity of the rotor; i_{dr0}, i_{qr0}, ϕ_{ds0} and ϕ_{qs0} the constant values of current and field variables, respectively; $\tau_{\mathrm{r}} = L_{\mathrm{r}} / R_{\mathrm{r}}$ refers to the rotor time constant and $\sigma = 1 - L_{\mathrm{m}}^2 / L_{\mathrm{s}} L_{\mathrm{r}}$ the leakage coefficient.

Solving i_{qr0} from Eq. (5) and substituting ϕ_{ds0} andi_{qr0} into Eq. (4), we obtain the electromagnetic torque of the motor at the steady-state as the following:

$$T_{\mathrm{e0}} = n_{\mathrm{p}} \frac{L_{\mathrm{m}}^2 U_{s0}^2}{L_{\mathrm{s}}^2 \omega_{\mathrm{s}} R_{\mathrm{r}}} \frac{s_0}{1 + (\sigma \tau_{\mathrm{r}} \omega_{\mathrm{s}} s_0)^2} \tag{6}$$

If the variation coefficient of mechanical angular velocity of the motor's rotor is ε during the steady-state operation of the vibration system, i.e., $\omega_{\mathrm{m}} = (1+\varepsilon)\omega_{\mathrm{m0}}$ and $s_0 = 1 - n_{\mathrm{p}} \omega_{\mathrm{m0}} (1+\varepsilon)/\omega_{\mathrm{s}}$. Substituting $s_0 = 1 - n_{\mathrm{p}} \omega_{\mathrm{m0}} (1+\varepsilon)/\omega_{\mathrm{s}}$ into Eq. (6) as well as applying the first-order Taylor expression around ω_{m0}, we obtain

$$T_e = T_{\mathrm{e0}} - k_{\mathrm{e0}} \varepsilon \tag{7}$$

where T_{e0} is the steady state electromagnetic torque expressed in Eq. (6); k_{e0} is called the velocity stiffness coefficient of the motor operating at the steady-state and expressed as:

$$k_{\mathrm{e0}} = n_{\mathrm{p}}^2 \frac{L_{\mathrm{m}}^2 U_{10}^2}{L_{\mathrm{s}}^2 \omega_{\mathrm{s}} R_{\mathrm{r}}} \frac{1 - \sigma^2 \tau_{\mathrm{r}}^2 \omega_{\mathrm{s}}^2 s_0^2}{(1 + \sigma^2 \tau_{\mathrm{r}}^2 \omega_{\mathrm{s}}^2 s_0^2)^2} \frac{\omega_{\mathrm{m0}}}{\omega_{\mathrm{s}}} \tag{8}$$

Equation (7) is called the electromagnetic torque of an induction motor at the quasi-steady-state operation in this paper.

4. Frequency capture and stability of synchronous operation

4.1. Equation of Frequency capture of the vibrating system

If the average phase and angular velocity of two eccentric rotors are φ and $\omega_m(t)$ respectively when a vibrating system operates at the steady-state, we have

$$\varphi = \varphi_0 + \int_{t_0}^{t} \omega_{\mathrm{m}}(t) \mathrm{d}t \tag{9}$$

where φ_0 denotes the average phase when the system begins to operate at the steady-state.

Assuming that the phase of eccentric rotor 1 leads φ by α and that of eccentric rotor 2 lags φ by α, then we have:

$$\begin{aligned}\varphi_1 &= \varphi + \alpha \\ \varphi_2 &= \varphi - \alpha\end{aligned} \tag{10}$$

Because the motion of the vibration is periodic, the change of mechanical angular velocity of the motors is also periodic. If the least common multiple period of rotors 1 and 2 is T_{a0}, the average value of two rotors' average velocity during the interval of T_{a0} must be a constant:

$$\omega_{\mathrm{m0}} = \frac{1}{T_{\mathrm{a0}}}\int_0^{T_{\mathrm{a0}}} \omega_{\mathrm{m}}(t)\mathrm{d}t = \text{constant} \tag{11}$$

Assuming the instantaneous variation coefficients of $\dot{\varphi}$ and $\dot{\alpha}$ are $\varepsilon_1$and ε_2(ε_1 and ε_2 are the functions of time t), respectively, i.e., $\dot{\varphi} = (1+\varepsilon_1)\omega_{\mathrm{m0}}$, $\dot{\alpha} = \varepsilon_2\omega_{\mathrm{m0}}$, differentiating Eq. (10) with respect to time t, we obtain

$$\begin{aligned}\dot{\varphi}_1 &= (1+\varepsilon_1+\varepsilon_2)\omega_{\mathrm{m0}} \\ \dot{\varphi}_2 &= (1+\varepsilon_1-\varepsilon_2)\omega_{\mathrm{m0}}\end{aligned} \tag{12}$$

If two motors operate at the same angular velocity, the average values of ε_1 and ε_2 over the single period $T_0 = 2\pi/\omega_{\mathrm{m0}}$ must be zero:

$$\begin{aligned}\bar{\varepsilon}_1 &= 0 \\ \bar{\varepsilon}_2 &= 0\end{aligned} \tag{13}$$

Because the slip of an induction motor is usually less than 0.08 at the steady-state [10], $\ddot{\varphi}_1$ and $\ddot{\varphi}_2$ in the first three Formulas of Eq. (1) can be neglected when a vibrating system operates at the steady-state [15]. Assuming the masses of two eccentric lamps, m_1 and m_2, are m_0 and ηm_0 $(0 < \eta \leqslant 1)$, respectively, and applying Eq. (12) into the first three Formulas of Eq. (1), we obtain

$$\begin{aligned}M\ddot{x} + f_x\dot{x} + k_x x &= m_0 r\omega_{\mathrm{m0}}^2[(1+\varepsilon_1+\varepsilon_2)^2\cos(\varphi+\alpha) - \eta(1+\varepsilon_1-\varepsilon_2)\cos(\varphi-\alpha)] \\ M\ddot{y} + f_y\dot{y} + k_y y &= m_0 r\omega_{\mathrm{m0}}^2[(1+\varepsilon_1+\varepsilon_2)^2\sin(\varphi+\alpha) + \eta(1+\varepsilon_1-\varepsilon_2)^2\sin(\varphi-\alpha)] \\ J\ddot{\psi} + f_\psi\ddot{\psi} + k_\psi\psi &= m_0 r\omega_{\mathrm{m0}}^2 l_0[-(1+\varepsilon_1+\varepsilon_2)^2\sin(\varphi+\alpha) + \eta(1+\varepsilon_1-\varepsilon_2)^2\sin(\varphi-\alpha)]\end{aligned} \tag{14}$$

For a non-resonant system, the operating frequency of the system is about (4 $\sim$ 5) times of its natural frequency and the damping constant is very small [15]. The amplitude of response of the system in x-direction caused by the exciter 1 can be expressed approximately as [15]:

$$x_0 = \frac{m_0 r}{M}\frac{1}{1-\omega_{\mathrm{nx}}^2/\omega_{\mathrm{m0}}^2(1+\varepsilon_1+\varepsilon_2)^2} \tag{15}$$

where $\omega_{\mathrm{n}x}$is the natural angular velocity of a vibrating system in x-direction.

Applying the first-order Taylor expression to Eq. (15) around ω_{m0}, we obtain

$$x_0 = \frac{m_0 r}{M}\frac{1}{1-\omega_{\mathrm{nx}}^2/\omega_{\mathrm{m0}}^2}[1 - \frac{2(\omega_{\mathrm{n}}/\omega_{\mathrm{m0}})^2}{1-\omega_{\mathrm{n}}^2/\omega_{\mathrm{m0}}^2}(\varepsilon_2+\varepsilon_1)] \tag{16}$$

When the induction motor operates at the steady-state, $|\varepsilon_1+\varepsilon_2| \leqslant 0.08$ and $\omega_{\mathrm{nx}}/\omega_{\mathrm{m0}} \leqslant 1/16$, therefore the second term at the right side of Eq. (16) can be neglected. Thus, the responses of Eq. (14) can be expressed as:

$$\begin{aligned}x &= -\frac{m_0 r}{m_x'}[\cos(\varphi+\alpha+\gamma_x) - \eta\cos(\varphi-\alpha+\gamma_x)] \\ y &= -\frac{m_0 r}{m_y'}[\sin(\varphi+\alpha+\gamma_y) + \eta\sin(\varphi-\alpha+\gamma_y] \\ \psi &= \frac{m_0 r l_0}{J'}[\sin(\varphi+\alpha+\beta+\gamma_\psi) - \eta\sin(\varphi+\beta-\alpha+\gamma_\psi)]\end{aligned} \tag{17}$$

where $m'_x = M - k_x/\omega_{\mathrm{m0}}^2$, $m'_y = M - k_y/\omega_{\mathrm{m0}}^2$ and $J' = J - k_\psi/\omega_{\mathrm{m0}}^2$ denote the calculated masses in x- and y-directions and the calculated moment of inertia of the machine body rotating about its centroid [16]; $\pi - \gamma_x$, $\pi - \gamma_y$ and $\pi - \gamma_\psi$ refer to the phase angles of the responses in x-, y- and ψ-directions, respectively.

Applying Eq. (17) into the last two Formulas of Eq. (1), and neglecting the high order terms of ε_1 and ε_2 as well as integrating them over $\varphi = 0 \sim 2\pi$, we have

$$
\begin{aligned}
&J_{01}\omega_{\mathrm{m0}}(\dot{\bar{\varepsilon}}_1 + \dot{\bar{\varepsilon}}_2) + f_{d1}\omega_{\mathrm{m0}}(1 + \bar{\varepsilon}_1 + \bar{\varepsilon}_2) = T_{\mathrm{e01}} - k_{\mathrm{e01}}(\bar{\varepsilon}_1 + \bar{\varepsilon}_2) - \bar{T}_{\mathrm{L1}} \\
&J_{02}\omega_{\mathrm{m0}}(\dot{\bar{\varepsilon}}_1 - \dot{\bar{\varepsilon}}_2) + f_{d2}\omega_{\mathrm{m0}}(1 + \bar{\varepsilon}_1 - \bar{\varepsilon}_2) = T_{\mathrm{e02}} - k_{\mathrm{e02}}(\bar{\varepsilon}_1 - \bar{\varepsilon}_2) - \bar{T}_{\mathrm{L2}}
\end{aligned} \tag{18}
$$

where

$$
\begin{aligned}
&T_{\mathrm{L1}} = d_{11}\dot{\bar{\varepsilon}}_1 + d_{12}\dot{\bar{\varepsilon}}_2 + d\bar{\varepsilon}_1 - d\bar{\varepsilon}_2 + d_{\mathrm{a}} + d_{\mathrm{f1}} \\
&T_{\mathrm{L2}} = d_{21}\dot{\bar{\varepsilon}}_1 + d_{22}\dot{\bar{\varepsilon}}_2 - d\bar{\varepsilon}_1 - d\bar{\varepsilon}_2 - d_{\mathrm{a}} + d_{\mathrm{f2}}
\end{aligned} \tag{19}
$$

and

$$
\begin{aligned}
d_{11} &= m_0^2 r^2 \omega_{\mathrm{m0}}(-W_{\mathrm{c0}} + \eta W_{\mathrm{c}}\cos 2\bar{\alpha})/2 \\
d_{12} &= m_0^2 r^2 \omega_{\mathrm{m0}}(-W_{\mathrm{c0}} - \eta W_{\mathrm{c}}\cos 2\bar{\alpha})/2 \\
d_{21} &= m_0^2 r^2 \omega_{\mathrm{m0}}(-\eta^2 W_{\mathrm{c0}} + \eta W_{\mathrm{c}}\cos 2\bar{\alpha})/2 \\
d_{22} &= m_0^2 r^2 \omega_{\mathrm{m0}}(\eta^2 W_{\mathrm{c0}} + \eta W_{\mathrm{c}}\cos 2\bar{\alpha})/2 \\
d_{\mathrm{f1}} &= m_0^2 r^2 \omega_{\mathrm{m0}}^2(W_{\mathrm{s0}} + \eta W_{\mathrm{s}}\cos 2\bar{\alpha})/2 \\
d_{\mathrm{f2}} &= m_0^2 r^2 \omega_{\mathrm{m0}}^2(\eta^2 W_{\mathrm{s0}} + \eta W_{\mathrm{s}}\cos 2\bar{\alpha})/2 \\
d &= 2d_{\mathrm{a}} = m_0^2 r^2 \omega_{\mathrm{m0}}^2 \eta W \sin 2\bar{\alpha} \\
W_{\mathrm{c0}} &= \cos\gamma_x/m'_x + \cos\gamma_y/m'_y + l_0^2\cos\gamma_\psi/J' \\
W_{\mathrm{c}} &= \cos\gamma_x/m'_x - \cos\gamma_y/m'_y + l_0^2\cos\gamma_\psi/J' \\
W_{\mathrm{s0}} &= \sin\gamma_x/m'_x + \sin\gamma_y/m'_y + l_0^2\sin\gamma_\psi/J' \\
W_{\mathrm{s}} &= -\sin\gamma_x/m'_x + \sin\gamma_y/m'_y - l_0^2\sin\gamma_\psi/J'
\end{aligned} \tag{20}
$$

Compared with the change of $\varphi(\dot{\varphi} = \omega_{\mathrm{m0}})$ with respect to time t, that of α, ε_1, ε_2, $\dot{\varepsilon}_1$ and $\dot{\varepsilon}_2$ are very small. Thus, these above five parameters are considered to be slow-changing parameters in this study. During the aforementioned integration over $\varphi = 0 \sim 2\pi$, they can be assumed to be the middle valves of their integration, i.e., $\bar{\alpha}\ldots$ and $\dot{\bar{\varepsilon}}_2$, respectively [8]. Because the damping constants of the system are very small, $\sin\gamma_x$, $\sin\gamma_y$ and $\sin\gamma_\psi$ can be considered to be zero in the coefficients of $\ldots$ and $\dot{\bar{\varepsilon}}_2$ in Eq. (20). Assembly of Eq. (18) in the following manner: adding two Formulas to get the first row, and subtracting two Formulas to obtain the second row as well as complementing the third row, $\dot{\bar{\alpha}} = \bar{\varepsilon}_2\omega_{\mathrm{m0}}$, we have

$$
\mathbf{A}\dot{\varepsilon} = \mathbf{B}\varepsilon + \mathbf{u} \tag{21}
$$

where $\dot{\varepsilon} = \{\bar{\varepsilon}_1 \quad \bar{\varepsilon}_2 \quad \bar{\alpha}\}^{\mathrm{T}}$, $\mathbf{u} = \{u_1 \quad u_2 \quad 0\}^{\mathrm{T}}$.

$$
\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \mathbf{0} \\ a_{21} & a_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b & b_{12} & \mathbf{0} \\ b_{21} & b & \mathbf{0} \\ bf0 & \omega_{\mathrm{m0}} & \mathbf{0} \end{bmatrix}
$$

and

$$
\begin{aligned}
a_{11} &= J_{01}\omega_{\mathrm{m0}} + J_{02}\omega_{\mathrm{m0}} + d_{11} + d_{21} \\
a_{12} &= J_{01}\omega_{\mathrm{m0}} - J_{02}\omega_{\mathrm{m0}} + d_{12} + d_{22} \\
a_{21} &= J_{01}\omega_{\mathrm{m0}} - J_{02}\omega_{\mathrm{m0}} + d_{11} - d_{21}
\end{aligned}
$$

$$a_{22} = J_{01}\omega_{\mathrm{m0}} - J_{02}\omega_{\mathrm{m0}} + d_{12} - d_{22}$$

$$b = -(K_1 + K_2), \quad b_{12} = -(K_1 - K_2 - 2d)$$

$$b_{22} = -(K_1 - K_2 + 2d), K_1 = k_{\mathrm{e01}} + f_{\mathrm{d1}}\omega_{\mathrm{m0}}$$

$$K_2 = k_{\mathrm{e02}} + f_{\mathrm{d2}}\omega_{\mathrm{m0}}$$

$$u_1 = T_{\mathrm{e01}} + T_{\mathrm{e02}} - d_{\mathrm{f1}} - d_{\mathrm{f2}} - (f_{\mathrm{d1}} + f_{\mathrm{d2}})\omega_{\mathrm{m0}}$$

$$u_2 = T_{\mathrm{e01}} - T_{\mathrm{e02}} - (d_{\mathrm{f1}} - d_{\mathrm{f2}}) - (f_{\mathrm{d1}} - f_{\mathrm{d2}})\omega_{\mathrm{m0}} - 2d_{\mathrm{a}}.$$

Equation (21) is called the Equation of Frequency Capture of the vibrating system.

4.2. *Conditions of implementing frequency capture*

Substituting Eq. (13) into Eq. (21) yields:

$$T_{\mathrm{e01}} + T_{\mathrm{e02}} - (d_{\mathrm{f1}} + d_{\mathrm{f2}}) - (f_{\mathrm{d1}} + f_{\mathrm{d2}})\omega_{\mathrm{m0}} = 0 \tag{22}$$

$$(T_{\mathrm{e01}} - T_{\mathrm{e02}}) - (d_{\mathrm{f1}} - d_{\mathrm{f2}}) - (f_{\mathrm{d1}} - f_{\mathrm{d2}})\omega_{\mathrm{m0}} - 2d_{\mathrm{a}} = 0 \tag{23}$$

In Eq. (22), the first two terms, $T_{\mathrm{e01}} + T_{\mathrm{e02}}$, are the sum of electromagnetic torques of two motors; the second two terms, $(d_{\mathrm{f1}} + d_{\mathrm{f2}})$, are the load torques that a vibrating system acts on two motors; the last term, $(f_{\mathrm{d1}} + f_{\mathrm{d2}})\omega_{\mathrm{m0}}$, are the sum of the damping torques of two motor axes. Equation (22) describes the torque balance of the vibrating system at the steady-state.

In Eq. (23), the first term, $(T_{\mathrm{e01}} - T_{\mathrm{e02}})$, is the difference of electromagnetic torque between two motors; the second term, $(d_{\mathrm{f1}} - d_{\mathrm{f2}})$, is that of the load torques that the vibrating system acts on two motors; the third term, $(f_{\mathrm{d1}} - f_{\mathrm{d2}})\omega_{\mathrm{m0}}$, is that of damping torque between two motor axes; and the last term, $2d_{\mathrm{a}}$, is the sum of the additional torque that the system acts on two motors, as shown in Eq. (19). If the electromagnetic torques and load torques (including the damping torque of the motor axis) of two motors are the same, their operation angular velocities must be the same. Thus, the last term, $2d_{\mathrm{a}}$, in Eq. (23) is the torque that the vibrating system acts on two motors to overcome the differences of electromagnetic torques and load torques between two motors, which is the key to implement the frequency capture and reach the synchronous operation of two motors. According to the expression of $2d_{\mathrm{a}}$in Eq. (20), we define the torque of frequency capture of the vibrating system as:

$$T_{\mathrm{Capture}} = m_0^2 r^2 \eta \omega_{\mathrm{m0}}^2 \left| W_{\mathrm{c}} \right| \tag{24}$$

and the difference in the residual torques of two motors is defined as

$$T_{\mathrm{Differnece}} = T_{\mathrm{Residual}}^1 - T_{\mathrm{Residual}}^2 \tag{25}$$

where T_{Residual}^1 and T_{Residual}^2 denote the residual torques of two motors operating at the steady-state, respectively, $T_{\mathrm{Residual}}^1 = T_{\mathrm{e01}} - d_{\mathrm{f1}} - f_{\mathrm{d1}}\omega_{\mathrm{m0}}$, $T_{\mathrm{Residual}}^2 = T_{\mathrm{e02}} - d_{\mathrm{f2}} - f_{\mathrm{d2}}\omega_{\mathrm{m0}}$.

Applying Eqs (24) and (25) into Eq. (23) and rearranging it, we obtain

$$2\alpha = \arcsin\frac{1}{D_{\mathrm{a}}} \tag{26}$$

$$D_{\mathrm{a}} = \frac{T_{\mathrm{Captrue}} sign(W_{\mathrm{c}})}{T_{\mathrm{Difference}}} \tag{27}$$

where D_{a} is called the synchronous index of the vibrating system [15,16].

From Eq. (26), it can be seen that the absolute value of the synchronization index must be greater than or equal to 1 to ensure a solution of 2α, i.e., $|D_{\mathrm{a}}| \geqslant 1$. Thus,

$$T_{\mathrm{Capture}} \geqslant |T_{\mathrm{Difference}}| \tag{28}$$

Inequation (28) demonstrates that the necessary condition of implementing the frequency capture is that the torque of frequency capture, T_{Capture}, is greater than or equal to the absolute value of difference in the residual torques of two motors, $|T_{\mathrm{Difference}}|$.

Substituting the relationship $s_0 = (\omega_{\mathrm{s}} - n_{\mathrm{p}}\omega_{\mathrm{m0}})/\omega_{\mathrm{s}}$, and T_{e01} and T_{e02} found by Eq. (6) into Eq. (22) and (23), we can obtain the nonlinear equations of $\bar{\alpha}$ and ω_{m0}. The solutions α^* and ω_{m0}^* of $\bar{\alpha}$ and ω_{m0} can be obtained by using numerical methods.

4.3. Conditions of Stability of synchronization operation

As shown in Eq. (29), we can derive a system of three first order differential equations by assembling the set of equations in the following manner: linearizing Eq. (21) around$\bar{\alpha} = \alpha^*$, neglecting the values of $\sin\gamma_x$, $\sin\gamma_y$ and $\sin\gamma_\psi$ as well as introducing Eqs (22), (23) and relationship $\dot{\bar{\alpha}} - \dot{\alpha}^* = \omega_{\mathrm{m0}}^* \bar{\varepsilon}_2$.

$$\dot{\mathbf{z}} = \mathbf{C}\mathbf{z} \tag{29}$$

where $\mathbf{C} = \mathbf{A}^{-1}\mathbf{B}'$, $\mathbf{z} = \{\bar{\varepsilon}_1 \quad \bar{\varepsilon}_2 \quad \bar{\alpha} - \alpha^*\}^{\mathrm{T}}$,

$$\mathbf{B}' = \begin{bmatrix} b & b_{12} & 0 \\ b_{21} & b & -2m_0^2 r^2 \omega_{\mathrm{m0}}^{*2} \eta W_{\mathrm{c}} \cos 2\alpha^* \\ 0 & \omega_{\mathrm{m0}}^* & 0 \end{bmatrix}$$

Assuming a exponential time-dependence form, $\mathbf{z} = \mathbf{v}\exp(\lambda t)$, in Eq. (29) and solving the determinant equation $\det(\mathbf{C} - \lambda \mathbf{I}) = \mathbf{0}$, we get a characteristic equation for the eigenvalue λ,

$$\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0 \tag{30}$$

where

$$c_1 = \frac{a_{12}b_{21} + a_{21}b_{12} - (a_{11} + a_{22})b}{a_{11}a_{22} - a_{12}a_{21}},$$

$$c_2 = \frac{2a_{11}m_0^2 r^2 W_{\mathrm{c}} \omega_{\mathrm{m0}}^{*3} \eta \cos 2\alpha^* + b^2 - b_{12}b_{21}}{a_{11}a_{22} - a_{12}a_{21}},$$

$$c_3 = -\frac{2bm_0^2 r^2 \eta W_{\mathrm{c}} \omega_{\mathrm{m0}}^{*3} \cos 2\alpha^*}{a_{11}a_{22} - a_{12}a_{21}}.$$

The zero solution of Eq. (29) is stable only if all roots of λ in Eq. (30) have the negative real parts. Using the Routh-Hurwitz criterion, Inequations (31) will satisfy the above requirements [7]:

$$c_1 > 0, c_3 > 0 \text{ and } c_1c_2 > c_3 \tag{31}$$

Substituting the expressions of a_{11}, a_{12}, a_{21} and a_{22} in Eq. (21) into the denominator of c_1, which will be denoted by $E(E = a_{11}a_{22} - a_{12}a_{21})$ in the following paper, we get

$$E = \omega_{\mathrm{m0}}^{*2}(4J'_{01}J'_{02} - m_0^2 r^4 \eta^2 W_{\mathrm{c}}^2 \cos^2 2\alpha^*) \geqslant \omega_{\mathrm{m0}}^{*2}(4J'_{01}J'_{02} - m_0^2 r^4 \eta^2 W_{\mathrm{c0}}^2) \tag{32}$$

where $J'_{01} = J_{01} - m_0^2 r^2 W_{\mathrm{c0}}/2$and $J'_{02} = J_{02} - \eta^2 m_0^2 r^2 W_{\mathrm{c0}}/2$ are called the relative moments of inertia of exciters 1 and 2, respectively.

The moment of inertia of the exciter is the sum of that of the eccentric lump and that of the motor's rotor. The moment of inertia of the motor's rotor is much smaller than that of the eccentric lumps in the system and it can be neglected. Then, the relative moments of inertia of the two exciters can be approximately expressed as:

$$\begin{aligned} J'_{01} &\approx m_0 r^2 (1 - m_0 W_{\mathrm{c0}}/2) \\ J'_{02} &\approx \eta m_0 r^2 (1 - \eta m_0 W_{\mathrm{c0}}/2) \end{aligned} \tag{33}$$

Inserting Eq. (33) into Inequation (32) will yield

$$E > 4m_0^2 r^4 \omega_{\mathrm{m0}}^{*2} \eta (1 - \frac{1+\eta}{2} m_0 W_{\mathrm{c0}}) \tag{34}$$

In a non-resonant vibrating system, the exciting frequency is usually (4 $\sim$ 5) times of the natural one and the masses of the eccentric lumps are much smaller than that of body of the vibrating machine [15,16]. If the structural parameters of the system can satisfy $m_0 W_{\mathrm{c0}} < 1$, we have

$$E = a_{11}a_{22} - a_{12}a_{21} > 0 \tag{35}$$

Inserting the expressions of a_{ij} ($i = 1, 2; \quad j = 1, 2.$), b, b_{12} and b_{21} in Eq. (21), $m_0 W_{\mathrm{c0}} < 1$ and Eq. (33) into the numerator of c_1, we obtain

$$a_{12}b_{21} + a_{21}b_{12} - (a_{11} + a_{22})b = 4(J'_{01}K_2 + J'_{02}K_1) > 0 \tag{36}$$

Thus, c_1 always meets the requirements of stability of synchronous operation if $m_0W_{c0} < 1$.

Applying Inequation (35) and the expression of b in Eq. (21) into the condition $c_3 > 0$, we obtain

$$W_c \cos 2\alpha^* > 0 \tag{37}$$

Introducing Eq. (33), Inequation (35) and the expressions of a_{ij} $(i = 1, 2; \quad j = 1, 2.)$, b, b_{12} and b_{21} in Eq. (21) into the condition $a_1a_2 > a_3$, we have

$$E_1W_c \cos 2\alpha^* > -E_2 \tag{38}$$

where

$$E_1 = [4J'^2_{01}K_2 + 4J'^2_{02}K_1 + m_0^4r_0^4\eta^2W_c^2(K_1 + K_2)\cos^2 2\alpha^*]m_0^2r^2\omega_{m0}^{*2}\eta$$

$$E_2 = 8(J'_{01}K_2 + J'_{02}K_1)(K_1 + K_2 + m_0^4r^4\omega_{m0}^{*2}\eta^2W_c^2\cos^2 2\alpha^*) + 4m_0^4r^4\omega_{m0}^{*2}W_c^2\eta^2(J'_{01}K_2 + J_{02}K_1)\cos^2 2\alpha^*$$

It can be found that Inequation (37) satisfies Inequation (38), which means that Inequation (37) and relationship $m_0W_{c0} < 1$ are the conditions of stability of the synchronous operation.

When $a_{11}a_{22} - a_{12}a_{21} < 0$, the stability condition $c_1 > 0$ requires $J'_{01}K_2 + J'_{02}K_1 < 0$, see equation (36), and $a_3 > 0$ requires $W_c \cos 2\alpha^* < 0$. If $J'_{01}K_2 + J'_{02}K_1 < 0$ and $W_c \cos 2\alpha^* < 0$, the left of inequation (38) is less than zero and its right is greater than zero, i.e., $a_1a_2 < a_3$. So $a_{11}a_{22} - a_{12}a_{21} < 0$ does not satisfy the stability condition of the synchronous operation. Therefore, $m_0W_{c0} < 1$ is the condition of global stability of synchronous operation. When $W_c > 0$, the stable interval of phase difference between two exciters is $2\alpha \in (-\pi/2, \pi/2)$; otherwise, it is $2\alpha \in (\pi/2, 3\pi/2)$.

5. Discussions

From the above theoretical analysis, it can be seen that the analytical approach employed in this paper converts the problem of synchronization of the two exciters in a vibrating system into that of existence and stability of zero solution for the Eq. (21). The small parameters ε_1 and ε_2 in Eq. (21) can describe the change of angular displacements of two coupled exciters in the system operation. Therefore, the Equation of Frequency Capture is equivalent to the average of motion equations of the two exciters in Eq. (1). The perturbations of average angular velocity and phase difference, $\varepsilon_1$andε_2, are the local small perturbation parameters.

From Eq. (28), it can be seen that if the torque of frequency capture of the vibrating system is equal to or greater than the difference in the residual toques of two motors, zero solutions of Eq. (21) must exist, and angular velocity and phase difference of the system operation can be calculated by numerical methods. If the rated angular velocities of the two motors are ω_{me1} and ω_{me2}, respectively, and the numerical solutions of Eqs (22) and (23) are ω^*_{m0} and α^*, the angular velocity ω^*_{m0} of the system operating at the steady-state must be equal to or greater than $\omega_{me} = \max\{\omega_{me1}, \omega_{me2}\}$, i.e., $\omega^*_{m0} \geqslant \omega_{me}$. The reason that the two motors can not synchronize is that there are the differences in electromagnetic torques and their load torques. While the torque of frequency capture can overcome these differences and make the two motors synchronous. From Eq. (19), it can be seen that one half of the torque of frequency capture and the sine of the phase difference between two exciters acts on one motor (leading phase) as load torque to decrease its angular velocity, as well as the other half acts on other motor (lagging phase) as driving torque to increase its angular velocity. The bigger the phase difference between two exciters is, the bigger this torque is. Therefore, the torque of frequency capture has an effect on limiting the increase of the phase difference between two exciters. When the phase difference, $2\bar{\alpha}$, reaches a certain value, the angular accelerations of two motors are zero at the same time and two motors operate at the synchronous state. This fact physically explains the mechanism of self-synchronization in the considered system. When the system operates at the steady-state, the torque of frequency capture does not do work.

The stability of synchronous operation is dependent on two conditions, one is $W_c \cos 2\alpha^* > 0$ and the other is $m_0W_{c0} < 1$. If the system satisfies Inequation (28), there must be a phase difference $2\alpha^*$ to meet the first condition of the stability of synchronous operation, $W_c \cos 2\alpha^* > 0$, i.e., when W_c >0, $2\alpha^* \in (-\pi/2, \pi/2)$; otherwise, $2\alpha^* \in (\pi/2, 3\pi/2)$. Therefore $W_c \cos 2\alpha^* > 0$ is the condition of phase difference for the stability of synchronous operation. To obtain the linear vibration of the machine body in y-direction, the spring constants in x- and y-directions are designed to be equal [15,16]. This can ensure that W_c >0 and $2\alpha^*$ is in the neighborhood of zero.

Based on the requirement that the denominator (E) of the coefficients of characteristic Eq. (30) needs to be greater than zero, i.e., $E > 0$, the condition, $m_0 W_c < 1$, is derived. There are two terms that have an effects on the sign of E in Eq. (32), $J'_{01} J'_{02}$ and $m_0^2 r^4 \eta^2 W_c^2 \cos^2 2\alpha$. It can be seen from Eq. (33) that the moments of inertia of two exciters will reduce and the reduction proportions are $m_0 W_{c0}/2$ and $\eta m_0 W_{c0}/2$, respectively. The condition, $m_0 W_0 < 1$, can ensure that the relative moments of two exciters, J'_{01} and J'_{02}, are all greater than zero and $4 J'_{01} J'_{02}$ is greater than the square of the coupling moment, $m_0^2 \eta^2 W_c^2 \cos^2 2\alpha$, see Eq. (32). These facts demonstrate that the stability of synchronous operation of two exciters is dependent on their relative moments of inertia and their coupling moment of inertia. The equation of frequency capture can describe the dynamic characteristics of relative motion of two coupled exciters. These special dynamic characteristics of the two coupled exciters will be discussed quantitatively in detail in our next paper.

6. Conclusions

From the theoretical investigation given in the above sections, the following remarks can be stressed:

(1) This paper proposes an analytical approach to study the problem of two non-identical coupled exciters in a vibrating system, which converts the problem of synchronization of two coupled exciters in a vibrating system into that of existence and stability of solution for Equation of Frequency Capture.

(2) The necessary condition of implementing the frequency capture and resulting in the synchronous operation of two exciters is that the torque of frequency capture is equal to or greater than the absolute value of the difference in the residual torques of two motors.

(3) One half of the torque of frequency capture and the sine of the phase difference between two exciters acts on one motor (leading phase) as load torque to decrease its angular velocity, and another half acts on other motor (lagging phase) as driving torque to increase its angular velocity. The torque of frequency capture plays the role of limiting the increase of the phase difference between the two exciters. When the phase difference reaches a certain value, the angular accelerations of two motors are zero at the same time and two motors rotate at the same angular velocity. When the system operates at the steady-state of synchronization, the torque of frequency capture does not do work.

(4) In the Equation of Frequency Capture of the system, the moments of inertia of two exciters reduce due to the motions of the system and the reduction proportions are $m_0 W_{c0}/2$ and $\eta m_0 W_{c0}/2$, respectively. The residual is called the relative moment of inertia. There is also a coupling moment of inertia between the two exciters. They have an effect on the stability of the synchronous operation. The condition of stability of synchronous operation is that the relative moments of two exciters are all greater than zero and four times product of the two relative moments is greater than the square of the coupling moment of inertia of two exciters.

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