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Reduced model in $\mathcal{H}_\infty$ vibration control using linear matrix inequalities

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Abstract. Many practical problems in structural dynamics are modeled with a high number of degrees of freedom in order to properly describe the structure. A formulation to design robust controllers is the $\mathcal{H}_\infty$ technique where the controller has the same order of the mathematical model and this becomes unpractical and infeasible for most practical problems where the number of degrees of freedom is not small. One way to overcome this difficulty is to employ a model reduction technique, and design a reduced order controller based on this reduced model. In this case, it is required that the reduced controller ensures a good performance also for the nominal model (reduced) and for the real model (non-reduced) of the structure. Since the reduced controller is designed based on a truncated dynamic model, the non-modeled vibration modes can be excited causing the spillover phenomena, which is a severe undesirable effect. This work investigates the behavior of a reduced order controller obtained based on a reduced model through the Guyan reduction. The $\mathcal{H}_\infty$ robust control and Linear Matrix Inequalities (LMI) formulations are employed to the problem of controlling a flexible structure subjected to an external disturbance. Some simulations are performed using a cantilever beam modeled by the finite element method. The results show that the Guyan reduced order model can be used to design a controller to the non-reduced model with success.

Keywords: Vibration control, linear matrix inequalities, $\mathcal{H}_\infty$ control, Guyan reduction

1. Introduction

Flexible structures models normally present a high number of degrees of freedom and respective resonances. The vibration control of these structures is a difficult problem because the mathematical models are, in general, reduced and only a limited range of frequency is considered [1,11]. In these cases, the controller is designed based on a reduced order model, but it should ensure a suitable performance and stability in frequency ranges related to real modes of vibration that were not considered in the mathematical model avoiding the spillover.

In general, for a suitable control design, the structures should be rich in terms of the instrumentation (sensors and actuators) and it is common that the control forces act in different points from those ones where the responses are measured, characterizing a non-collocated control problem. Usually, piezoelectric devices are employed since they can behavior as sensors and as actuators, and they do not affect significantly the structure [24,30].

The flexible structures vibration control problem presents dynamic uncertainties that are caused by the modes truncation between the structure model and the real structure, and correspond to the differences between the real structure and the reduced model in terms of frequency response. Another kind of uncertainty is the parametric that is related to variations on estimated or identified parameters of the structures, such as mass, stiffness and mainly damping. Experimental difficulties can also be responsible for uncertainties due to measurements limitations. Although robust controllers are designed using a reduced model (nominal model) of the structure, it should be able

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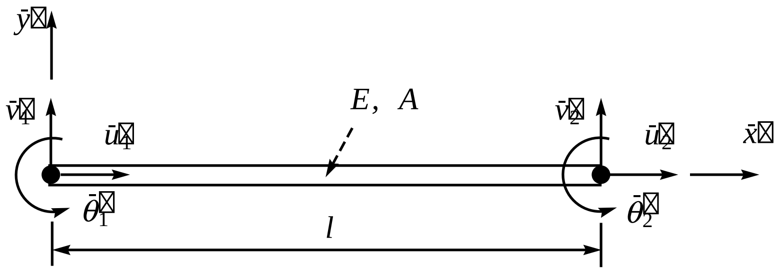

Fig. 1. Two-dimensional beam finite element.

to control the real structure under parametric and dynamic uncertainties [3,21,26,31] ensuring stability and a good performance.

The mathematical model of the structure can be obtained efficiently through the Finite Element Method (FEM) [2] or by an experimental identification [7]. The FEM is particularly suitable when the structure is not available for a direct measurement in the system design phase.

The $\mathcal{H}_\infty$ control [29] consists of a frequency domain method where the peak of singular value of the transfer matrix between the disturbance input and the performance output is minimized [14,28] and corresponds to an increase in the stability margins [9,17]. In other words, the closed loop system minimizes the external disturbance effects and the uncertainty effects for a desired performance output. State feedback control law [27] is common in the literature, but the output feedback allows the design of a controller based on the output since the states can not be available for measurement in most problems. In this work, output feedback formulation is employed.

Besides the disturbance rejection, the control force is an important constraint [8] to allow a practical implementation. In the $\mathcal{H}_\infty$ control, the control signal can also be considered in the performance output in order to obtain a feasible solution of practical implementation.

In this work, the $\mathcal{H}_\infty$ robust control problem is solved using Linear Matrix Inequalities (LMI). LMI started to be studied in 1890 with Lyapunov, but only in 1940 Lur'e and Postnikov performed applications in control engineering. Nowadays, LMI can be used to represent several kinds of control problems. The essentials of LMI are that the problem is formulated according to the minimization of a linear objective subjected to linear matrix inequalities as constraints. This is a semi-definite programming problem, which is convex and can be solved efficiently by interior point algorithms [4,23].

In practice, the design and implementation of a controller for flexible structures of high order is, in most cases, an infeasible problem. Several interesting and practical problems present a great number of degrees of freedom and it is desirable to design a reduced order controller in order to control the structure efficiently. One approach to handle with this difficulty is to obtain a reduced order controller based on a reduced model. The Guyan reduction [15,2] is a classical technique that can be used to provide a reduced model to be used to design a reduced order $\mathcal{H}_\infty$ controller.

This paper investigates the behavior of a reduced $\mathcal{H}_\infty$ controller designed using the Guyan reduction for a simple beam structure. In order to obtain the controller, the problem is formulated and solved using LMI techniques. The results are briefly discussed and show the potential of the adopted approach.

2. Mathematical model of the structure

The finite element method can be used to obtain the structure model. It consists of dividing the structure in finite elements connected by nodes with the respective degrees of freedom. The interest values in the interior of the element can be obtained through the nodal solutions by using interpolation functions. The element stiffness and mass matrices can be calculated by an integration over the finite element, and the global structure matrices can be obtained by the assembly operation [2,18].

The two-dimensional Hermitian beam element [2,18] with six degrees of freedom (see Fig. 1) is used in this work to test the control formulation.

The mass and stiffness matrices for this finite element are respectively:

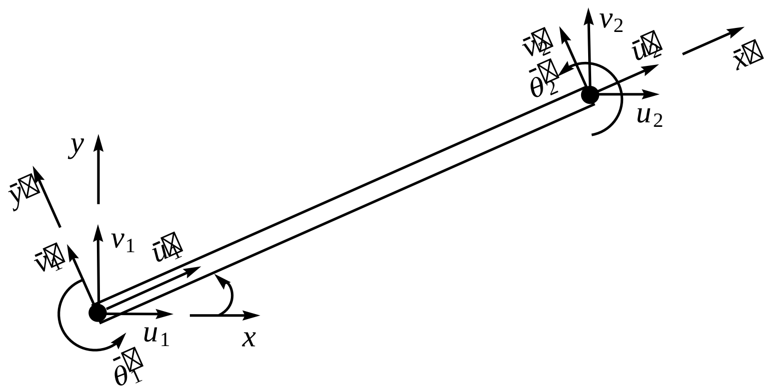

Fig. 2. Finite element in global coordinates system.

$$\bar{\mathbf{K}} = \frac{E}{l^3}\begin{bmatrix} Al^2 & 0 & 0 & -Al^2 & 0 & 0 \\ 0 & 12I & 6Il & 0 & -12I & 6Il \\ 0 & 6Il & 4Il^2 & 0 & -6Il & 2Il^2 \\ -Al^2 & 0 & 0 & Al^2 & 0 & 0 \\ 0 & -12I & -6Il & 0 & 12I & -6Il \\ 0 & 6Il & 2Il^2 & 0 & -6Il & 4Il^2 \end{bmatrix}, \tag{1}$$

$$\bar{\mathbf{M}} = \rho Al\begin{bmatrix} 1/3 & 0 & 0 & 1/6 & 0 & 0 \\ 0 & 13/35 & 11l/210 & 0 & 9/70 & -13l/420 \\ 0 & 11l/210 & l^2/105 & 0 & 13l/420 & -l^2/140 \\ 1/6 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 9/70 & 13l/420 & 0 & 13/35 & -11l/210 \\ 0 & -13l^2/420 & -l^2/140 & 0 & -11l/210 & l^2/105 \end{bmatrix}, \tag{2}$$

where ρ, A, I, E, l are material density, cross-section area, cross-section inertia moment, Young's modulus and element length.

The local element matrices defined in Eq. (1) need to be transformed from the local coordinate system ($\bar{u}_1$, $\bar{v}_1$, $\bar{\theta}_1$, $\bar{u}_2$, $\bar{v}_2$, $\bar{\theta}_2$) to the global coordinate system (u_1, v_1, θ_1, u_2, v_2, θ_2) if the element is rotated as in Fig. 2. This rotation is performed according to the Eq. (3) [18]:

$$\begin{Bmatrix} \bar{u}_1 \\ \bar{v}_1 \\ \bar{\theta}_1 \\ \bar{u}_2 \\ \bar{v}_2 \\ \bar{\theta}_2 \end{Bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta & 0 & 0 & 0 & 0 \\ -\sin\beta & \cos\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\beta & \sin\beta & 0 \\ 0 & 0 & 0 & -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{Bmatrix}. \tag{3}$$

After the mass and stiffness matrices for all finite elements of the mesh were obtained, these matrices should be assembled to obtain the structure mass and stiffness matrices. The boundary conditions can then be applied.

One common difficulty in structures modeling is the damping determination. Damping can be determined experimentally through specific measurements in the structure identification phase. Although, when the structure is not available for identification, the damping should be estimated. In some cases it is usual to consider proportional (to mass and stiffness) damping because the mathematical treatment is simplified [2,7].

The dynamic equation of a linear vibration problem [11,16] with n degrees of freedom can written as

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{D}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{f}, \tag{4}$$

where $\mathbf{M}$, $\mathbf{D}$, $\mathbf{K}$, $\mathbf{f}$ and $\mathbf{q}$ are mass matrix, damping matrix, stiffness matrix, external applied forces vector and displacement vector respectively.

Defining the state vector as $\mathbf{x} = [\mathbf{q} \;\; \dot{\mathbf{q}}]^T$ it is possible to write:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{f} \end{bmatrix}, \tag{5}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{D} \end{bmatrix}. \tag{6}$$

In control problems there is the interest to analyse the effects of each force input individually. The vector $\mathbf{f}$ can be written in terms of a base $\mathbf{s}_i$ (responsible for the input selection) as:

$$\mathbf{f} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} f_1 + \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} f_2 + \ldots + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} f_n = \sum_i^n \mathbf{s}_i f_i.$$

Related to a specific force component i it is possible to write

$$\begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1} \end{bmatrix} \mathbf{s}_i f_i = \mathbf{B} f_i, \tag{7}$$

where

$$\mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1} \end{bmatrix} \mathbf{s}_i.$$

The Eq. (5) can be written as

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B} f_i, \\ \mathbf{q} &= \mathbf{C}\mathbf{x}, \end{aligned} \tag{8}$$

where $\mathbf{B}$ is the input matrix (input force selection) and $\mathbf{C}$ is the output matrix (output selection).

The state-space model is the basis of several modern control techniques.

3. $\mathcal{H}_\infty$ control and linear matrix inequalities

The $\mathcal{H}_\infty$ norm of a stable transfer matrix $\mathbf{G}(jw)$ is defined as the greatest singular value of this matrix regarding all frequencies w [14,28], i.e.,

$$||\mathbf{G}(jw)||_\infty = \sup_w \bar{\sigma}[\mathbf{G}(jw)], \tag{9}$$

where $\bar{\sigma}$ represents the greatest singular value of $\mathbf{G}(jw)$.

In the optimal $\mathcal{H}_\infty$ design, the controller is selected to minimize the $\mathcal{H}_\infty$ norm of the transfer matrix between the disturbance inputs, $\mathbf{w}$, and the desired outputs, $\mathbf{z}$. This minimization is pursuit because it reduces the effects of the disturbances in the performance outputs. In other words, the main objective of $\mathcal{H}_\infty$ control is to guarantee the system performance in presence of external disturbances [6].

The solution of $\mathcal{H}_\infty$ problem can be found in two main ways: $i)$ solving the associated Riccati equations, or $ii)$ solving an optimization problem that has LMI as constraint equations [9,20]. The formulation and solution using LMI is a more recent point of view. It can be considered more general because many of control problems can be solved using this approach and other constraint equations can be added to the problem.

In the last years, LMI formulations have been widely used in control applications because LMI formulated problems are convex and convexity is a property in mathematical programming area that allows the use of very efficient algorithms and computer methods [10,23], particularly the interior point methods. This is an interesting numerical feature of the problem, because if the iterative process ends before the final optimal solution was found, the obtained solution can be regarded as a sub-optimal solution since it is a feasible one.

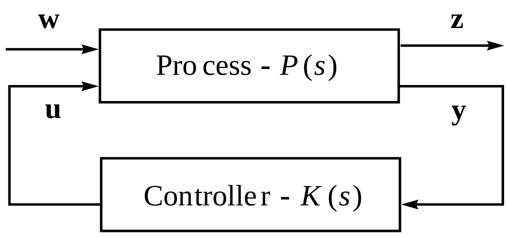

Fig. 3. Process and $\mathcal{H}_\infty$ controller scheme.

The $\mathcal{H}_\infty$ problem can be formulated [6,14,28] according to Fig. 3 with the corresponding equations:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}_1\mathbf{w} + \mathbf{B}_2\mathbf{u}, \tag{10}$$

$$\mathbf{z} = \mathbf{C}_1\mathbf{x} + \mathbf{D}_{12}\mathbf{u}, \tag{11}$$

$$\mathbf{y} = \mathbf{C}_2\mathbf{x} + \mathbf{D}_{21}\mathbf{w}, \tag{12}$$

where $\mathbf{u}$ is the control signal vector, $\mathbf{w}$ is the external disturbance vector, $\mathbf{y}$ is the signal sent to controller and $\mathbf{z}$ is the desired output. Matrix $\mathbf{B}_1$ is the input matrix related to the exogenous inputs, $\mathbf{B}_2$ is the input matrix related to the control input, $\mathbf{C}_1$ is the output matrix related to the states, $\mathbf{D}_{12}$ relates the control force to the performance output, and $\mathbf{D}_{21}$ relates the exogenous inputs to the signal sent to the controller.

This representation is usual and normally $\mathbf{D}_{21} = \mathbf{0}$. The relation between the inputs and the outputs in Fig. 3 can be written based on a transfer matrix, i.e.,

$$\begin{bmatrix} \mathbf{z} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{u} \end{bmatrix}, \tag{13}$$

where $\mathbf{P}_{11}$ is the transfer function between $\mathbf{w}$ input and $\mathbf{z}$ output, $\mathbf{P}_{12}$ is the transfer function between $\mathbf{u}$ input and $\mathbf{z}$ output, $\mathbf{P}_{21}$ is the transfer function between $\mathbf{w}$ input and $\mathbf{y}$ output and $\mathbf{P}_{22}$ is the transfer function between $\mathbf{u}$ input and $\mathbf{y}$ output.

It is also possible to write

$$\mathbf{z} = \mathbf{P}_{11}\mathbf{w} + \mathbf{P}_{12}\mathbf{u}, \tag{14}$$

$$\mathbf{y} = \mathbf{P}_{21}\mathbf{w} + \mathbf{P}_{22}\mathbf{u}. \tag{15}$$

The controller transfer matrix is $\mathbf{K}(s)$ and the control law can be written as

$$\mathbf{u} = \mathbf{K}(s)\mathbf{y}. \tag{16}$$

The relation between $\mathbf{w}$ and $\mathbf{z}$ in closed loop system can be obtained substituting Eqs (15) into (16) and (16) into (14), i.e.,

$$\mathbf{z} = [\mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(\mathbf{I} - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21}]\mathbf{w}. \tag{17}$$

The expression Eq. (17) is called lower linear fractional transformation [28] and it is represented as

$$\mathbf{F}_l(\mathbf{P}, \mathbf{K}) = \mathbf{P}_{11} + \mathbf{P}_{12}\mathbf{K}(\mathbf{I} - \mathbf{P}_{22}\mathbf{K})^{-1}\mathbf{P}_{21}, \tag{18}$$

where $\mathbf{F}_l(\mathbf{P}, \mathbf{K})$ is the transfer matrix between the disturbance input $\mathbf{w}$ and the performance output $\mathbf{z}$.

In $\mathcal{H}_\infty$ design it is desired to find the controller $\mathbf{K}(s)$ in order to minimize the $\mathcal{H}_\infty$ norm of $\mathbf{F}_l(\mathbf{P}, \mathbf{K})$. This corresponds to the minimization of the peak value of the singular diagram of the transfer matrix between the disturbances input $\mathbf{w}$ and the performance output $\mathbf{z}$ in the frequency domain [28], i.e.,

$$\min_{w} \; \|\mathbf{F}_l(\mathbf{P}, \mathbf{K})\|_\infty. \tag{19}$$

A sub-optimal controller can be found when this norm is lower than a real value $\gamma > 0$, i.e.,

$$\|\mathbf{F}_l(\mathbf{P}, \mathbf{K})\|_\infty < \gamma. \tag{20}$$

A common way to solve the sub-optimal $\mathcal{H}_\infty$ problem consists of determining an initial value of γ, the Riccati equations are solved and the condition Eq. (20) is verified. If this condition is satisfied, γ is reduced and the process is repeated until the restriction Eq. (20) is not satisfied. Another point of view, which is employed in this work, is to formulate an optimization problem that presents LMI constraints.

It is important to mention that the control force is taken account in the performance vector $\mathbf{z}$ through the matrix $\mathbf{D}_{12}$, and it can be checked for the final solution. If the control force is not suitable, matrix $\mathbf{D}_{12}$ can be changed in order to define a new weight for the control force in the performance objective.

An output feedback controller formulation based on LMI is presented next.

The dynamic controller state-space model can be written as:

$$\dot{\mathbf{x}}_c = \mathbf{A}_c\mathbf{x}_c + \mathbf{B}_c\mathbf{y}, \tag{21}$$

$$\mathbf{u} = \mathbf{C}_c\mathbf{x}_c, \tag{22}$$

where $\mathbf{y}$ is the controller input vector, $\mathbf{u}$ is the controller output vector, $\mathbf{x}_c$ is the controller state vector and $\mathbf{A}_c$, $\mathbf{B}_c$, $\mathbf{C}_c$ are the controller matrices.

For a closed loop analysis it is possible to obtain the extended state space model from Eqs (10), (11), (12), (21) and (22), i.e.,

$$\dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{A}}\tilde{\mathbf{x}} + \tilde{\mathbf{B}}\mathbf{w}, \tag{23}$$

$$\mathbf{z} = \tilde{\mathbf{C}}\tilde{\mathbf{x}}, \tag{24}$$

where:

$$\tilde{\mathbf{x}} = \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_c \end{bmatrix}, \ \tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_2\mathbf{C}_c \\ \mathbf{B}_c\mathbf{C}_2 & \mathbf{A}_c \end{bmatrix}, \ \tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_c\mathbf{D}_{21} \end{bmatrix}, \ \tilde{\mathbf{C}} = \begin{bmatrix} \mathbf{C}_1 \ \mathbf{D}_{12}\mathbf{C}_c \end{bmatrix}.$$

It is possible to demonstrate that $\mathcal{H}_\infty$ control problem can be written in terms of a minimization problem [4,20] as:

$$minimize\ \gamma$$

$$subjected\ to$$

$$\tilde{\mathbf{A}}^T\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\tilde{\mathbf{A}} + \gamma^{-2}\tilde{\mathbf{P}}\tilde{\mathbf{B}}\tilde{\mathbf{B}}^T\tilde{\mathbf{P}} + \tilde{\mathbf{C}}^T\tilde{\mathbf{C}} < 0, \tag{25}$$

$$\tilde{\mathbf{P}} > 0, \tag{26}$$

where the symbol $<$ means that the expression (25) is negative-definite and the symbol $>$ in (26) means that the matrix $\tilde{\mathbf{P}}$ is positive-definite. The expression (25) replaces the Riccati equation when it is desirable to solve the $\mathcal{H}_\infty$ problem using LMI.

The matrix inequality Eq. (25) is non-linear in $\tilde{\mathbf{P}}$. Applying Schur Complement [4] it is possible to obtain:

$$\left[\begin{array}{cc|c} \tilde{\mathbf{A}}^T\tilde{\mathbf{P}} + \tilde{\mathbf{P}}\tilde{\mathbf{A}} & \tilde{\mathbf{C}}^T & \tilde{\mathbf{P}}\tilde{\mathbf{B}} \\ \tilde{\mathbf{C}} & -\mathbf{I} & \mathbf{0} \\ \hline \tilde{\mathbf{B}}^T\tilde{\mathbf{P}} & \mathbf{0} & -\gamma^2\mathbf{I} \end{array}\right] < 0. \tag{27}$$

Using a congruence transformation [4,13], the linear matrix inequality Eq. (27) can be left and right-multiplied respectively by

$$\begin{bmatrix} \tilde{\mathbf{T}}^T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \tilde{\mathbf{T}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix},$$

and this results in the equation

$$\begin{bmatrix} \tilde{\mathbf{T}}^T(\tilde{\mathbf{A}}^T\tilde{\mathbf{P}}+\tilde{\mathbf{P}}\tilde{\mathbf{A}})\tilde{\mathbf{T}} & \tilde{\mathbf{T}}^T\tilde{\mathbf{C}}^T & \tilde{\mathbf{T}}^T\tilde{\mathbf{P}}\tilde{\mathbf{B}} \\ \tilde{\mathbf{C}}\tilde{\mathbf{T}} & -\mathbf{I} & \mathbf{0} \\ \tilde{\mathbf{B}}^T\tilde{\mathbf{P}}\tilde{\mathbf{T}} & \mathbf{0} & -\gamma^2\mathbf{I} \end{bmatrix} < 0 \tag{28}$$

or

$$\begin{bmatrix} \tilde{\mathbf{T}}^T(\tilde{\mathbf{A}}^T\tilde{\mathbf{P}}+\tilde{\mathbf{P}}\tilde{\mathbf{A}})\tilde{\mathbf{T}} & \tilde{\mathbf{T}}^T\tilde{\mathbf{C}}^T & \tilde{\mathbf{T}}^T\tilde{\mathbf{P}}\tilde{\mathbf{B}} \\ \tilde{\mathbf{C}}\tilde{\mathbf{T}} & -\mathbf{I} & \mathbf{0} \\ \tilde{\mathbf{B}}^T\tilde{\mathbf{P}}\tilde{\mathbf{T}} & \mathbf{0} & -\mu\mathbf{I} \end{bmatrix} < 0, \tag{29}$$

where:

$$\mu = \gamma^2, \quad \tilde{\mathbf{P}} = \begin{bmatrix} \mathbf{X} & \mathbf{U}^T \\ \mathbf{U} & \hat{\mathbf{X}} \end{bmatrix}, \quad \tilde{\mathbf{P}}^{-1} = \begin{bmatrix} \mathbf{Y} & \mathbf{V}^T \\ \mathbf{V} & \hat{\mathbf{Y}} \end{bmatrix}, \quad \tilde{\mathbf{T}} = \begin{bmatrix} \mathbf{Y} & \mathbf{I} \\ \mathbf{V} & \mathbf{0} \end{bmatrix}.$$

It is possible to simplify Eq. (29) defining the following terms:

$$\tilde{\mathbf{P}}\tilde{\mathbf{P}}^{-1} = \mathbf{I} = \begin{bmatrix} \mathbf{X}\mathbf{Y}+\mathbf{U}^T\mathbf{V} & \mathbf{X}\mathbf{V}^T+\mathbf{U}^T\hat{\mathbf{Y}} \\ \mathbf{U}\mathbf{Y}+\hat{\mathbf{X}}\mathbf{V} & \mathbf{U}\mathbf{V}^T+\hat{\mathbf{X}}\hat{\mathbf{Y}} \end{bmatrix}, \tag{30}$$

$$\tilde{\mathbf{P}}^{-1}\tilde{\mathbf{P}} = \mathbf{I} = \begin{bmatrix} \mathbf{Y}\mathbf{X}+\mathbf{V}^T\mathbf{U} & \mathbf{Y}\mathbf{U}^T+\mathbf{V}^T\hat{\mathbf{X}} \\ \mathbf{V}\mathbf{X}+\hat{\mathbf{Y}}\mathbf{U} & \mathbf{V}\mathbf{U}^T+\hat{\mathbf{Y}}\hat{\mathbf{X}} \end{bmatrix}. \tag{31}$$

It is also possible to verify that

$$\mathbf{X}\mathbf{Y}+\mathbf{U}^T\mathbf{V} = \mathbf{Y}\mathbf{X}+\mathbf{V}^T\mathbf{U} = \mathbf{I}$$

$$\mathbf{U}\mathbf{Y}+\hat{\mathbf{X}}\mathbf{V} = \mathbf{V}\mathbf{X}+\hat{\mathbf{Y}}\mathbf{U} = \mathbf{0}.$$

Besides, it is also possible to simplify (29) using $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$, $\tilde{\mathbf{C}}$, $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{T}}$, i.e.,

$$\tilde{\mathbf{C}}\tilde{\mathbf{T}} = \begin{bmatrix} \mathbf{C}_1\mathbf{Y}+\mathbf{D}_{12}\mathbf{F} & \mathbf{C}_1 \end{bmatrix}, \tag{32}$$

$$\tilde{\mathbf{B}}^T\tilde{\mathbf{P}}\tilde{\mathbf{T}} = \begin{bmatrix} \mathbf{B}_1^T & \mathbf{B}_1^T\mathbf{X}+\mathbf{D}_{21}^T\mathbf{L}^T \end{bmatrix}, \tag{33}$$

$$\tilde{\mathbf{T}}^T\tilde{\mathbf{P}}\tilde{\mathbf{A}}\tilde{\mathbf{T}} = \begin{bmatrix} \mathbf{A}\mathbf{Y}+\mathbf{B}_2\mathbf{F} & \mathbf{A} \\ \mathbf{M} & \mathbf{X}\mathbf{A}+\mathbf{L}\mathbf{C}_2 \end{bmatrix}, \tag{34}$$

where:

$$\mathbf{F} = \mathbf{C}_c\mathbf{V},$$

$$\mathbf{L} = \mathbf{U}^T\mathbf{B}_c,$$

$$\mathbf{M} = \mathbf{X}\mathbf{A}\mathbf{Y}+\mathbf{X}\mathbf{B}_2\mathbf{F}+\mathbf{L}\mathbf{C}_2\mathbf{Y}+\mathbf{U}^T\mathbf{A}_c\mathbf{V}.$$

Considering that $\tilde{\mathbf{T}} > 0$ and remembering that $\tilde{\mathbf{P}} > 0$, the equations in Eq. (31) can be used to replace some terms in Eq. (35), i.e.,

$$\tilde{\mathbf{P}} > 0 \Leftrightarrow \tilde{\mathbf{T}}^T\tilde{\mathbf{P}}\tilde{\mathbf{T}} > 0,$$

$$\begin{bmatrix} \mathbf{Y} & \mathbf{V}^T \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{X} & \mathbf{U}^T \\ \mathbf{U} & \hat{\mathbf{X}} \end{bmatrix} \begin{bmatrix} \mathbf{Y} & \mathbf{I} \\ \mathbf{V} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{Y} & \mathbf{I} \\ \mathbf{I} & \mathbf{X} \end{bmatrix} > 0. \tag{35}$$

Therefore, the expression Eq. (35) should be satisfied in order to allow the solution of problem Eq. (25).

Substituting the terms of Eqs (33) and (34) in the expression Eq. (29), it is possible to formulate the $\mathcal{H}_\infty$ problem with output feedback as a minimization problem subjected to LMI constraints [13,20], i.e,

$$\begin{cases} minimize\ \mu \\ subjected\ to: \\ \qquad \begin{bmatrix} \mathbf{N}_{11} & \mathbf{N}_{12} & \mathbf{N}_{13} & \mathbf{N}_{14} \\ \mathbf{N}_{12}^T & \mathbf{N}_{22} & \mathbf{N}_{23} & \mathbf{N}_{24} \\ \mathbf{N}_{13}^T & \mathbf{N}_{23}^T & \mathbf{N}_{33} & \mathbf{N}_{34} \\ \mathbf{N}_{14}^T & \mathbf{N}_{24}^T & \mathbf{N}_{34}^T & \mathbf{N}_{44} \end{bmatrix} < 0, \quad \begin{bmatrix} \mathbf{Y} & \mathbf{I} \\ \mathbf{I} & \mathbf{X} \end{bmatrix} > 0, \end{cases} \tag{36}$$

where:

$$\mathbf{N}_{11} = \mathbf{AY} + \mathbf{B}_2\mathbf{F} + \mathbf{YA}^T + \mathbf{F}^T\mathbf{B}_2^T,$$

$$\mathbf{N}_{12} = \mathbf{A} + \mathbf{M}^T,$$

$$\mathbf{N}_{13} = \mathbf{YC}_1^T + \mathbf{F}^T\mathbf{D}_{12},$$

$$\mathbf{N}_{14} = \mathbf{B}_1,$$

$$\mathbf{N}_{22} = \mathbf{XA} + \mathbf{LC}_2 + \mathbf{A}^T\mathbf{X} + \mathbf{C}_2^T\mathbf{L}^T,$$

$$\mathbf{N}_{23} = \mathbf{C}_1^T,$$

$$\mathbf{N}_{24} = \mathbf{XB}_1 + \mathbf{LD}_{21},$$

$$\mathbf{N}_{33} = -\mathbf{I},$$

$$\mathbf{N}_{34} = \mathbf{0},$$

$$\mathbf{N}_{44} = -\mu\mathbf{I},$$

and $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{L}$, $\mathbf{F}$, $\mathbf{M}$, μ are the unknowns of the problem.

Since $\mathbf{XY} + \mathbf{U}^T\mathbf{V} = \mathbf{I}$, it is possible to consider $\mathbf{U}^T$ arbitrary and calculate $\mathbf{V}$. Thus $\mathbf{A}_c$, $\mathbf{B}_c$ and $\mathbf{C}_c$ (the controller matrices) can be obtained from:

$$\mathbf{F} = \mathbf{C}_c\mathbf{V}, \tag{37}$$

$$\mathbf{M} = \mathbf{XAY} + \mathbf{XB}_2\mathbf{F} + \mathbf{LC}_2\mathbf{Y} + \mathbf{U}^T\mathbf{A}_c\mathbf{V}, \tag{38}$$

$$\mathbf{L} = \mathbf{U}^T\mathbf{B}_c. \tag{39}$$

The optimization problem Eq. (36) has a linear objective function (μ) under LMI constraint equations and can be solved using available LMI solvers such as the function *mincx* of Matlab [10].

It is possible to note that the controller order is the same of the system model (matrices $\mathbf{A}$ and $\mathbf{A}_c$ have the same dimensions), justifying the interest in model reduction techniques. Another aspect that can be considered is that the control signal should be checked for the solution of the sub-optimal problem Eq. (20), and the weight matrix $\mathbf{D}_{12}$ adjusted if required.

4. Guyan model reduction

The design and implementation of a controller with the same order of a high order model is generally unpractical. For flexible structures the high number of degrees of freedom is common, which requires a reduction of the model [19].

A well known reduction technique in structural mechanics is the Guyan reduction [15]. In this technique the degrees of freedom are classified in masters and slaves degrees of freedom. Considering an undamped system, the mass and stiffness matrices can be put in the following form:

$$\begin{bmatrix} \mathbf{M}_{mm} & \mathbf{M}_{ms} \\ \mathbf{M}_{sm} & \mathbf{M}_{ss} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_m \\ \ddot{\mathbf{q}}_s \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{mm} & \mathbf{K}_{ms} \\ \mathbf{K}_{sm} & \mathbf{K}_{ss} \end{bmatrix} \begin{bmatrix} \mathbf{q}_m \\ \mathbf{q}_s \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix}, \tag{40}$$

where $\mathbf{q}_m$ denotes the masters degrees of freedom and $\mathbf{q}_s$ denotes the slaves degrees of freedom.

One can consider that there are not applied forces in the slaves degrees of freedom and that the inertia forces in the slaves are negligible compared to the masters degrees of freedom. According to this hypothesis, it is possible to express the slave degrees of freedom as a function of the masters [2,15], i.e.,

$$\mathbf{q}_s = -\mathbf{K}_{ss}^{-1}\mathbf{K}_{sm}\mathbf{q}_m, \tag{41}$$

$$\mathbf{q} = \begin{bmatrix} \mathbf{q}_m \\ \mathbf{q}_s \end{bmatrix} = \mathbf{W}\mathbf{q}_m = \begin{bmatrix} \mathbf{I} \\ -\mathbf{K}_{ss}^{-1}\mathbf{K}_{sm} \end{bmatrix} \mathbf{q}_m. \tag{42}$$

where

$$\mathbf{W} = \begin{bmatrix} \mathbf{I} \\ -\mathbf{K}_{ss}^{-1}\mathbf{K}_{sm} \end{bmatrix}. \tag{43}$$

Therefore, it is possible to obtain the reduced mass and the reduced stiffness matrices:

$$\mathbf{W}^T\mathbf{M}\mathbf{W} = \mathbf{M}_{mm} - \mathbf{K}_{sm}^T\mathbf{K}_{ss}^{-1}\mathbf{M}_{sm} - \mathbf{M}_{ms}\mathbf{K}_{ss}^{-1}\mathbf{K}_{sm} + \mathbf{K}_{sm}^T\mathbf{K}_{ss}^{-1}\mathbf{M}_{ss}\mathbf{K}_{ss}^{-1}\mathbf{K}_{sm},$$

$$\mathbf{W}^T\mathbf{K}\mathbf{W} = \mathbf{K}_{mm} - \mathbf{K}_{ms}\mathbf{K}_{ss}^{-1}\mathbf{K}_{sm}.$$

In the case of proportional damping, a reduced damping matrix can also be obtained using the reduction transformation:

$$\mathbf{W}^T\mathbf{D}\mathbf{W} = \mathbf{W}^T(\eta\mathbf{K} + \alpha\mathbf{M})\mathbf{W} = \eta\mathbf{W}^T\mathbf{K}\mathbf{W} + \alpha\mathbf{W}^T\mathbf{M}\mathbf{W},$$

where η and α are the proportionality coefficients related to stiffness and mass respectively.

The reduced model presents the order the order related to the number of masters degrees of freedom and can be used for the design of a reduced order controller.

5. Numerical results

The performance of the $\mathcal{H}_\infty$ controller designed using the LMI formulation was evaluated to control the vibration of a cantilever beam by simulation.

The mass and stiffness matrices were obtained by the finite element method for the cantilever beam (Fig. 4), and a proportional damping matrix with $\alpha = 0.00001$ and $\eta = 0.00001$ were used in the simulations.

The beam model has 15 degrees of freedom (full model) and it was reduced to 9 degrees of freedom (reduced model) using the Guyan reduction. See Fig. 4 for details.

In order to compare the dynamic response of the full and the reduced models, the singular value diagrams were determined, see Fig. 5. It is possible to verify that, in this case, the reduced model can be considered a good approximation to the full model. The $\mathcal{H}_\infty$ norms of the full and of the reduced models are -22.7715 dB and -22.7698 respectively.

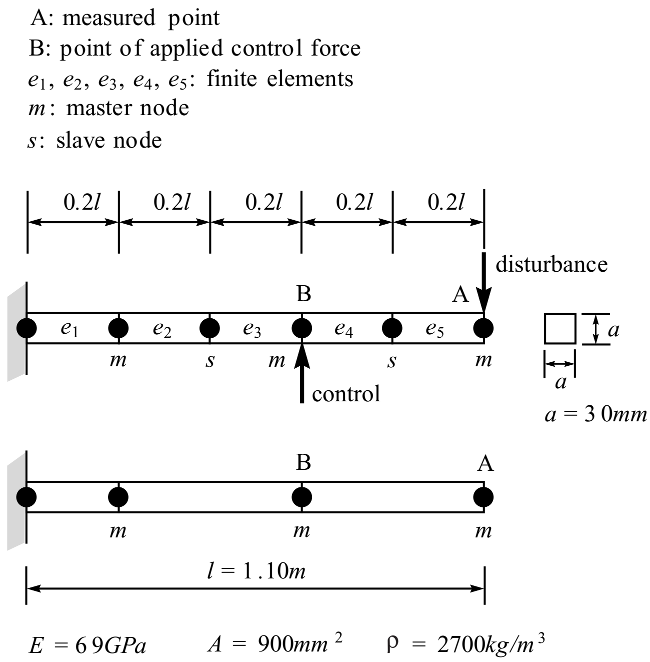

Fig. 4. Structure with 15 degrees reduced to 9 master degrees of freedom.

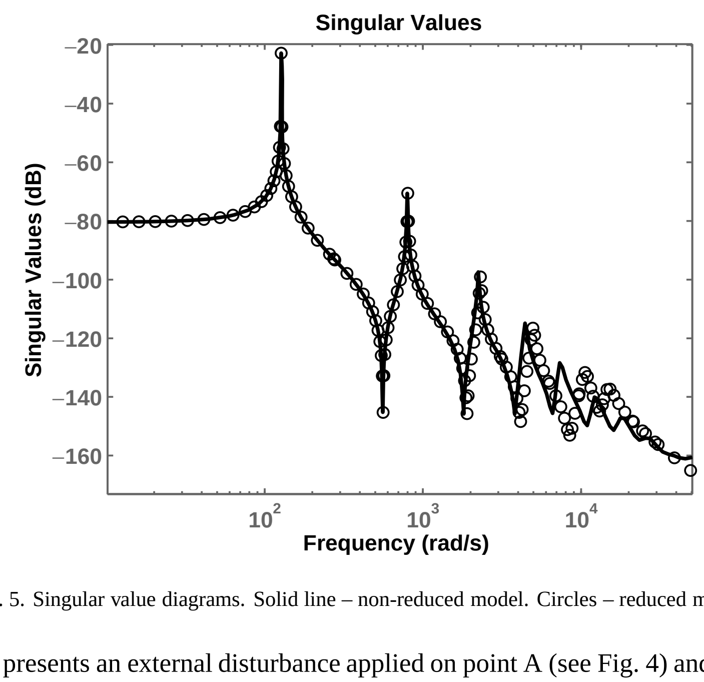

Fig. 5. Singular value diagrams. Solid line – non-reduced model. Circles – reduced model.

The flexible structure presents an external disturbance applied on point A (see Fig. 4) and the control acts at point B. The desired performance outputs are the vertical displacement of point A and the weighted control force. The signal used for feedback is the velocity of point B.

The matrices of the $\mathcal{H}_\infty$ formulation, based on Eqs (10), (11) and (12) for the reduced model are:

$\mathbf{A}$: obtained from the FEM according to Eq. (6),

$$\mathbf{B}_1 = \begin{bmatrix} 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \cdots \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \cdots \end{bmatrix}$$

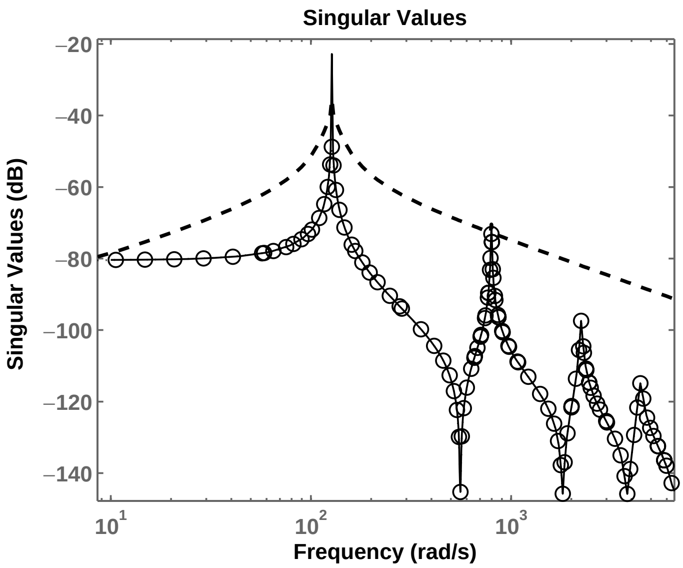

Fig. 6. Singular value diagrams. Continuous line – system without control. Dotted line – closed loop disturbance to control signal. Circles – closed loop disturbance to displacement. Reduced order controller.

$$\begin{matrix} \cdots -0.0163 -1.9707\; 0 -0.0182 -3.3384\; 0\; 1.9484\; 49.6161 \\ \cdots \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \end{matrix}\Bigg]^T,$$

$$\mathbf{B}_2 = \big[0\,0\,0\,0\,0\,0\,0\,0\,0 \cdots$$

$$\cdots 0 -0.0481 -9.0694\; 0\; 1.7778\; 3.3027\; 0\; 1.0342\; 28.5362\big]^T,$$

$$\mathbf{C}_1 = \begin{bmatrix} 0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \end{bmatrix},$$

$$\mathbf{C}_2 = \big[0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0\,0\big],$$

$$\mathbf{D}_{12} = \begin{bmatrix} 0 \\ 0.001 \end{bmatrix} \quad \text{and} \quad \mathbf{D}_{21} = \big[0\;1\big].$$

The matrix $\mathbf{D}_{12}$ was chosen in order to limit the control effort and avoid the saturation of actuators.

The controller based on the reduced model (9 degrees of freedom) was employed to control the full model (15 degrees of freedom). It is possible to verify from Fig. 6 that the peak of singular value was reduced by the controller. The full model without control presented -22.8 dB as peak singular value and the controlled system with the reduced controller presented -35.4 dB as a peak value.

The poles of the controller based on the reduced model are:

$$10^4 \times \begin{bmatrix} -0.7920 \pm 3.9004i \\ -0.3066 \pm 2.4571i \\ -0.0269 \pm 0.7333i \\ -0.1077 \pm 1.4638i \\ -0.0564 \pm 1.0598i \\ -0.0123 \pm 0.4961i \\ -0.0027 \pm 0.2312i \\ -0.0003 \pm 0.0802i \\ -0.0010 \pm 0.0127i \end{bmatrix}.$$

Table 1
$\mathcal{H}_\infty$ norm, iterations number and control force for each case

Case	$\|\mathcal{H}_\infty\|$	Iterations	Max. control force
Without control	-22.8 dB	–	–
With reduced controller	-35.4 dB	101	8.8e-3 N
With full order controller	-35.6 dB	142	7.7e-3 N

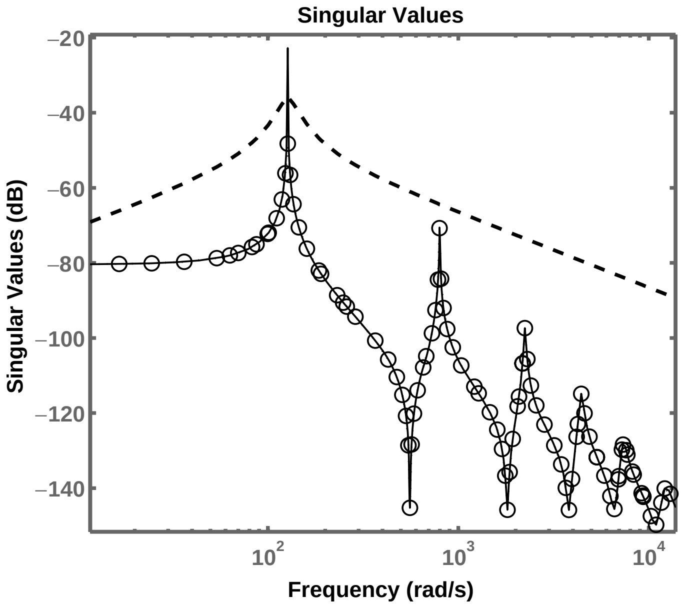

Fig. 7. Singular value diagrams. Continuous line – system without control. Dotted line – closed loop disturbance to control signal. Circles – closed loop disturbance to displacement. Full order controller.

A controller designed with the same order of the full model was also used to control beam in order to have a comparison. It can be verified in Fig. 7 that the full order controller presented a better result than the reduced controller as expected since the controller of full order does not have any information truncation. The peak of singular value obtained with the full order controller was -35.6 dB.

Table 1 presents a comparison between the $\mathcal{H}_\infty$ norm, the number of iterations in the minimization problem and the maximum control force for the reduced and full order controller.

The solution was obtained using the LMI solver of Matlab [10]. The convergence criterion in the *mincx* function was based on the relative variation of μ, i.e.,

$$\frac{\mu_{k+1} - \mu_k}{\mu_k} < 0.1, \tag{44}$$

where k is the number of iterations.

Figure 8 shows the vertical displacement at point A of the beam without control and controlled with the reduced controller when a random disturbance (normal distribution with mean zero, variance and standard deviation one) of amplitude 100 was applied. It is possible to verify that the disturbance effect was minimized for the controlled system. Figure 10 shows generated control signal for this case and it is possible to see that the control signal values are not high.

Figure 9 shows the vertical displacement at point A of the beam without control and controlled by the full order controller when the same disturbance was applied. It is possible to verify the disturbance effect reduction. Figure 11 shows the control signal for this case, which is also not high.

The number of minimization iterations for the full order controller design were 142 and the optimum value of μ was 2.7586e-004. For the reduced controller, the number of iterations were 101 and the optimum value of μ is 2.7695e-004. The difference in terms of iterations can be considered significant and is a measure of the computational

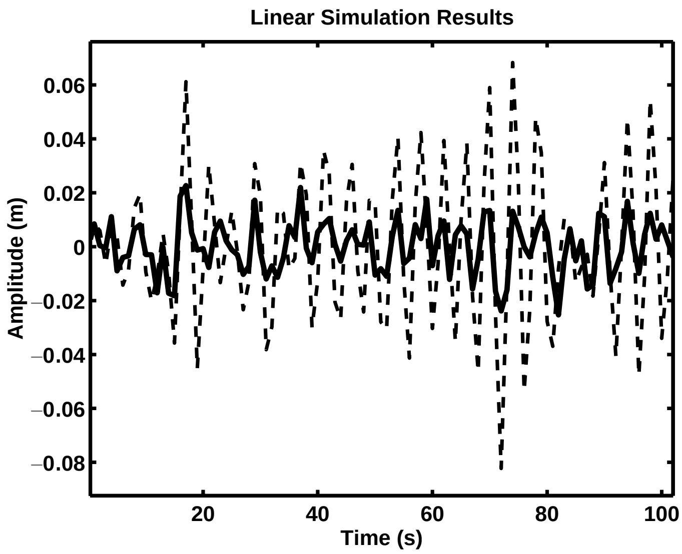

Fig. 8. Beam free-end displacement. Dashed line – without control. Solid line – controlled (reduced order).

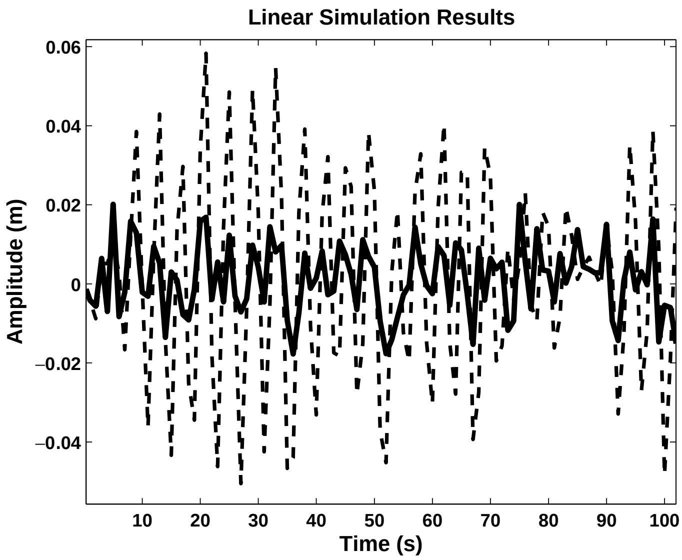

Fig. 9. Beam free-end displacement. Dashed line – without control. Solid line – controlled (full order).

effort. The computational elapsed time to obtain the full order controller was about 3 hours and about 5 minutes for the reduced order controller in a *Pentium* IV 1.8 GHz computer. In terms of the objective value, the difference can be considered not significant and it indicates that the model reduction did not change very much the system characteristics.

6. Concluding remarks

It was verified in this work some aspects of the flexible structures control problem using a reduced order controller based on the Guyan reduction. The formulation used was the $\mathcal{H}_\infty$ control under the point of view of Linear Matrix Inequalities.

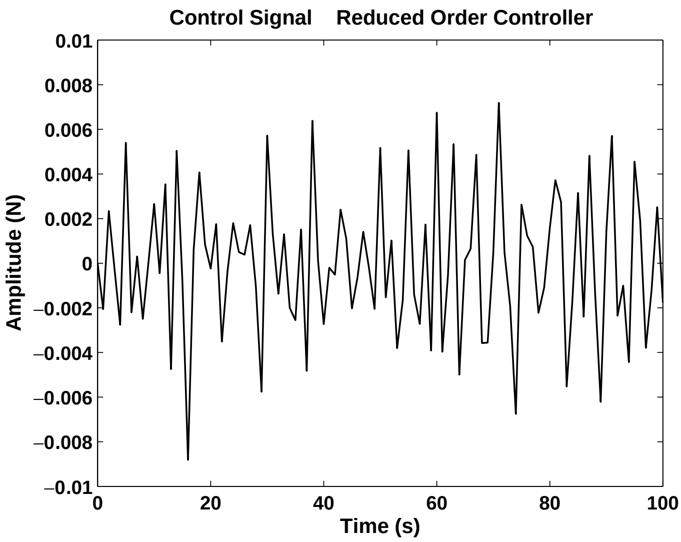

Fig. 10. Control signal – reduced order controller.

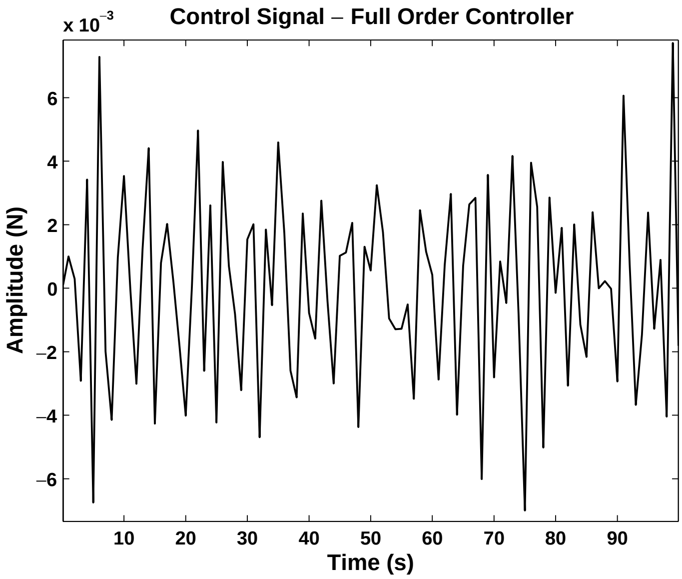

Fig. 11. Control signal – full order controller.

The reduced order controller (reduction from 15 degrees of freedom to 9 masters degrees of freedom for the beam example) was efficient in the $\mathcal{H}_\infty$ norm reduction. A reduction of 12.60 dB from disturbance input to performance output transfer function was found.

One important aspect to be verified is the stability of the closed loop system. Since the model reduction truncates some information, a risk of instability always exists. Other constraint equations can be imposed in the minimization problem in order to ensure pole placements in specific regions. This can also be put in terms of LMI [5,25].

In terms of computational processing effort, the full order controller demanded a higher computational processing time (3 hours) compared to the reduced order controller (5 minutes). This shows the importance of reduction techniques. The example tested in this work can be considered very small, and even in this case, the processing time for the full order model was very high. For a problem of a significant number of degrees of freedom, the full order design will be certainly unpractical and efficient reduction techniques are essential.

As a future perspective, it is possible to consider the use of a spatial $\mathcal{H}_\infty$ norm [12] in order to account simultaneous vibration points and the integrated design of the control and the structure to have a more global optimization [22,32].

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Responsibility notice

The authors are the only responsible for the printed material included in this paper.

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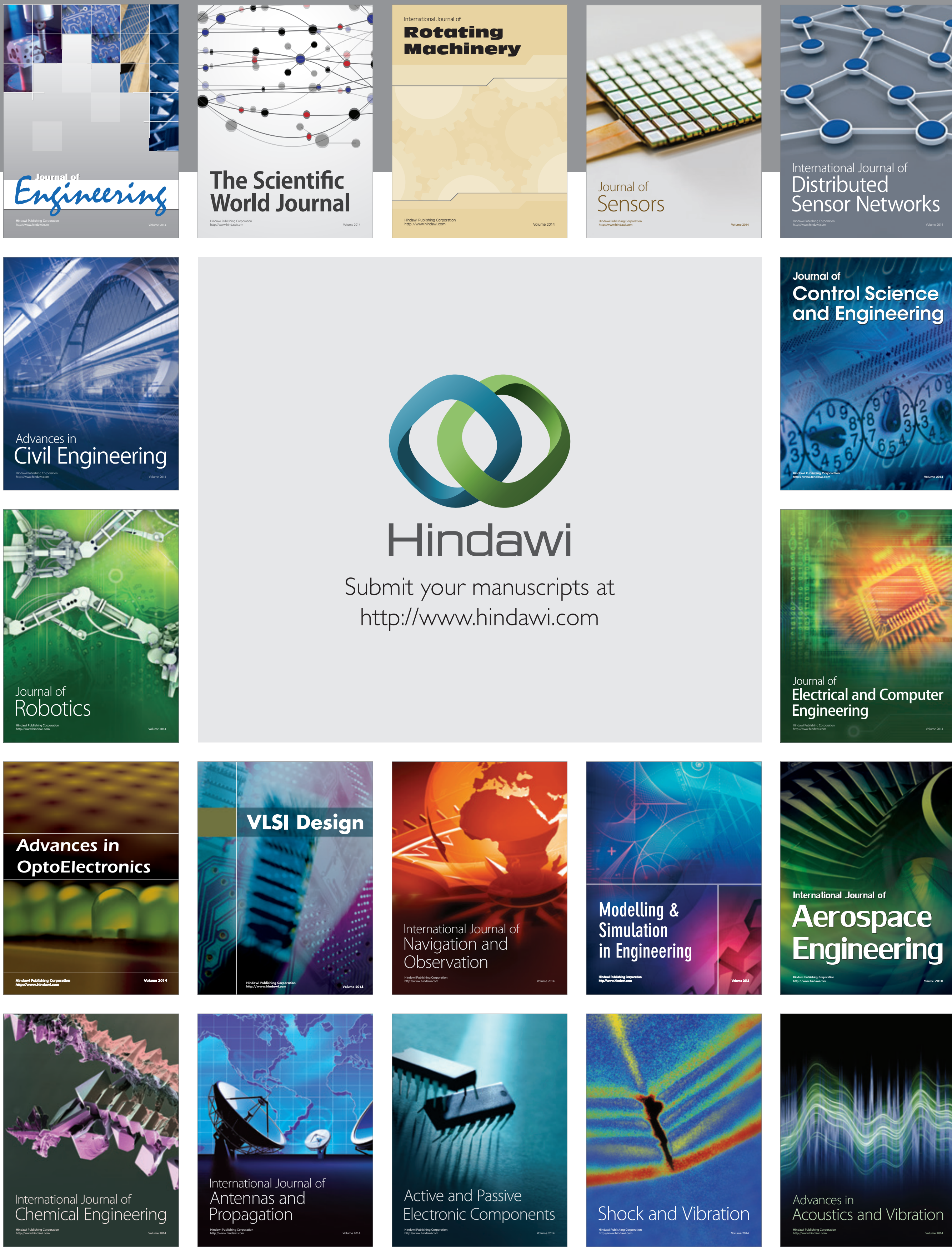
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