On the optimal shock isolation of a system with one and a half degrees of freedom

D.V. Balandin [a], N.N. Bolotnik [b] and W.D. Pilkey [c,*]

[a] *Research Institute for Applied Mathematics and Cybernetics of Nizhny Novgorod State University, 10 Uljanov St., Nizhny Novgorod 603005, Russia*

[b] *Institute for Problems in Mechanics of the Russian Academy of Sciences, 101-1 Prospect Vernadskogo, Moscow 117526, Russia*

[c] *Department of Mechanical and Aerospace Engineering, Thornton Hall, University of Virginia, Charlottesville, VA 22903-2442, USA*

The limiting performance of shock isolation of a system with one and a half degrees of freedom is studied. The possibility of using a single-degree-of-freedom model for this analysis is investigated. The error of such an approximation is estimated. Numerical examples are presented.

1. Introduction

The most advanced achievements in the theory of optimum impact and shock isolation have been made for a single-degree-of-freedom model. This model consists of a rigid body (the object to be isolated) connected to a moving base by means of an isolator (controller). It is assumed that the body and the base move rectilinearly along the same line. The base is subjected to a shock disturbance modeled by an acceleration pulse applied to it. In simpler cases, the acceleration pulse is assumed to be precisely prescribed. The limiting performance analysis of such systems implies the determination of an optimal open-loop control force acting between the base and the body that would minimize the peak magnitude of the displacement of the body, provided that the peak force transmitted to it does not exceed a prescribed value. Also, a formulation is used in which the peak force transmitted to the body is to be minimized, with the constraint being imposed on the peak displacement. For a comprehensive presentation of the state-of-the-art of the theory of optimal shock isolation, see [1]. This book contains an extensive bibliography of relevant publications. See also [2,4].

Any realistic body, possessing elastic and dissipative properties, is not perfectly rigid. When utilizing methods developed for a rigid body model in the practice of isolation system design, one should have an estimate of the accuracy provided by the single-degree-of-freedom approximation in terms of the performance index. This estimate depends on the elastic and dissipative characteristics of the body to be isolated. In the present paper, we obtain such an estimate for the case where the body is modeled by a mass to which a Voigt element is attached.

The problem treated in the present paper can be related to such problems as the design of optimal isolation systems for automobiles. A typical example would be the design of padding intended to isolate the legs of an automobile driver in a frontal crash. The goal could be to prevent fracture of the right leg of the driver. At the instant of crash, the driver is usually pressing on the brake pedal with his or her right leg. The collision of the automobile with another car or with a fixed obstacle can cause a large axial impact loading of the leg which can lead to fracture. In order to reduce hazardous consequences of the impact, the floor pan of an automobile near the brake pedal can be made compliant to help isolate the leg from high impact loadings. In other words, the automobile is equipped with a shock isolator, such as padding. The mechanical behavior of the human leg under axial shock loading in automobile crashes is sometimes modeled by a mass with a Voigt viscoelastic element [3]. This effective mass takes into account the influence of the masses of the upper leg and lower leg, while the Voigt element models viscoelastic properties of the lower leg which is especially vulnerable in frontal crashes. The results presented in this paper permit the estimation of the minimum rattlespace for the isolator ensuring that the peak force

*Corresponding author: Tel.: +1 804 924 3291; Fax: +1 804 924 3020; E-mail: wdp@virginia.edu.

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transmitted to the leg does not exceed a limiting value beyond which the lower leg could be fractured.

2. Description of the mechanical system

Consider a mechanical system consisting of a heavy body to which a Voigt spring-damper element is attached. See Fig. 1. The masses of the spring and damper are neglected. The system is attached to a movable base by means of a shock isolator (controller) generating the control force applied to the free end of the Voigt element. The base and the body are assumed to move along a straight line. The motion of this system is governed by the system of equations

$$m(\ddot{x} + \ddot{z}) + C(\dot{x} - \dot{y}) + K(x - y) = 0, \qquad (2.1)$$

$$C(\dot{x} - \dot{y}) + K(x - y) = -F, \qquad (2.2)$$

where m is the mass of the body, z is the displacement of the base with respect to an inertial reference frame, x is the displacement of the body relative to the base, y is the displacement of the free end of the Voigt element with respect to the base, C and K are the damping and stiffness coefficients, and F is the control force. Equations (2.2) and (2.2) can be equivalently represented in the form

$$m\ddot{x} = F - m\ddot{z}, \qquad (2.3)$$

$$C(\dot{x} - \dot{y}) + K(x - y) = -F. \qquad (2.4)$$

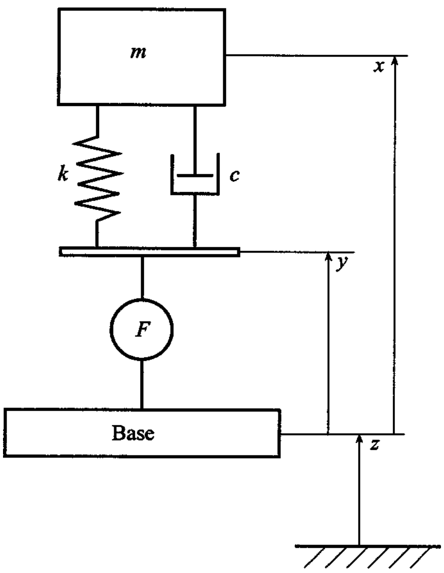

Fig. 1. A heavy body with a Voigt element attached to a moving base by an isolator.

Note that Eq. (2.3) coincides with the equation of motion of the body relative to the moving base, with the control force being applied directly to the body, i.e., the body does not "feel" the presence of the Voigt element. This is a consequence of the assumption that the mass of the entire system is concentrated in the body.

The system of Eqs (2.3) and (2.4) is a simple model of a viscoelastic structure isolated from shock applied to the base. The variable y may be interpreted as the displacement of this structure relative to the base and the difference $x - y$ as its deformation. In accordance with Eq. (2.4), the control force is equal in magnitude to the internal force developed in the viscoelastic structure. The time history of the base acceleration, $\ddot{z}$, is treated as the external disturbance.

We introduce the notation

$$u = -\frac{F}{m},\ v = -\ddot{z},\ c = \frac{C}{m},\ k = \frac{K}{m} \qquad (2.5)$$

to represent Eqs (2.3) and (2.4) in the form

$$\ddot{x} + u = v, \qquad (2.6)$$

$$c(\dot{x} - \dot{y}) + k(x - y) = u. \qquad (2.7)$$

3. Limiting performance analysis of the isolation system

3.1. Problem formulation

Consider two problems of limiting isolation capabilities (limiting performance problems) for the system of Eqs (2.6) and (2.7). We choose the maximum magnitude of the displacement of the free end of the Voigt element,

$$J_1(u) = \max_{t\in[0,\infty)} |y(t)|, \qquad (3.1)$$

and the maximum magnitude of the absolute acceleration of the body,

$$J_2(u) = \max_{t\in[0,\infty)} |u(t)|, \qquad (3.2)$$

to be the performance criteria.

Problem 1. For a system described by Eqs (2.6) and (2.7) subject to the initial conditions,

$$x(0) = 0,\ \dot{x}(0) = 0,\ y(0) = 0, \qquad (3.3)$$

and a prescribed disturbance $v(t)$, define an optimal open-loop control $u^0(t)$ such that

$$J_2(u^0) = \min_u J_2(u), \tag{3.4}$$

provided

$$J_1(u) \leqslant D, \tag{3.5}$$

where D is a prescribed positive number.

In terms of the viscoelastic structure, Problem 1 involves the minimization of the maximum magnitude of the internal force developed in this structure with the constraint being imposed on the peak displacement of it. The initial conditions of Eq. (3.3) imply that at the initial time instant, the position and velocity of the center of mass of the viscoelastic structure are specified ($x(0) = 0,\ \dot{x}(0) = 0$) and the structure is undeformed ($x(0) - y(0) = 0$). Note that the initial rate of deformation, $\dot{x} - \dot{y}$ is not prescribed. This is a consequence of the assumption that the mass of the entire system is concentrated at one point. In accordance with Eq. (2.7), the rate of deformation is uniquely expressed in terms of the deformation and the control force applied to the system.

Problem 2. For a system described by Eqs (2.6) and (2.7) subject to the initial conditions,

$$x(0) = 0,\ \dot{x}(0) = 0,\ y(0) = 0, \tag{3.6}$$

and a prescribed disturbance $v(t)$, define an optimal open-loop control $u_0(t)$ such that

$$J_1(u_0) = \min_u J_1(u), \tag{3.7}$$

provided

$$J_2(u) \leqslant u_*, \tag{3.8}$$

where u_* is a prescribed positive number.

Problem 2 implies the minimization of the displacement of the body with the constraint being imposed on the internal force developed in it.

3.2. Duality of optimization problems

Problems 1 and 2 are very similar. The only difference is that in Problem 1, the peak force transmitted to the body is minimized, while the peak displacement is subjected to a constraint and in Problem 2 these criteria change place. Solutions of these two problems are also closely related.

In Problem 1, the optimal control u^0 depends on the maximum displacement, D, allowed for the body. To indicate this fact, we will write $u^0 = u^0_D$. In Problem 2, the optimal control u_0 depends on the maximum magnitude, u_*, of the absolute acceleration allowed for the body, i.e., $u_0 = u_0^{u_*}$.

Suppose we have solved Problem 2 for all values of u_* for some interval $[u_-, u_+]$. Then we can construct the function $g(u_*)$

$$g(u_*) = J_1\big(u_0^{u_*}\big), \tag{3.9}$$

showing the minimum value of the peak displacement of the body, depending on the maximum value of force allowed to be transmitted to it. The function $g(u_*)$ is nonincreasing. This follows from the fact that the greater the u_*, the larger the set of admissible controls and, hence, the lower the minimum value of the performance index. Normally, this function is continuous, monotonically decreasing, and has the shape shown in Fig. 2. The range of the function $g(u_*)$ defined for $u_* \in [u_-, u_+]$ is the interval $[D_-, D_+]$, where

$$D_- = g(u_+), \qquad D_+ = g(u_-). \tag{3.10}$$

In the literature on shock isolation, the function defined by Eq. (3.9) is usually referred to as the *limiting performance characteristic* or the *trade-off curve*.

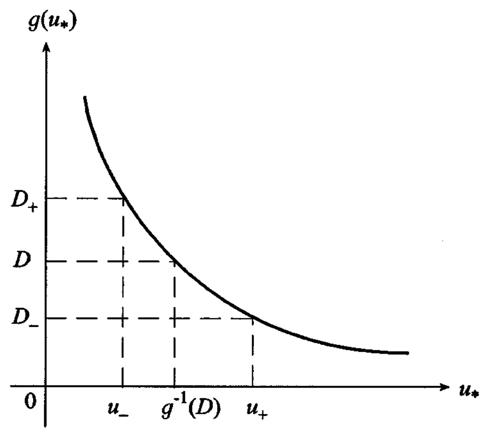

Fig. 2. General shape of the trade-off curve and illustration of the duality property.

If the trade-off curve constructed by solving Problem 2 for all $u_* \in [u_-, u_+]$ and the corresponding optimal controls $u_0^{u_*}$ are known we can obtain the solution of Problem 1 for any $D \in [D_-, D_+]$ without solving the latter problem directly. This solution is defined as

$$J_2\big(u_D^0\big) = g^{-1}(D), \quad u_D^0 = u_0^{g^{-1}(D)}, \tag{3.11}$$

where the function $g^{-1}(\cdot)$ is the inverse of $g(\cdot)$. As is apparent from Eq. (3.11), the optimal control, u_D^0, for Problem 1 with a specific value of D coincides with an optimal control, $u_0^{u_*}$, for Problem 2 with $u_* = g^{-1}(D)$. This is the essence of the duality property of optimization problems. Note that this property is not specific only to limiting performance problems for shock isolation systems. This is the general case for optimization problems with two performance criteria, one of which is to be minimized, while the other is subject to the constraint.

The simplest method of calculating the quantity $g^{-1}(D)$ is a graphical method using the trade-off curve $g(u_*)$ shown in Fig. 2. To calculate this quantity one should draw a line through the point $(0, D)$ parallel to the u_*-axis on the coordinate plane gu_* and find the point of intersection of this line with the trade-off curve. The abscissa of the intersection point is the desired value $g^{-1}(D)$.

In the subsequent sections, we deal mostly with Problem 2, since it is easier to analyze theoretically.

4. Estimates for the solution of Problem 2

4.1. Maximum deformation

To estimate the maximum deformation we introduce the notation

$$\xi = x - y \tag{4.1}$$

to represent Eq. (2.7) in the form

$$c\dot{\xi} + k\xi = u. \tag{4.2}$$

The solution of this equation with the zero initial condition, $\xi(0) = 0$, is expressed as

$$\xi(t) = \frac{1}{c}\int_0^t \exp\left(-\frac{k}{c}(t-\tau)\right) u(\tau)\,\mathrm{d}\tau, \tag{4.3}$$

which implies the estimate

$$\begin{aligned}\big|\xi(t)\big| &\leqslant \frac{u_*}{c}\int_0^t \exp\left(-\frac{k}{c}(t-\tau)\right)\mathrm{d}\tau \\ &= \frac{u_*}{k}\left[1 - \exp\left(-\frac{k}{c}t\right)\right],\end{aligned} \tag{4.4}$$

where u_* is the maximum magnitude allowed for the control variable u. Equation (4.4) implies a simple upper estimation for the deformation magnitude,

$$\big|\xi(t)\big| \leqslant \frac{u_*}{k}. \tag{4.5}$$

Hence, the deformation of the Voigt element does not exceed the static deformation produced by the maximum allowable control force.

4.2. Problem of optimal shock isolation for a single-degree-of-freedom system

Consider the problem of optimal isolation for a single-degree-of-freedom system governed by Eq. (2.6), with the external disturbance coinciding with that of Problem 2. Denote the peak displacement for this system by $\widetilde{J}_1$, i.e.,

$$\widetilde{J}_1(u) = \max_{t\in[0,\infty)} \big|x(t)\big|, \tag{4.6}$$

where $x(t)$ is the solution of the differential equation

$$\ddot{x} + u = v \tag{4.7}$$

with the initial conditions

$$x(0) = 0, \quad \dot{x}(0) = 0. \tag{4.8}$$

Problem 3. For the system governed by Eq. (4.7) with the initial conditions of Eq. (4.8), find an optimal control $\widetilde{u}_0$ such that

$$\widetilde{J}_1(\widetilde{u}_0) = \min_u \widetilde{J}_1(u), \tag{4.9}$$

provided

$$\big|u(t)\big| \leqslant u_*. \tag{4.10}$$

Problem 3 is the well-known fundamental problem of optimal shock isolation of a single-degree-of-freedom system. The solution of this problem is available for a number of typical shapes of the external disturbance. See, for example, [1,2,4].

4.3. Comparison of the optimal values of the performance indices in Problems 2 and 3

The absolute value of the difference between the optimal values $J_1(u_0)$ and $\widetilde{J}_1(\widetilde{u}_0)$ of the performance indices in Problems 2 and 3, respectively, is estimated by

$$\left|J_1(u_0) - \widetilde{J}_1(\widetilde{u}_0)\right| \leqslant \frac{u_*}{k}. \tag{4.11}$$

To prove this inequality, we use Eq. (4.1) which leads to the relations

$$x = \xi + y, \tag{4.12}$$

$$y = x - \xi. \tag{4.13}$$

The relations of Eqs (4.12) and (4.13) yield the inequalities

$$\left|x(t)\right| \leqslant \left|y(t)\right| + \left|\xi(t)\right| \tag{4.14}$$

and

$$\left|y(t)\right| \leqslant \left|x(t)\right| + \left|\xi(t)\right|, \tag{4.15}$$

respectively.

Taking into account the estimate of Eq. (4.5), we obtain

$$\left|x(t)\right| \leqslant \left|y(t)\right| + \frac{u_*}{k}, \tag{4.16}$$

$$\left|y(t)\right| \leqslant \left|x(t)\right| + \frac{u_*}{k}, \tag{4.17}$$

where $x(t)$ and $y(t)$ define a solution of the system of Eqs (2.6) and (2.7) with the initial conditions of Eq. (3.6) for a control function $u(t)$. From Eqs (4.16) and (4.17), it follows that

$$\max_{t\in[0,\infty)} \left|x(t)\right| \leqslant \max_{t\in[0,\infty)} \left|y(t)\right| + \frac{u_*}{k}, \tag{4.18}$$

$$\max_{t\in[0,\infty)} \left|y(t)\right| \leqslant \max_{t\in[0,\infty)} \left|x(t)\right| + \frac{u_*}{k}. \tag{4.19}$$

The behavior of the variable x in this solution coincides with that of the variable x in the solution of Eq. (4.7) with the initial conditions of Eq. (4.8) for the same control $u(t)$. This follows from the fact that Eqs (2.6) and (4.7), as well as the initial conditions for x in Eqs (3.6) and (4.8), coincide. With reference to the definitions of Eqs (3.1) and (4.6) for the criteria $J_1(u)$ and $\widetilde{J}_1(u)$, one can rewrite Eqs (4.18) and (4.19) in the form

$$\widetilde{J}_1(u) \leqslant J_1(u) + \frac{u_*}{k}, \tag{4.20}$$

$$J_1(u) \leqslant \widetilde{J}_1(u) + \frac{u_*}{k}. \tag{4.21}$$

Substitute $u = u_0$ (the optimal control for Problem 2) into Eq. (4.20) to obtain

$$\widetilde{J}_1(u_0) \leqslant J_1(u_0) + \frac{u_*}{k}. \tag{4.22}$$

Since $\widetilde{u}_0$ is the optimal control for Problem 3, we have the inequality

$$\widetilde{J}_1(\widetilde{u}_0) \leqslant \widetilde{J}_1(u_0). \tag{4.23}$$

The relations of Eqs (4.22) and (4.23) imply that

$$\widetilde{J}_1(\widetilde{u}_0) - J_1(u_0) \leqslant \frac{u_*}{k}. \tag{4.24}$$

Substitute now $u = \widetilde{u}_0$ (the optimal control for Problem 3) into Eq. (4.21) to obtain

$$J_1(\widetilde{u}_0) \leqslant \widetilde{J}_1(\widetilde{u}_0) + \frac{u_*}{k}. \tag{4.25}$$

Since u_0 is the optimal control for Problem 2, we have the inequality

$$J_1(u_0) \leqslant J_1(\widetilde{u}_0). \tag{4.26}$$

It follows from Eqs (4.25) and (4.26) that

$$J_1(u_0) - \widetilde{J}_1(\widetilde{u}_0) \leqslant \frac{u_*}{k}. \tag{4.27}$$

Equations (4.24) and (4.27) imply the relation of Eq. (4.11).

For practical use, it is convenient to represent the estimate of Eq. (4.11) in the form

$$\frac{|J_1(u_0) - \widetilde{J}_1(\widetilde{u}_0)|}{\widetilde{J}_1(\widetilde{u}_0)} \leqslant \frac{u_*}{k\widetilde{J}_1(\widetilde{u}_0)}. \tag{4.28}$$

5. Graphical estimation technique

5.1. Preliminary considerations

It is useful to represent the estimate of Eq. (4.11) graphically. Let us rewrite Eq. (4.11) in the form

$$\left|J_1\left(u_0^{u_*}\right) - \widetilde{J}_1\left(\widetilde{u}_0^{u_*}\right)\right| \leqslant \frac{u_*}{k} \tag{5.1}$$

indicating the dependence of the optimal controls u_0 (for Problem 2) and $\widetilde{u}_0$ (for Problem 3) on u_*. The function $\widetilde{J}_1(\widetilde{u}_0^{u_*})$ defines the trade-off curve for Problem 3 (for the single-degree-of-freedom model). This curve is assumed to be known. The function $J_1(u_0^{u_*})$ defines the trade-off curve for Problem 2. This curve is unknown. The inequality of Eq. (5.1) implies that the trade-off curve for Problem 2 lies within a corridor which can be defined as

$$G_-(u_*) \leqslant J_1\left(u_0^{u_*}\right) \leqslant G_+(u_*), \tag{5.2}$$

where the boundary functions $G_-(u_*)$ and $G_+(u_*)$ have the form

$$G_-(u_*) = \widetilde{J}_1\left(\widetilde{u}_0^{u_*}\right) - \frac{u_*}{k},$$
$$G_+(u_*) = \widetilde{J}_1\left(\widetilde{u}_0^{u_*}\right) + \frac{u_*}{k}. \tag{5.3}$$

The plots of these functions are easy to construct if the trade-off curve $\widetilde{J}_1(\widetilde{u}_0^{u_*})$ is known. We will refer to the corridor defined by Eq. (5.2) as the *trade-off corridor*, since all trade-off curves for Problems 1 and 2 lie within it.

5.2. *The case of a half-sine pulse disturbance*

We choose to illustrate the construction of the corridor defined by Eq. (5.2) for the case where the external disturbance has the shape of a half-sine pulse

$$v = \begin{cases} a\sin\left(\frac{\pi}{T_*}t\right), & \text{if } 0 \leqslant t \leqslant T_*, \\ 0, & \text{if } t > T_*. \end{cases} \tag{5.4}$$

5.2.1. *Trade-off curve for a single-degree-of-freedom system*

Figure 3 shows the trade-off curve $\widetilde{J}_1(\widetilde{u}_0^{u_*})$ for a single-degree-of-freedom system subjected to the half-sine pulse disturbance of Eq. (5.4). This curve is represented in normalized (dimensionless) variables, with the optimal value of the peak displacement, $\widetilde{J}_1(\widetilde{u}_0^{u_*})$, being normalized by the quantity

$$d = aT_*^2/\pi \tag{5.5}$$

and the maximum magnitude of the absolute acceleration allowed for the body, u_*, by the maximum acceleration of the base, a. In other words, Fig. 3 shows the plot of the function $J_*(w_*)$, where

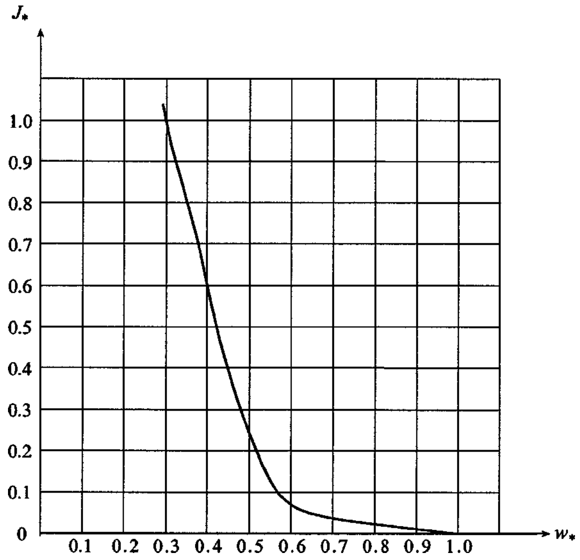

Fig. 3. Trade-off curve for a single-degree-of-freedom system subject to a half-sine pulse disturbance.

$$J_* = \frac{\widetilde{J}_1(\widetilde{u}_0^{u_*})}{d}, \qquad w_* = \frac{u_*}{a}. \tag{5.6}$$

Figure 3 provides a unified limiting performance characteristic for the single-degree-of-freedom system of Problem 3 subjected to the disturbance of the form of Eq. (5.4). Using this curve, one can determine the minimum value of the peak displacement, $\widetilde{J}_1(\widetilde{u}_0^{u_*})$, for any specific set of the input data, which involve the quantities u_*, a, and T_*. To this end, one should

- Calculate the quantity $w_* = u_*/a$.
- Find the corresponding value $J_*(w_*)$ using the curve of Fig. 3.
- Calculate the quantity $\widetilde{J}_1(\widetilde{u}_0^{u_*}) = J_*(w_*)d$.

The trade-off curve of Fig. 3 can be used also for solving the problem which is dual to Problem 3, i.e., one of the minimization of the maximum magnitude of the absolute acceleration experienced by the body, provided the peak displacement does not exceed a prescribed quantity D. We will denote the desired minimum of the peak absolute acceleration by $\widetilde{J}_2(\widetilde{u}_D^0)$. To determine the value $\widetilde{J}_2(\widetilde{u}_D^0)$ for a specific set of the input data (D, a, and T_*), one should

- Calculate the quantity $J_*^0 = D/d$.
- Find the corresponding value w_*^0 using the trade-off curve of Fig. 3.
- Calculate the desired quantity $\widetilde{J}_2(\widetilde{u}_D^0) = w_*^0 a$.

5.2.2. Trade-off corridor

Divide Eq. (5.2) by the quantity d of Eq. (5.5). Then, with reference to Eq. (5.3), we obtain

$$g_-(w_*) \leqslant J(w_*) \leqslant g_+(w_*), \tag{5.7}$$

where

$$J(w_*) = \frac{J_1(u_0^{u_*})}{d}, \tag{5.8}$$

$$g_-(w_*) = J_*(w_*) - \varepsilon w_*,$$

$$g_+(w_*) = J_*(w_*) + \varepsilon w_*. \tag{5.9}$$

The parameter ε in Eq. (5.9) is defined as

$$\varepsilon = \frac{\pi}{kT_*^2}. \tag{5.10}$$

The inequalities of Eq. (5.7) represent the trade-off corridor in a normalized form. The shape of this corridor depends only on one dimensionless parameter ε. Qualitatively, the trade-off corridor is shown in Fig. 4; the trade-off curve for the single-degree-of-freedom system is depicted by the solid line.

5.2.3. Determination of the relative error for Problem 2

The estimate of the relative error of the approximation of the original system by a single-degree-of-freedom system for Problem 2 (Eq. 4.28) can be represented as

$$\frac{|J_1(u_0^{u_*}) - \widetilde{J}_1(\widetilde{u}_0^{u_*})|}{\widetilde{J}_1(\widetilde{u}_0^{u_*})} \leqslant \frac{\varepsilon w_*}{J_*(w_*)}. \tag{5.11}$$

One can calculate the right-hand side of Eq. (5.11) for any set of the input data, involving the quantities u_*, a, T_*, and k, by using the trade-off curve for the single-degree-of-freedom system. To this end, one should

- Calculate $w_* = u_*/a$ and $\varepsilon = \pi/(kT_*^2)$.
- Determine $J_*(w_*)$ using the trade-off curve.
- Calculate the ratio on the right-hand side of Eq. (5.11).

5.2.4. Construction of estimates for Problem 1

Using the trade-off corridor shown in Fig. 4, one can also estimate the relative error of the approximation of the original system by the single-degree-of-freedom system for Problem 1, which is dual to Problem 2. The input data for Problem 1 involve the quantities D, a, T_*, and k. To obtain the desired estimate one should

- Calculate the parameter $\varepsilon = \pi/(kT_*^2)$.
- Plot the curves $g_-(w_*)$ and $g_+(w_*)$ together with the trade-off curve, $J_*(w_*)$, for the single-degree-of-freedom system.
- Calculate the quantity $J_*^0 = D/d$, where d is defined by Eq. (5.5).
- Using the plots of the functions $g_-(w_*)$, $g_+(w_*)$, and $J_*(w_*)$, determine the values $w_* = w_*^-$, $w_* = w_*^+$, and $w_* = w_*^0$ such that $g_-(w_*^-) = J_*^0$, $g_+(w_*^+) = J_*^0$, $J_*(w_*^0) = J_*^0$. This stage is illustrated by Fig. 5.

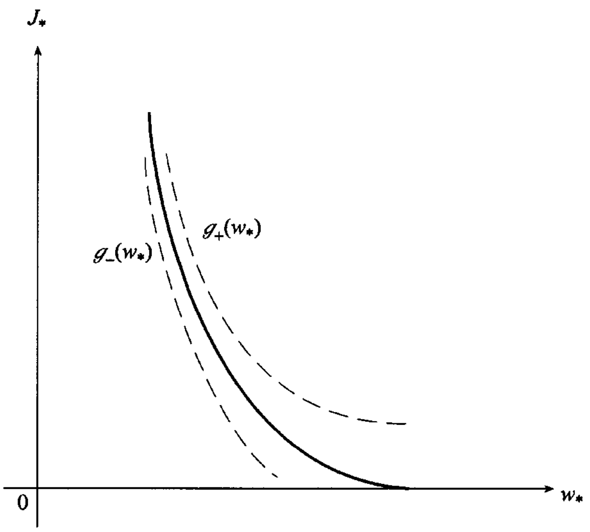

Fig. 4. Trade-off corridor.

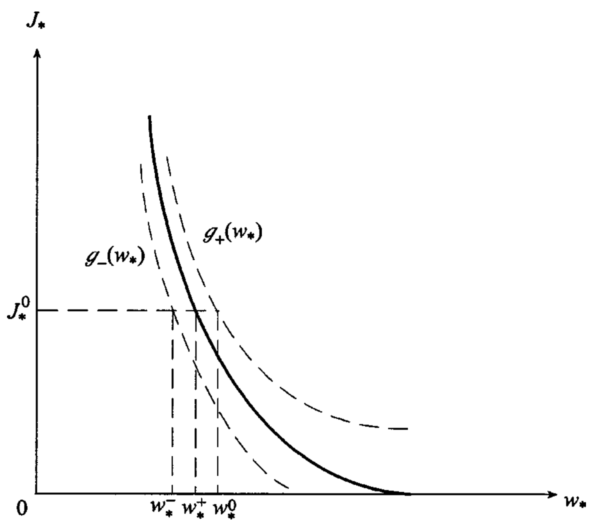

Fig. 5. Error estimate for the problem of the minimization of the peak value of the transmitted force.

- Estimate the absolute error as

$$\left|J_2\left(u_D^0\right) - \widetilde{J}_2\left(\widetilde{u}_D^0\right)\right| \leqslant a\Delta, \tag{5.12}$$

where

$$\Delta = \max\left\{w_*^0 - w_*^-,\ w_*^+ - w_*^0\right\}. \tag{5.13}$$

- Estimate the relative error as

$$\frac{|J_2(u_D^0) - \widetilde{J}_2(\widetilde{u}_D^0)|}{\widetilde{J}_2(\widetilde{u}_D^0)} \leqslant \frac{\Delta}{w_*^0}. \tag{5.14}$$

6. Numerical example

6.1. *Parameters of the system*

Let the external disturbance be a half-sine pulse of the form of Eq. (5.4), with

$$a = 500\,\frac{\text{m}}{\text{s}^2}, \qquad T_* = 0.07\ \text{s}. \tag{6.1}$$

The parameters of the model are identified as

$$m = 11.5\ \text{kg}, K = 963\,\frac{\text{kN}}{\text{m}}, \tag{6.2}$$

which corresponds to

$$k = \frac{K}{m} = 84 \cdot 10^3\ \text{s}^{-2}. \tag{6.3}$$

6.2. *Estimates for Problem 2*

Set the maximum absolute acceleration to be experienced by the body to be

$$u_* = 217\,\frac{\text{m}}{\text{s}^2}, \tag{6.4}$$

which corresponds to the maximum force allowed to be transmitted to the body of

$$F_* = mu_* = 2500\ \text{N}. \tag{6.5}$$

The solution of Problem 3 (with a single-degree-of-freedom model) for these parameters yields

$$\widetilde{J}_1(\widetilde{u}_0) = 36\ \text{cm}. \tag{6.6}$$

One can readily obtain this quantity using the trade-off curve of Fig. 3. To do so, substitute the values of a, T_*, and u_* into Eqs (5.5) and (5.6) to obtain

$$d = 78\ \text{cm}, \qquad w_* = 0.43. \tag{6.7}$$

Then, using the curve of Fig. 3, find the value of J_* for $w_* = 0.43$. This yields

$$J_* = 0.46. \tag{6.8}$$

By multiplying this value by d, we obtain the value of Eq. (6.6) for the minimum of the peak displacement.

Substitution of this solution and the parameters of Eqs (6.3) and (6.4) into Eq. (4.28) leads to the estimate

$$\frac{|J_1(u_0) - \widetilde{J}_1(\widetilde{u}_0)|}{\widetilde{J}_1(\widetilde{u}_0)} \leqslant 0.007. \tag{6.9}$$

6.3. *Estimates for Problem 1*

We identify the maximum peak displacement as

$$D = 36\ \text{cm}. \tag{6.10}$$

Following the technique described in Section 4.2.4, we determine the estimate for the relative error (Eq. 5.14) to be

$$\frac{|J_2(u_D^0) - \widetilde{J}_2(\widetilde{u}_D^0)|}{\widetilde{J}_2(\widetilde{u}_D^0)} \leqslant 0.008, \tag{6.11}$$

with

$$\widetilde{J}_2\left(\widetilde{u}_D^0\right) = 217\,\frac{\text{m}}{\text{s}^2}.$$

7. Conclusions

If the static deformation of the Voigt element produced by the maximum force allowed to be transmitted to the body is substantially less than the absolute minimum of the peak displacement of the rigid single-degree-of-freedom system model, then the rigid model provides a good approximation to the original system. The larger the stiffness of the Voigt element, the more accurate the rigid body approximation is in terms of the performance index. In this case, the optimal control for the single-degree-of-freedom system is a near-optimal control for the original system with one and a half degrees of freedom.

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