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Frequency and Spatial Shaping of Inputs for Multiaxis Shaker Testing

Controlled amplitude and phase relationships between multiaxial shaker inputs (i.e., spatial shaping) provides for more realistic simulation of a service environment than does conventional frequency shaping alone. Spatial shaping is described in terms of a basic mathematical model relating test article response (absolute and relative motions) to excitation by the shaker. Advantages and objectives are viewed through spectral relationships. The objective of simulating dynamic responses as in service is shown to be the duplication of the resultant cross-modal response for all important modes, even if the sources of excitation in service are unknown.

INTRODUCTION

In recent years, multiaxis shakers have been proposed and built for environmental and stress screen testing of structures (Bonnet, 1986; Chang and Frydman, 1990; Thompson, 1986; Bausch and Good, 1992, 1995). It is generally agreed that multiaxis shakers provide the potential for better control and more realistic test environments than single axis shakers. However, the extent of this potential to provide real benefit has never been clearly defined. Here we provide a theoretical basis for multiaxis shaker testing, where controlling multiple axis excitations simultaneously is defined as spatial shaping. The theoretical basis provides a reference for clarifying and realizing the potential of multiaxis shakers and a basis for design of the control systems for multiaxis shakers.

Significant previous work was done to design multiple axis shaker control (Fisher and Posehn, 1977; Smallwood, 1978, 1982a, 1982b; Greenfield, 1983; Smith et al., 1980; Hobbs and Mercado, 1984; Lehman, 1985; Frydman, 1988; Stroud and Hamma, 1988). Among these works, methods for defining multiple inputs with prescribed spectral relationships were developed and methodologies for adjusting those inputs to achieve given spectral outputs were developed. Here we focus not on control methodologies, but upon the theoretical basis of structural dynamic interactions to define the objectives, possibilities, and limitations of structural testing and control using a single platform, multiaxis shaker.

Dynamic environmental or shaker testing has long taken advantage of the independence of dynamic modes and their natural separation according to frequency. Dynamic excitation has thus been shaped according to frequency in order to excite individual modes and/or separate the effects of different modes. A test environment is thus typically defined in the form of a vibration amplitude as a function of frequency, with the intent that certain frequencies will excite particu-

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lar modes that may cause failure. The development of multiaxis shakers now creates the possibility of *spatial shaping* as well as the traditional *frequency shaping* of the test environment. Controlled excitation of more modes independently and simultaneously makes possible the creation of a test environment that more accurately simulates the service environment and is less likely to result in overtest and undertest associated with less controlled modes and degradation due to the greater handling and exposure associated with sequential testing on successive axes. Better simulation enables more accurate prediction of survival or failure in service and more reliable monitoring of a test article being tested periodically for degradation with age.

BASIC DYNAMIC MODEL FOR SHAKER EXCITATION

The following model will be useful in describing frequency and spatial shaping. Although the parameters of a model of a system under test will typically not be known during a given test, the model will be helpful in allowing us to quantify the objectives and possibilities of multiaxis shaker testing. Consider the positive definite system with position relative to a fixed datum defined by the vector $\mathbf{x}$ and natural motion defined by the equation set

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}.$$

Now suppose the datum to which $\mathbf{x}$ is referred is not fixed. Let the position of the datum be the vector $\mathbf{x}_d$, so that the total motion at each degree of freedom of the system is defined by $\mathbf{x}_t = \mathbf{x} + \mathbf{B}\mathbf{x}_d$, where $\mathbf{B}\mathbf{x}_d$ represents the rigid body motion of the system corresponding to the motion of the datum. If motion of the datum is assumed to affect the kinetic energy of the system but not the potential energy (strain energy not affected by rigid body motion) or dissipative forces (no dissipation from rigid body motion), the natural motion will be described by the equation set

$$\mathbf{M}(\ddot{\mathbf{x}} + \mathbf{B}\ddot{\mathbf{x}}_d) + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}$$

or

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = -\mathbf{M}\mathbf{B}\ddot{\mathbf{x}}_d$$

or

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{C}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = -\mathbf{M}\mathbf{B}\mathbf{a}_d, \quad (1)$$

where $\mathbf{a}_d$ represents the acceleration of the datum. This set of equations can be used then to model the motions relative to a shaker, $\mathbf{x}$, of the system subject to the shaker input accelerations, $\mathbf{a}_d$. It assumes that the motions of the shaker are small enough that gravitational and dissipative forces associated with rigid body motion are constant with respect to the motion coordinates of the system.

Consider first the undamped system for which normal modes exist. Although this restriction will be relaxed later in this development, it is easier to illustrate the basic concepts using normal modes. Assuming the natural frequencies, ω_i, and corresponding mode shapes, ϕ_i, we form the modal matrix

$$\Phi = [\phi_1\,\phi_2 \cdots \phi_n],$$

and we transform to modal coordinates such that $\mathbf{x} = \Phi\mathbf{q}$ by substituting into Eq. (1) with $\mathbf{C} = \mathbf{0}$,

$$\mathbf{M}\Phi\ddot{\mathbf{q}} + \mathbf{K}\Phi\mathbf{q} = -\mathbf{M}\mathbf{B}\mathbf{a}_d.$$

Premultiplying by the transpose of the modal matrix, Φ^t, we have

$$\Phi^t\mathbf{M}\Phi\ddot{\mathbf{q}} + \Phi^t\mathbf{K}\Phi\mathbf{q} = -\Phi^t\mathbf{M}\mathbf{B}\mathbf{a}_d. \quad (2)$$

Because the magnitudes of the mode shapes are arbitrary, for convenience we will normalize the mode shapes such that $\Phi^t\mathbf{M}\Phi = \mathbf{I}$, and Eq. (2) becomes

$$\ddot{\mathbf{q}} + \mathbf{W}\mathbf{q} = -\Phi^t\mathbf{M}\mathbf{B}\mathbf{a}_d, \quad (3)$$

where $\mathbf{W}$ is the diagonal matrix with the square of the natural frequencies down the diagonal. Equation (3) is an uncoupled set of equations of the form

$$\ddot{q}_i + \omega_i^2 q_i = f_i, \quad (4)$$

where the modal forces, f_i, are defined as the terms in the vector $\mathbf{f} = -\Phi^t\mathbf{M}\mathbf{B}\mathbf{a}_d$. We therefore view the matrix, $-\Phi^t\mathbf{M}\mathbf{B}$, as the transform that indicates how the accelerometer excitation vector, $\mathbf{a}_d$, affects (excites) each of the modes. In the frequency domain, letting the transforms be represented by capitals, Eq. (4) can be written

$$Q_i = \frac{1}{\omega_i^2 - \omega^2} F_i = \Delta(\omega) F_i, \quad (5)$$

which can be used to form the matrix set

$$\mathbf{Q} = \Delta_x(\omega) \mathbf{F}, \quad (6)$$

where $\Delta_x(\omega)$ is the diagonal matrix with the terms $1/(\omega_i^2 - \omega^2)$ at each diagonal element Δ_{xii}.

Substituting for the original displacement vector, $\mathbf{x} = \Phi\mathbf{q}$, results in

$$\mathbf{X} = \Phi\mathbf{Q} = \Phi\Delta_x(\omega) \mathbf{F}$$

or

$$\mathbf{X} = \Phi\mathbf{Q} = -\Phi\Delta_x(\omega)\Phi'\mathbf{MBA}_d. \quad (7)$$

Here we identify the matrix

$$\mathbf{H}_x(\omega) = -\Phi\Delta_x(\omega)\Phi'\mathbf{MB} \quad (8)$$

as the matrix of transfer functions relating the input accelerations from the shaker to the output displacements of the system relative to the shaker table displacement. To find the matix relating (relative) acceleration outputs, we multiply each term of the matrix by $-\omega^2$, resulting in

$$\mathbf{H}_a(\omega) = \Phi\Delta_a(\omega)\Phi'\mathbf{MB}, \quad (9)$$

where $\Delta_a(\omega)$ is the diagonal matrix with the terms $\omega^2/(\omega_i^2 - \omega^2)$ at each diagonal element, and the relative accelerations, $\mathbf{A_x}$, are found from the input accelerations, $\mathbf{A_d}$, by the relationship

$$\mathbf{A_x} = \mathbf{H}_a(\omega) \mathbf{A_d}. \quad (10)$$

If we define the vector $\mathbf{A}$ as the vector of absolute acceleration outputs, we note that $\mathbf{A} = \mathbf{A}_x + \mathbf{BA}_d$, so that

$$\begin{aligned}\mathbf{A} = \mathbf{A}_x + \mathbf{BA}_d &= \mathbf{H}_a(\omega) \mathbf{A}_d + \mathbf{BA}_d \\ &= (\mathbf{H}_a(\omega) + \mathbf{B}) \mathbf{A}_d = \mathbf{H}(\omega) \mathbf{A}_d.\end{aligned} \quad (11)$$

where we defined the frequency response function matrix

$$\mathbf{H}(\omega) = (\mathbf{I} + \Phi\Delta_a(\omega)\Phi'\mathbf{M}) \mathbf{B} = \mathbf{KB}, \quad (12)$$

where

$$\mathbf{K} = (\mathbf{I} + \Phi\Delta_a(\omega)\Phi'\mathbf{M}). \quad (13)$$

Now if we postmultiply both sides of Eq. (13) by Φ we have

$$\mathbf{K}\Phi = \Phi + \Phi\Delta_a(\omega)\Phi'\mathbf{M}\phi,$$

and note that $\Phi'\mathbf{M}\Phi = I$,

$$\begin{aligned}\mathbf{K}\Phi &= \Phi + \Phi\Delta_a(\omega) = \Phi (I + \Delta_a(\omega)) \\ &= \Phi\Delta(\omega),\end{aligned} \quad (14)$$

where $\Delta(\omega)$ is the diagonal matrix with the elements

$$\Delta_{ii} = \frac{\omega_i^2}{\omega_i^2 - \omega^2} = \frac{1}{1 - \left(\frac{\omega}{\omega_i}\right)^2}. \quad (15)$$

Postmultiplying both sides of (14) by $\mathbf{M}\Phi'$ results in

$$\mathbf{K}\Phi\mathbf{M}\Phi' = \mathbf{K} = \Phi\Delta(\omega)\mathbf{M}\Phi', \quad (16)$$

which substituted into (12) results in

$$\mathbf{H}(\omega) = \Phi\Delta(\omega)\mathbf{M}\Phi'\mathbf{B}. \quad (17)$$

Equation (17) relates the system physical properties and the response function. The last three matrices in Eq. (17) define the way in which the shaker inputs excite each mode. The geometric relationships between the mass distribution and the shaker coordinates is defined by the matrix $\mathbf{B}$. $\mathbf{M}\Phi'\mathbf{B}$ describes the distribution of the system inertia relative to the mode shpaes and the geometric relationship between the shaker and system coordinates. It tells us that to excite any mode, we must shake at least some mass "particles" in the direction of the mode shape for that mode. $\Delta(\omega)$ defines the dynamic amplification of the motion associated with each mode. The mode shape Φ distributes the modal motion to each physical coordinate. It describes how much of each modal displacement we will see at each location (or direction). We note from (14) that for $\omega \ll \omega_i$ for all i (i.e., for frequencies much lower than the system natural frequencies), $\Delta(\omega) = \mathbf{I}$ and $\mathbf{H}(\omega) = \Phi\mathbf{M}\Phi'\mathbf{B} = \mathbf{B}$, and the system moves as a rigid body.

To look at the frequency response at position i subject to input j, the term in the frequency response function can be written

$$h_{ij} = \sum_{k=1}^{n} \phi_i^k \Delta_k \left(\sum_{m=1}^{M} m_m \phi_{mj}^k \right),$$

where ϕ_i^k is the kth mode shape at position i, and ϕ_{mj}^k is the component of the kth mode shape at the location of mass particle m_m in the direction of the input j. The term in parentheses could be written as an integral over all mass particles, and Δ_k could be considered the dynamic amplification factor for the kth mode and is the only frequency dependent term in the frequency response function.

Let us define

$$\theta_{kj} = \left(\sum_{m=1}^{M} m_m \phi_{mj}^k \right) \tag{18}$$

as the kth modal mass influence coefficient in the jth direction, then

$$h_{ij} = \sum_{k=1}^{n} \phi_i^k \Delta_k \theta_{kj}. \tag{19}$$

The controllability (by the jth input) of the kth mode is defined by the term θ_{kj} whereas the observability (at the ith output) is defined by the term θ_i^k.

For systems with damping (nonproportional), normal modes do not exist; but the mode shapes become complex and the concepts above can be extended by considering the phase relationships between components and the spatial locations in the mode shapes. For these cases, the modal dynamic amplification factors become functions of position i and j and take the form

$$\Delta_{ij}^k(\omega) = \frac{1 + j\left(\zeta_k + \sqrt{1-\zeta_k^2}\tan(\varphi_i^k + \varphi_j^k)\left(\frac{\omega}{\omega_k}\right)\right)}{1 - \left(\frac{\omega}{\omega_k}\right)^2 + j\zeta_k\left(\frac{\omega}{\omega_k}\right)},$$

where ω_k is the modal undamped natural frequency and ζ_k is the modal damping coefficient. The mode shapes become in general complex and represent both amplitude and phase components of modal displacements; and the angles φ_i^k and φ_j^k are the phase angles of the ith and jth components, respectively, of the kth mode shape. For lightly damped systems, the imaginary term in the numerator above is small compared to the real term, and the modal dynamic amplification factor is only weakly dependent on position.

For convenience in our discussion it will be assumed that the dynamic amplification factor is a function of frequency only, and we will use the symbol $\Delta_k(\omega)$. The modal dynamic amplification factors $\Delta_k(\omega)$ are the key to frequency shaping of the inputs to excite particular modes, whereas the mode shapes and modal mass influence coefficients are key to spatial shaping of inputs to excite particular modes. For cases where damping is large, so that ζ_k and $\tan(\varphi_i^k + \varphi_j^k)$ are not much less than unity, spatial shaping will also affect the dynamic amplification factors. Although the mathematical uncoupling is not as straightforward, the concepts of spatial and frequency shaping are nevertheless applicable.

Relative vs. Absolute Acceleration Outputs

Equation (10) defined the relationship between the base acceleration inputs and the relative acceleration vector $\mathbf{A}_\mathbf{x}$. In many cases, it may be more desirable to produce a prescribed relative motion within the structure than to produce a prescribed absolute motion, because stresses and strain are a function of the relative rather than the absolute motions within the structure. In this case, we expand Eqs. (9) and (10) and look at the frequency response at position i subject to input j, where each term in the frequency response function can be written

$$h_{a_{ij}} = \sum_{k=1}^{n} \Phi_i^k \Delta_a^k \left(\sum_{m=1}^{M} \phi_m^k m_m b_{mj} \right),$$

where again ϕ_i^k is the kth mode shape at position i, and b_{mj} is the corresponding term in the $\mathbf{B}$ matrix defined earlier. Again, the term in parentheses could be written as an integral over all mass particles, and Δ_a^k could be considered the dynamic amplification factor for the kth mode; and it is the only frequency dependent term in the frequency response function. Here we will define

$$\chi_{kj} = \left(\sum_{m=1}^{M} \phi_m^k m_m b_{mj} \right) \tag{20}$$

as the kth modal mass influence coefficient in the jth direction and

$$\Delta_a^k(\omega) = \left(\frac{\omega}{\omega_k}\right)^2 \Delta_k(\omega) \tag{21}$$

as the kth modal relative amplification factor.

Then

$$h_{aij} = \sum_{k=1}^{n} \phi_i^k \Delta_a^k \chi_{kj}. \qquad (22)$$

Interpretation of Eqs. (20) and (22) parallels that of Eqs. (18) and (19) given earlier, except that the outputs are relative accelerations rather than absolute accelerations. Controlling the relative motions within a structure rather than the absolute motions will control the flexible body modes independent of the rigid body modes. Because the shaker table can usually not duplicate the rigid body modes, this may be desirable. Because a multiaxis shaker makes spatial shaping possible as well as frequency shaping, the controlled excitation of multiple modes within a given frequency band is possible, thus enabling a dynamic shaker test to more closely duplicate the dynamic service environment of the object to be tested.

SPECTRAL RELATIONSHIPS

Because all terms will be in the frequency domain, here we adopt the nomenclature that vectors will be lower case and matrices will be upper case. Consider the system where **x** is a vector of system inputs (accelerations), **y** a vector of outputs, and the transfer function matrix is defined as the matrix **H** such that

$$\mathbf{y} = \mathbf{H}\mathbf{x}. \qquad (23)$$

Here we will use **y** as a vector of generalized outputs, which could be either relative or absolute accelerations, or other variables of interest; and **x** is a vector of generalized inputs. Defining the spectral matrices $\mathbf{S_x} = \mathbf{xx}^*$, $\mathbf{S_y} = \mathbf{yy}^*$ where $\mathbf{x}^*$ and $\mathbf{y}^*$ are the transpose of the conjugates of **x** and **y**, respectively,

$$\mathbf{y}^* = \mathbf{x}^*\mathbf{H}^*,$$

so that

$$\mathbf{S_y} = \mathbf{yy}^* = \mathbf{Hxx}^*\mathbf{H}^* = \mathbf{HS_xH}^*, \qquad (24)$$

gives the relationship between the input spectral matrix and the output spectral matrix. For the case where we have a single output to be controlled by multiple inputs, the output

$$y = \mathbf{h}_r\mathbf{x}, \qquad (25)$$

where $\mathbf{h}_r$ is a row vector of transfer functions, and the output autospectrum is defined by the relation

$$S_y = \mathbf{h}_r\mathbf{S}_x\mathbf{h}_r^*. \qquad (26)$$

For example, if **x** is a 3-dimensional vector (as for a 3 degree of freedom shaker), let

$$\mathbf{S}_x = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \quad \mathbf{h}_r = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix},$$

and the output autospectrum, S_y [Eq. (26)] can be found as

$$S_y = \sum_{i=1}^{3}\sum_{j=1}^{3} h_i h_j^* S_{ij} = h_1 h_1^* S_{11} + 2[|h_1 h_2^* S_{12}| + |h_1 h_3^* S_{13}| + |h_2 h_3^* S_{23}|], \qquad (27)$$

because $S_{21} = S_{12}^*$, $S_{31} = S_{13}^*$, and $S_{32} = S_{23}^*$.

If we write an expression for the transfer function relating the response at y to the jth input as [from Eq. (19)]

$$h_{yj} = \sum_{k=1}^{n} \phi_y^k \Delta_k \theta_{kj}, \qquad (28)$$

then substituting into (27) we have

$$\begin{aligned} S_y &= \sum_{i=1}^{3}\sum_{j=1}^{3} h_{yi} h_{ij}^* S_{ij} \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3} \left(\sum_{k=1}^{n} \phi_y^k \Delta_k \theta_{ki}\right)\left(\sum_{m=1}^{n} \phi_y^m \Delta_m \theta_{mj}\right)^* S_{ij} \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{n}\sum_{m=1}^{n} \phi_y^k \Delta_k \theta_{ki} \cdot \phi_y^{m*} \Delta_m^* \theta_{mj}^* S_{ij} \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{n}\sum_{m=1}^{n} \phi_y^k \phi_y^{m*} \Delta_k \Delta_m^* \theta_{ki} \theta_{mj}^* S_{ij}, \end{aligned} \qquad (29)$$

where it is noted that if the system is undamped such that normal modes exist, the mode shapes, modal dynamic amplification factors, and modal influence coefficients are all real. The conjugate forms are retained in Eqs. (29), however, for generality. If normal modes do not exist, each of the terms in Eq. (29) may be complex, including the mode shapes, dynamic amplification functions, and modal mass influence coefficients. For each mode, the dynamic amplification factors are functions only of frequency, the mode shapes are functions only of positions where the outputs are measured, and the modal mass influence coefficients are functions only of the excitation direction (or location).

For multiple outputs, say y and z, S_z can be found from inerchanging z for y in (29), and the output cross spectrum relating y and z can be found from the relation

$$\begin{aligned} S_{yz} &= \sum_{i=1}^{3}\sum_{j=1}^{3} h_i h_{zj}^* S_{ij} \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{n}\sum_{m=1}^{n} \phi_y^k \phi_z^{m*} \Delta_k \Delta_m^* \theta_{ki} \theta_{mj}^* S_{ij}. \end{aligned} \tag{30}$$

Interpretation and Generalization of Equations

Because Eq. (29) can be considered a particular case of Eq. (30), our discussion here will focus upon interpretation of Eq. (30). Each output spectrum characterizes the statistical relationship between two variables y and z [or y and itself (frequency content) in the case of an autospectrum]. The objective of a shaker test is typically to reproduce (statistically) some kind of system response. For example, we may wish to reproduce the stresses throughout a structure that are experienced by the structure when "in service." To do this would require that autospectra and cross spectra for all possible values of y and z are the same as occur during service. This will be true if (1) all modal properties are the same during the test as during service, and (2) if all modes are excited at the same level during the test as in service.

Rewriting Eq. (30) by interchanging the order of the summation and rearranging,

$$\begin{aligned} S_{yz} &= \sum_{i=1}^{3}\sum_{m=1}^{n} \phi_y^k \phi_z^{m*} \Delta_k \Delta_m^* \sum_{i=1}^{3}\sum_{j=1}^{3} \theta_{ki} \theta_{mj}^* S_{ij} \\ &= \sum_{k=1}^{n}\sum_{m=1}^{n} \phi_y^k \phi_z^{m*} R_{km}, \end{aligned} \tag{31}$$

where $R_{km} = \Delta_k \Delta_m^* \sum_{i=1}^{3} \sum_{j=1}^{3} \theta_{ki} \theta_{mj}^* S_{ij}$ is defined as the cross-modal response for modes k and m (automodal response when $k = m$).

Assuming that condition (1) is true so that the mode shapes, dynamic amplification factors, and modal mass influence coefficients are unchanged from service to test, the control of the test [creating condition (2)] requires control of the input spectra, S_{ij}, such that the cross-modal responses,

$$R_{km} = \Delta_k \Delta_m^* \sum_{i=1}^{3}\sum_{j=1}^{3} \theta_{ki} \theta_{mj}^* S_{ij}, \tag{32}$$

to the shaker inputs are the same as the cross-modal responses,

$$\tilde{R}_{km} = \Delta_k \Delta_m^* \sum_{r=1}^{I_s}\sum_{s=1}^{I_s} \theta_{kr} \theta_{ms}^* S_{rs}, \tag{33}$$

to the in service inputs, where the number of inputs, I_s, and input spectra in service, S_{rs}, are unknown. If $R_{km} > \tilde{R}_{km}$, one or both of the modes experience overstress (for modes k and m; for a nondominant mode, system stresses may still be low; for "significant" modes, system overstress would be likely), and the test is too severe (as compared to in service). If $\tilde{R}_{km} < R_{km}$, undertesting occurs. We are thus left to ask: Under what conditions is it possible, and what procedure will allow us to make R_{km} sufficiently close to $\tilde{R}_{km}$ to conduct a meaningful test?

It is clear that under all conditions, it would be impossible to always choose the S_{ij} in Eq. (32) such that $R_{km} = \tilde{R}_{km}$ for all m and n. There are too many more degrees of freedom in $\tilde{R}_{km}$ than R_{km}. At first glancee, the cause may look hopeless. However, the nature of the modal characteristics provides a very practical help, allowing us to make $R_{km} \approx \tilde{R}_{km}$ for many cases.

If we examine the terms in Eqs. (32) and (33), we note that the dynamic amplification factor for each mode (Δ_k for the kth mode) is relatively small at frequencies distant from the natural frequency of the mode. Therefore the product, $\Delta_k \Delta_m^*$, is small except near the natural frequencies for modes k and m, and each of the terms

$$\sum_{i=1}^{3}\sum_{j=1}^{3} \theta_{ki} \cdot \theta_{mj}^* S_{ij} = T_{km}, \tag{34}$$

defined as the cross-modal (automodal when $k = m$) excitation, need only approximate the cross-modal excitation in service,

$$\tilde{T}_{km} = \sum_{r=1}^{I_s}\sum_{s=1}^{I_s} \theta_{kr} \cdot \theta_{ms}^* S_{rs}, \tag{35}$$

in the frequency range where the cross-modal amplification product $\Delta_k \Delta_m^*$ is not negligible. With a 3 degree of freedom shaker, it will in general be possible to make $T_{km} \approx \tilde{T}_{km}$ over a frequency range where three modes or less have significant "contribution." With more degrees of freedom, more modes could be potentially excited in the way they were excited in service; and the potential to perform the test in such a way as to not undertest or overtest the structure increases, particularly if the structure has a high modal frequency density. In other words, the objective is to determine the input spectra, S_{ij}, over the frequency band of interest such that all modes are excited as they were in service. The magnitude of the spectra relate to the level of the inputs at each frequency, whereas the phase of the spectra can be adjusted to excite or suppress modes depending upon the phase of their modal mass influence coefficients. That is, the excitation by 2 degrees of freedom may cancel each other for a particular mode because of the relative phase of the excitation for those 2 degrees of freedom. Similarly, 2 degrees of freedom may reinforce each other (by proper phase selection) to excite another mode more heavily than either degree of freedom by itself.

Practical Design of Spectra

In theory, the spectra $S_{ij}(f)$ are frequency functions defined as the transforms of functions of time defined over an infinite range of time ($-\infty \leq t \leq \infty$). In other words, they are statistical descriptors that are averaged over infinite time. As a practical matter, we must create a finite portion of a time function for a given test. It is therefore typical to create time records for a number of finite blocks of time. The time records for each pair of axes has a fixed relationship at each frequency within each block of time. The relationship between axes, however, is varied from block to block; so as the blocks are averaged, the desired spectra are approached as more and more blocks are averaged. A coherence function defines the degree of variation of the relationship (phase) between a pair of axes among the time blocks. If the two axes always have the same phase relationship (in each block), the two axes have a coherence of 1 (perfect coherence). If the phase relationship from block to block varies uniformly (one value in the range as likely as any other) from 0 to 2π, the coherence will be zero; and the cross spectrum between those two axes (averaged over an infinite number of blocks) approaches zero (no coherence).

Flexibility in design and control of R_{km} increases as the number of degrees of freedom of the shaker increases. For an N degree of freedom shaker, Eq. (32) can be written in the form

$$R_{km} = \Delta_k \Delta_m^* \sum_{i=1}^{N} \sum_{j=1}^{N} \theta_{ki} \theta_{mj}^* S_{ij}. \tag{36}$$

Thus when modes are closely spaced in frequency, it generally becomes more possible to make $R_{km} \approx \tilde{R}_{km}$, for all k and m, as the number of degrees of freedom, N, gets larger. Each added degree of freedom increases the number of independent spectra, S_{ij}, which can be shaped to achieve this objective.

In general, a single input cannot be shaped to produce a response at a single point that was caused (created) by several (other) inputs. If the response at a point is dominated by a single mode, however, a single input can be shaped to excite that mode at the same level as it was excited by the multiple inputs, thus recreating the modal response and approximating the point response. If 6 degrees of freedom are available as inputs (as from a shaker with 6 controlled degrees of freedom), then up to six modes could be excited to specified levels, thus recreating the desired response over a band of frequencies. Furthermore, in principle, if such bands are significantly separated in frequency, up to six additional modes per such frequency band can be excited simultaneously. Input coherence functions as well as autospectra must be shaped properly to take full advantage of this capability. Future investigation and experience is expected to provide guidelines for evaluating practical limitations given relevant properties of a shaker and test object.

If the outputs to be reproduced are the spectra of relative accelerations, Eq. (22) can be used to recreate Eq. (28) in the form

$$h_{ayj} = \sum_{k=1}^{n} \phi_i^k \Delta_a^k \chi_{kj} \tag{37}$$

and the rest of the development above follows by substituting for $\Delta_k \rightarrow \Delta_a^k$ and $\theta_{kj} \rightarrow \chi_{kj}$ into Eqs. (29)–(36). Because stresses are related to relative motions and the shaker will generally be unable to duplicate absolute motions of the test article, it is recommended that tests normally be conducted with the objective of reproducing relative rather than absolute motions within the structure. The equations above were developed in the form of absolute motions, consistent with common practice; but by substituting the modal amplification factors and modal influence coefficients for relative motion transfer functions the equations will apply to this recommended change.

It was noted that modal properties, including both modal amplification factors and mode shapes, of any dynamic system are functions of the boundary conditions seen by the system. Thus, adequate representation in a test of the dynamic environment in service requires sufficient attention to reproduction of service boundary conditions during the test. Although this requirement applies in any shaker test, whether single axis or multiaxis, realization of the inherent added capability afforded by a multiaxis shaker depends on increased attention to boundary conditions relative to the modes of interest (Smith and Staffanson, 1995).

SUMMARY AND CONCLUSIONS

In summary, we developed equations of motion appropriate for modeling systems subject to multiaxis shaker testing and showed their frequency domain solution form. These frequency response functions are sums of terms representing each dynamic mode of the system. Each of these modal terms is the product of a modal shape function, a modal dynamic amplification factor, and a modal mass influence coefficient. The modal dynamic amplification factors (one for each mode) are functions of frequency only and provide the basis for frequency shaping to provide shaker input that excites or suppresses modes that lie in given regions of frequency. The modal mass influence coefficients are functions of position and orientation (space) only and provide the basis for spatial shaping of inputs to excite modes according to their spatial properties. The possibility of spatial shaping is the advantage afforded by a multiple degree of freedom shaker, whereas a single degree of freedom shaker is limited to frequency shaping and less selectivity of modes within the excited frequency bands. Spatial shaping discriminates modes within the frequency band.

Because shaker tests are typically controlled by observing and shaping spectra, the relationships between input spectra and output spectra for a system under test were developed. These relationships were determined in terms of the above-mentioned mode shapes, amplification factors, and mass influence coefficients, and define the auto- and cross-modal response functions, auto- and cross-modal amplification products, and the auto- and cross-modal excitation functions. Because the modal amplification products are small except in the regions of the natural frequencies of their modes, spatial shaping can be applied independently and need only be applied in such regions isolated in frequency. The objective of a shaker test designed to simulate stresses to be experienced in service is shown to be the duplication of the resultant auto- and cross-modal responses for all important modes, even if the particular sources in service are unknown.

The mathematical relations are expressed in terms of relative as well as absolute motion, because shaker tests should be designed in most cases to reproduce relative motion within the test article.

Noted also is the importance of preserving boundary conditions when the dynamic response of the test article is expected to represent the dynamic response in a service environment.

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