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Positive Real Zeros in Flexible Beams

Feedback control of flexible structures naturally involves actuators and sensors that often cannot be placed at the same point in the structure. It has been widely recognized that this noncollocation can lead to difficult control problems and, in particular, difficulty in achieving high robustness to variation in the dynamic properties of the structure. This problem has previously been traced to transmission zeros in the dynamic transfer function between sensor location and actuator location, especially those lying on the positive real axis in the complex plane. In this article, the physical significance of these zeros is explored and the dynamic properties of beams that give rise to real positive zeros are contrasted to those of torsional and compressive systems that do not. © *1995 John Wiley & Sons, Inc.*

INTRODUCTION

In considering active control of flexible structures, interest has been focused on the elemental problems of vibration dissipation in uniform beams in tension, torsion, or lateral shear (Rosenthal, 1984; Spector and Flashner, 1989). These three problems are similar in that they all arise in the same physical structure (the slender beam or shaft); they are all conveniently studied as nondissipative; and they all present a difficult control problem when the sensor and actuator are not placed at the same point along the beam. However, despite the similarities, the last case of transverse motion (perpendicular to the beam axis) is fundamentally different from the two former cases. Perhaps the most important manifestation of this difference is that noncollocated actuator/sensor transfer functions for transverse beams exhibit positive (and negative) real zeros (Spector and Flashner, 1989; Lefante, 1992) whereas the torsional and tensile transfer functions do not (Rosenthal, 1984). These positive real zeros naturally attract root loci into the right half of the complex plane, producing systems with very limited stability margins. In addition, Herzog and Bleuler (1992) demonstrate that right half plane zeros introduce fundamental limitations to the disturbance rejection that can be achieved.

This fundamental difference can be identified in a number of different guises: the form of the governing differential equations is a clear point of contrast; but the present work attempts to provide a clear and simple physical interpretation. The underlying structural property that distinguishes the transverse beam from the torsional and tensile beam is the number of energy transfer mechanisms across sectional interfaces in the beam. In the torsional and tensile beams, only one mechanism is available for energy transfer from one section of the beam to another: torsional forces for the former and tensile forces for the latter. By contrast, transverse beams provide two distinct mechanisms for energy transfer: bending moment and shear force.

The availability of multiple, independent energy transfer mechanisms at sectional interfaces allows energy to move across these interfaces

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even when one of the degrees of freedom associated with the interface is identically zero. This capability is the underlying mechanism that makes it possible for transverse beams to exhibit real (both positive and negative) transmission zeros.

Transfer function zeros associated with this coupling between multiple energy transfer mechanisms is actually very much a part of everyday experience. The center of percussion is a special case of an input–output configuration with a zero at the origin of the complex plane (see Ginsberg and Genin, 1977). For players of ball sports like cricket or baseball, the center of percussion concept makes it possible to hold the bat at a certain point so that the impact of hitting the ball does not shock the player's hands. Automobile manufacturers, who early placed the front wheels of the car directly under the radiator cap for aesthetic reasons, realized in the 1930s that it is much better to place the front and rear wheels at mutual centers of percussion to prevent the influence of road bumps at one axle from affecting traction at the other. Although the idea of the center of percussion is normally developed for a rigid beam and represents a transmission zero at the origin of the complex plane, the concept is readily generalized in flexible beams leading to transmission zeros on the real axis.

In outline, the remainder of the article demonstrates why torsional and compressive beams, and other similar dynamic systems, have only imaginary zeros. It then shows that transverse beams do not share the characteristic that in these systems precludes real zeros. Having thus established a necessary (but not sufficient) condition for real zeros, the existence of such zeros and the nature of the associated forced response are demonstrated by a simple example.

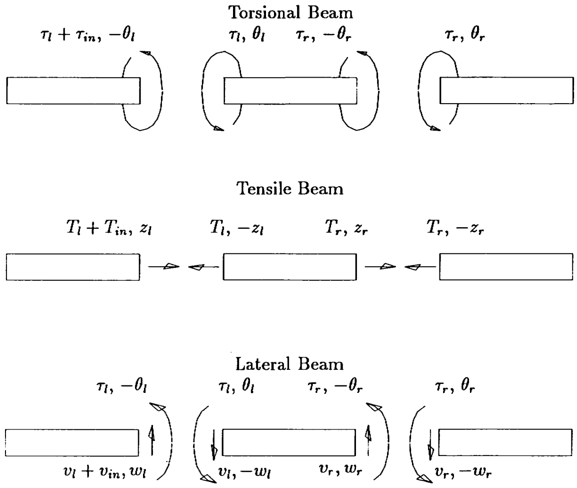

FIGURE 1 Interface continuity in section cuts: torsional, axial, and lateral beams.

CONNECTIVITY

If a beam in torsion or tension is cut to inspect an interface as in Figure 1, it is seen that the geometric matching condition involves only one degree of freedom: rotation in the torsional problem or axial extension in the tensile problem. Further, each interface transmits only a single force: torsion for the torsional problem and tension in the tensile problem. This force is continuous across the cut unless a concentrated external force is applied at the cut point. In studying the transfer function from a point of input to a measurement point, this force discontinuity occurs only at the point of input: elsewhere in the beam the force must be the same on either side of a cut.

Many dynamic systems display this kind of connectivity, referred to in the sequel as *singly connected.* Note that a singly connected system is necessarily linelike. Loops that would permit energy to flow from one point to another along more than one path render a system *multiply connected.* Such loops, however need not be overtly manifested by the shape of the structure for it to be multiply connected. The lateral beam, also illustrated in Figure 1, has two types of force and two degrees of freedom at the interface: moment and shear forcing mechanisms, rotation and translation degrees of freedom. Consequently, the lateral beam is a multiply connected system although it is linelike in appearance.

TRANSFER FUNCTION ZEROS

The input–output relationship of any linear dynamic system can be concisely represented in terms of a transfer function (D'Azzo and Houpis, 1981). This transfer function defines the response of a given output point to forces applied at a given input point. Of course, any continuous system can have any number of input–output pairs (infinitely many) and each pair has a distinct transfer function. Mathematically, the transfer function is a function of time that when convolved with the input function, produces the output function:

$$w(t) = \int_0^t g(t - \tau)f(\tau)\, d\tau$$

where the initial conditions of the dynamic system have all been assumed zero. This relationship is usually most conveniently treated in the frequency domain through the Laplace transform where time domain convolution becomes simple multiplication:

$$W(s) = G(s)F(s) : W(s) \doteq \mathcal{L}(w(t)),$$
$$G(s) \doteq \mathcal{L}(g(t)), \quad F(s) \doteq \mathcal{L}(f(t)).$$

The Laplace transform of the transfer function is commonly characterized in terms of a set of zeros and poles. The poles, which can be shown to be common to all of the various transfer functions of the dynamic system, are those complex frequencies s for which the magnitude of $G(s)$ is unbounded. The poles are the same as the system eigenvalues. The zeros, which are *not* common to the various transfer functions but are, instead, characteristic of the specific input–output pair, are those frequencies at which the magnitude of the transfer function is zero.

Of particular importance in understanding transfer function zeros is that, if s_0 is a zero of the transfer function from a point x_{in} to a point x_{out}, then the forced response at the point x_{out} to an input $e^{s_0 t}$ at x_{in} is zero. To see this, compute the Laplace transform of the system response at the output:

$$W(s) = G(s)\frac{1}{s - s_0}.$$

Assuming that s_0 is a zero of $G(s)$ and is not a pole of $G(s)$, factor the zero dependency out:

$$W(s) = \hat{G}(s)(s - s_0)\frac{1}{s - s_0}$$
$$= \hat{G}(s) : \hat{G}(s_0) \quad \text{is finite.}$$

Because the response $w(t) = \mathcal{L}^{-1}\hat{G}(s)$ contains no terms in $e^{s_0 t}$, the response is only due to initial conditions: the forced response is zero. If s_0 is complex, then it is easily demonstrated that the complex conjugate $\bar{s}_0$ is also a zero so inputs that are a linear combination of $e^{-s_0 t}$ and $e^{-\bar{s}_0 t}$ also produce zero forced response. This permits real valued functions of time when the zeros are complex.

This zero forced response to an input $e^{s_0 t}$ is the key to understanding the transfer function zeros of the systems under consideration in the present work. Of particular importance is the fact that the overall response of the system to an input $e^{s_0 t}$ may not be zero even if the initial conditions are zero and s_0 is a zero of the transfer function. However, there does exist a set of initial conditions that are *compatible* with the input $e^{s_0 t}$ such that the output $w(t)$ is zero for all time $t > 0$. Thus, in seeking to measure the transfer function zero, or experimentally detect it, one would have to not only apply the right input $e^{s_0 t}$, but also provide the appropriate initial conditions to see that the response w(t) is identically zero. Note that such initial conditions only exist for nonzero inputs $e^{s_0 t}$ if s_0 is a zero of the transfer function.

SINGLY CONNECTED SYSTEMS

A key property of conservative singly connected systems is that they can only have imaginary transfer function zeros between any given input point and any other output point. To prove this, we explore the response of such a system at a point x_{out} to an input $e^{s_0 t}$ applied at a point x_{in} where, for convenience of discussion, it is assumed that $x_{in} < x_{out}$ in the sense of some assigned coordinate system. The essence of the proof is that the transmission zero serves as an energy blocking mechanism in singly connected systems and that, as a result, either the forced response to exponentials with a nonzero real part is either everywhere zero or everywhere nonzero. In the latter case, such exponents cannot be transfer function zeros; the former case is demonstrated to be nonphysical. Hence, such systems can only have imaginary zeros, corresponding to standing sinusoidal waves with nodes at the output points.

THEOREM 1: *In a singly connected, linear, conservative system, if s_0 is a transfer function zero, then the input $e^{s_0 t}$ applied at x_{in} cannot supply energy to that section of the system that lies beyond the point of output x_{out}, when the initial conditions are chosen so that the only response of the system is the forced response.*

PROOF. Cut the structure at the point x_{out}. Because the forced response w(x_{out}) = 0, the energy transferred from the input to the section beyond the cut must be zero:

$$P(x_{out}) = f(x_{out})\dot{w}(x_{out}) = f(x_{out}) \times 0 = 0.$$

Note that this result is dependent upon the initial conditions. That is, if the initial conditions are not consistent with the forced response, then we cannot assume that $w(x_{out}) = 0$ for all time. However, it is sufficient to assume that the initial conditions are chosen to be consistent with the forced response because the system is linear and the transfer function zeros are a characteristic of the forced response.

THEOREM 2: *If s_0 is a transfer function zero of a singly connected linear conservative system between the input point x_{in} and the output point x_{out}, then the motion of the section $x > x_{in}$ must be either constant or sinusoidal.*

PROOF. If the response had a nonzero real exponent then the energy in the section would be either increasing or decreasing. However, the section is conservative and has no energy exchange across the interface so its energy must be constant. From this it is immediately concluded that the motion in the section beyond the output point must be either zero or sinusoidal: motions that conserve energy.

THEOREM 3: *If s_0 is a transfer function zero of a singly connected linear conservative system between the input point x_{in} and the output point x_{out} and $\Re(s_0) \neq 0$, then s_0 is also a zero of the transfer function between x_{in} and any other point $x > x_{out}$.*

PROOF. From Theorem 2, the motion $w(x)$: $x > x_{out}$ is either sinusoidal or constant: it contains no exponentially increasing or decreasing functions. Therefore, it contains no terms with the same time dependence as the input. A set of initial conditions can be found for which $w(x) = 0$ for any $x > x_{out}$.

THEOREM 4: *If s_0 is a transfer function zero of a singly connected continuous linear conservative system between the input point x_{in} and the output point x_{out} and $\Re(s_0) \neq 0$, then it is a zero of the transfer function between x_{in} and any other point $x > x_{in}$.*

PROOF. By Theorem 3, the forced response to the input corresponding to a zero at s_0 must be zero everywhere that $x > x_{out}$. Continuity of the system (in the sense that $w_x = \partial w/\partial x$ is differentiable) at the point x_{out} therefore implies that the forced response is also zero everywhere between x_{in} and x_{out}. Therefore, s_0 is also a transfer function zero to every point intervening between x_{in} and x_{out}.

THEOREM 5: *If s_0 is a transfer function zero of a single connected linear conservative system between the input point x_{in} and the output point x_{out} and $\Re(s_0) \neq 0$, then it is a zero of the transfer function between x_{in} and any other point in the structure.*

PROOF. Again, appealing to only the weakest continuity of the system at the point x_{in}, $w(x)$ is differentiable at x_{in} (where the only discontinuity is in the forcing function). The entire beam must have zero forced response to the input $e^{s_0 t}$.

THEOREM 6: *A singly connected, continuous, linear, conservative dynamic system cannot have a transfer function zero, s_0 from any point x_{in} to any other point x_{out} for which the real part of s_0 is nonzero.*

PROOF. From Theorem 5, either s_0 is a transfer function zero from a force applied at the point x_{in} to *every* point or it is not a zero to any point. The case where s_0 is a zero for every system transfer function originating at x_{in} is the trivial case: the system must be clamped to ground at that point.

Thus, we conclude that a conservative linear system that is only singly connected must not have any transfer functions that have real parts. A similar result is suggested by Knospe and Lefante (1993) who demonstrate that the zeros of such systems are poles of subsystems, all of which would have to lie on the imaginary axis in the absence of energy sources or sinks.

COROLLARY 1: *A necessary (but not sufficient) condition for a conservative linear continuous system to have transfer function zeros with a nonzero real part is that the system must be multiply connected.*

PROOF. By Theorem 6, singly connected linear conservative continuous systems cannot have transfer functions with a nonzero real part; therefore, multiple connectivity is a necessary but insufficient condition. This is simply a statement of the contrapositive.

UNIFORM TRANSVERSE BEAM

In the case of a transverse beam, we can solve the differential equation to show that the forced response to an exponential input can be zero at one or more points along the beam. Therefore, the beam has a transmission zero from the input to these points of zero forced response. These zeros are, of course, possible because there is an alternate energy transfer mechanism at the node point that permits energy to be transferred to the remaining system (beam) beyond the node.

Here, the differential equation for an Euler–Bernoulli beam of unit length with a lateral forcing input $v(t)$ at a point x_{in} is (Lalanne et al., 1983):

$$w_{xxxx} + w_{tt} = 0 \qquad (1)$$

with boundary conditions

$$w_{xx} = w_{xxx} = 0 \quad \text{at} \quad x = 0,\, 1 \qquad (2)$$

and forced shear boundary

$$\lim_{\varepsilon \to 0} w_{xxx}\big|_{x_{in}-\varepsilon} - w_{xxx}\big|_{x_{in}+\varepsilon} = v(t). \qquad (3)$$

To examine the forced response to a real zero, $s_0 = \sigma$ where σ is real, assume that

$$w(x,\, t) = \lambda(x)e^{\sigma t} = ae^{\beta}xe^{\sigma t}.$$

Then (1) requires that

$$\beta^4 = -\sigma^2 = |\sigma|^2 e^{(2n+1)\pi} = |\beta|^4 e^{4\gamma}.$$

That is,

$$|\beta| = \sqrt{|\sigma|}$$

and

$$\gamma = \frac{2n+1}{4}\pi \quad n = 0, 1, 2, 3.$$

Thus, the forced response in each section $[0,\, x_{in})$ and $(x_{in},\, 1]$ is given by

$$w(x,\,t) = \begin{cases} e^{\sigma t}(e^{\omega x}(a_L\cos\omega x + b_L\sin\omega x) \\ \quad + e^{-\omega x}(c_L\cos\omega x + d_L\sin\omega x)) & : 0 < x < x_{in} \\ e^{\sigma t}(e^{\omega(1-x)}(a_R\cos\omega(1-x) + b_R\sin\omega(1-x)) \\ \quad + e^{-\omega(1-x)}(c_R\cos\omega(1-x) + d_R\sin\omega(1-x))) & : x_{in} < x < 1 \end{cases} \qquad (4)$$

in which

$$\omega = \sqrt{\frac{\sigma}{2}}.$$

For the left-hand section, the boundary conditions are

$${}_Lw_{xxx}\big|_{x=0} = {}_Lw_{xx}\big|_{x=0} = 0$$

and for the right-hand section, the boundary conditions are

$${}_Rw_{xxx}\big|_{x=1} = {}_Rw_{xx}\big|_{x=1} = 0.$$

Applying these boundary conditions to (4) reveals that $b = d$ and $c = a - 2b$. Consequently, the forced response solution on either side of the input point is

$$w(x,\,t) = \begin{cases} e^{\sigma t}(a_L\cos\omega x(e^{\omega x} + e^{-\omega x}) + b_L\sin\omega x(e^{\omega x} + e^{-\omega x}) \\ \quad - 2\, b_L{}^{-\omega x}\cos\omega x) & : 0 < x < x_{in} \\ e^{\sigma t}(a_R\cos\omega(1-x)\,(e^{\omega(1-x)} + e^{-\omega(1-x)}) \\ \quad + b_R\sin\omega(1-x)(e^{\omega(1-x)} + e^{-\omega(1-x)}) \\ \quad - 2\, b_R e^{-\omega(1-x)}\cos\omega(1-x)) & : x_{in} < x < 1. \end{cases} \qquad (5)$$

The continuity conditions at the point $x = x_{in}$ are

$${}_Lw_{xxx}\big|_{x=x_{in}} = {}_Rw_{xxx}\big|_{x=x_{in}} + ve^{\sigma t} \qquad (6)$$

$${}_Lw_{xx}\big|_{x=x_{in}} = {}_Rw_{xx}\big|_{x=x_{in}} \qquad (7)$$

$${}_Lw_{x}\big|_{x=x_{in}} = {}_Rw_{x}\big|_{x=x_{in}} \qquad (8)$$

$${}_Lw\big|_{x=x_{in}} = {}_Rw\big|_{x=x_{in}}. \qquad (9)$$

These last four equations permit solution for the four unknown coefficients (a_L, b_L, a_R, and b_R)

needed to represent the response in the two sections on either side of the input interface. A final equation determines where the actual zeros lie:

$$w|_{x=x_{\text{out}}} = 0.$$

This equation can be solved for x_{out} in terms of σ and x_{in}:

$$x_{\text{out}} = f_1(\sigma, x_{\text{in}});$$

for σ in terms of x_{in} and x_{out}:

$$\sigma = f_2(x_{\text{in}}, x_{\text{out}});$$

or for x_{in} in terms of σ and x_{out}:

$$x_{\text{in}} = f_3(\sigma, x_{\text{out}}).$$

The first relationship determines all of the output points for which σ is a zero to forces applied at x_{in}, the second finds all of the zeros for the transfer function from x_{in} to x_{out}, and the last finds all of the input points for which σ is a zero for output measured at x_{out}.

In examining this solution, it is sufficient to note that Eq. (6) through (9) are linear equations in the unknowns with real coefficients. Therefore, unless the continuity conditions are indeterminate, the forced response problem has a solution (5) composed of bounded real coefficients.

The character of this solution and the conditions under which transmission zeros arise can be understood by examining the left half of the solution domain of (5). Seeking those points where the force response is identically zero requires that

$$e^{\sigma t}(a_L \cos \omega x(e^{\omega x} + e^{-\omega x}) + b_L \sin \omega x(e^{\omega x} + e^{-\omega x}) - 2b_L e^{-\omega x} \cos \omega x) = 0. \quad (10)$$

After some rearrangement of terms, the required identity is

$$\tan \omega x = \frac{1}{1 + e^{2\omega x}} \frac{a_L}{b_L} \quad 0 < x < x_{\text{in}}. \quad (11)$$

A sufficient (but not necessary) condition for there to be solutions to (11) is that

$$\omega x_{\text{in}} \geq \pi.$$

Thus, it should come as no great surprise that these solutions exist: the transfer function from x_{in} to x_{out} will have zeros on the real axis. It is important to recognize that these zeros are associated with a rather natural and obvious physical response of lateral beams and are not due to an obscure mechanism like propagation delay as suggested by Spector and Flashner (1989).

As an example, assume that the input point lies at $x_{\text{in}} = 0.75$ and that the test zero is $\sigma = 50$ so that $\omega = 5$. Clearly, this should produce a transfer function zero for some output point between 0 and 0.75 because $5 \times 0.75 \geq \pi$. Figure 2 is a plot of the beam's forced response to the input e^{50t} applied at the point $x_{\text{in}} = 0.75$. It is clearly seen that the response passes through zero at the point $x = 0.286$.

CONCLUSIONS

The transverse response of beams subject to lateral shear is clearly distinguished from the tensile or torsional response of the same structure: the former can exhibit transfer function zeros on the real axis of the complex plane while the latter cannot. The underlying characteristic that permits such zeros in transverse beams is the existence of multiple energy transfer mechanisms acting at any given sectional interface of the beam.

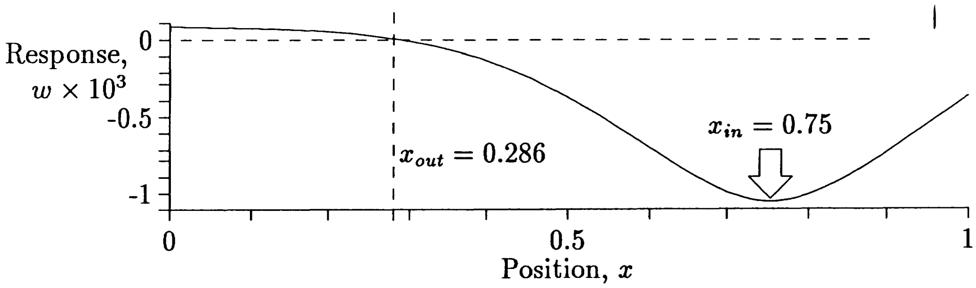

FIGURE 2 Uniform beam forced response to e^{50t} at $x_{\text{in}} = 0.75$.

This is the same mechanism that gives rise to the very useful center of percusion in rigid beams and pendula. Because the tensile and torsional beams only have a single energy transfer mechanism, they are fundamentally incapable of exhibiting any but imaginary transfer function zeros.

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