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Research Article

Gaussian Regularized Periodic Nonuniform Sampling Series

Feng Wang,[1] Congwei Wu,[2] and Liang Chen[3]

[1]*Department of Mathematics, Pingxiang University, Pingxiang 337000, China*
[2]*Department of Mathematics, Xi'an High-Tech Institute, Xi'an 710025, China*
[3]*College of Science, Jiujiang University, Jiujiang 332000, China*

Correspondence should be addressed to Liang Chen; chenliang3@mail2.sysu.edu.cn

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The periodic nonuniform sampling plays an important role in digital signal processing and other engineering fields. In this paper, we introduce the Gaussian regularization method to accelerate the convergence rate of periodic nonuniform sampling series. We prove that the truncation error of the Gaussian regularized periodic nonuniform sampling series decays exponentially. Numerical experiments are presented to demonstrate our result.

1. Introduction

In signal processing, the Paley–Wiener space is defined by

$$\mathscr{B}_\delta(\mathbb{R}) := \left\{ f(x) \in C(\mathbb{R}) \cap L_2(\mathbb{R}): f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} e^{iwt} \widehat{f}(w) \mathrm{d}w \right\}. \tag{1}$$

For each $f \in \mathscr{B}_\delta(\mathbb{R})$ with $\delta \le \pi$, the periodic nonuniform sampling formula is of the form [1, 2]

$$f(t) = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{M} f(\tau_{m,n}) \psi_{m,n}(t), \tag{2}$$

where

$$\psi_{m,n}(t) := \frac{M \prod_{k=1}^{M} \sin\left((\pi/M)\left(t - \tau_{k,n}\right)\right)}{\pi\left(t - \tau_{m,n}\right) \prod_{k=1,k\neq m}^{M} \sin\left((\pi/M)\left(t_m - t_k\right)\right)}, \tag{3}$$

$$\tau_{m,n} := t_m + nM, \quad 0 \le t_1 < t_2 < \cdots < t_M < M, \; n \in \mathbb{Z}. \tag{4}$$

Unlike Lagrangian nonuniform sampling, periodic nonuniform sampling does not require 1/4 condition (see [3]). The engineering background of periodic nonuniform sampling is Time-interleaved Analog-to-Digital Converters (TIADC) [4], which uses several low sampling rate analog-to-digital converters for parallel sampling to achieve high-speed data acquisition. TIADC is widely used in radar, communications, and other fields. Because the system has the mismatch error of the sampling clock, it leads to the generation of periodic nonuniform sampling data. Periodic nonuniform sampling has attracted considerable attention both in applied mathematics [5–9] and engineering [10–16].

We are concerned with the practical situation when only finitely many sample data are available. To reconstruct the values $f(t)$ for $t \in [-M, M]$, we shall use the localized data $\tau_{m,n}$, $n \le N$. Truncating the periodic nonuniform sampling series leads to a convergence rate of the order $O(1/\sqrt{N})$ [6]. In order to improve the convergence rate, the case of oversampling is considered (namely, bandwidth δ is strictly less than π); Strohmer and Tanner [9] proposed the Gevrey regularized periodic nonuniform sampling series which achieves a truncation error of the order $O(\exp(-\lambda N^{1/\alpha}))$, where λ is some positive constant and $\alpha > 1$. This method provides high-order accuracy to approximate band-limited functions. However, most of Gevrey functions are hardly expressed explicitly, and the decay of the truncation error is not strictly exponential.

On the contrary, the Gaussian regularization method has been successfully used in Shannon sampling [17–21] and Hermite sampling [22–24]. Thanks to its simplicity and high convergence rates. In this note, we apply the Gaussian regularization method to the periodic nonuniform sampling series:

$$S_{f,N}(t) := \sum_{n=-N}^{N}\sum_{m=1}^{M} f\left(\tau_{m,n}\right)\psi_{m,n}(t)g_N\left(t-\tau_{m,n}\right), \quad (5)$$

where

$$g_N(t) := \exp\left(\frac{-(\pi-\delta)t^2}{2(N-1)M}\right). \quad (6)$$

The following theorem shows the corresponding truncation error is exponentially decaying as the number of sample data increases to infinity.

Theorem 1. *Let* $0<\delta<\pi$, $f\in\mathscr{B}_\delta(\mathbb{R})$, $M,N\in\mathbb{N}$, *then*

$$\sup_{t\in[-M,M]}\left|f(t)-S_{f,N}(t)\right| \le C_\delta M\sqrt{NM}\mu \cdot \exp\left(\frac{-M(\pi-\delta)(N-1)}{2}\right)\|f\|_{L^2(\mathbb{R})}, \quad (7)$$

where C_δ *is some constant which depends on* δ *and*

$$\mu = \max_{1\le m\le M}\left|\frac{1}{\prod_{k=1,k\ne m}^{M}\sin\left((\pi/M)\left(t_m-t_k\right)\right)}\right|. \quad (8)$$

From the above estimate, we can see that if the sampling points are too close, which means the data degradation occurs, the error will become very large.

We give the proof in Section 2. The original proof (based on Fourier analysis [18] or complex analysis [25]) for Gaussian regularized Shannon sampling may not be directly extended to this problem. We give an elementary proof that applies not only to Gaussian regularized periodic nonuniform sampling but also to other Gaussian regularization sampling methods such as Gaussian regularized Lagrangian nonuniform sampling. In Section 3, some numerical experiments are performed to illustrate our result.

Proof of Theorem 1. We begin with a decomposition

$$f(t)-S_{f,N}(t) := E_1(t)+E_2(t), \quad (9)$$

where

$$E_1(t) := f(t)-\sum_{n=-\infty}^{\infty}\sum_{m=1}^{M} f\left(\tau_{m,n}\right)\psi_{m,n}(t)g_N\left(t-\tau_{m,n}\right),$$
$$E_2(t) := \sum_{|n|>N}\sum_{m=1}^{M} f\left(\tau_{m,n}\right)\psi_{m,n}(t)g_N\left(t-\tau_{m,n}\right). \quad (10)$$

For $t\in[-M,M]$, observe that

$$\left|E_2(t)\right| \le M\mu\|f\|_{L^2(\mathbb{R})}\sum_{|n|>N} g_N((n-1)M) \le M\mu\|f\|_{L^2(\mathbb{R})}\int_{(N-1)M}^{\infty} g_N(x)\mathrm{d}x \le \frac{M\mu\|f\|_{L^2(\mathbb{R})}}{\pi-\delta}e^{-((N-1)M(\pi-\delta))/2}, \quad (11)$$

where we have used the Cauchy–Schwartz inequality

$$f(t) \le \frac{1}{\sqrt{2\pi}}\|\widehat{f}\|_{L^1([-\pi,\pi])} \le \|\widehat{f}\|_{L^2([-\pi,\pi])} = \|f\|_{L^2(\mathbb{R})}, \quad (12)$$

and elementary inequality

$$\int_a^{\infty}\exp\left(\frac{-x^2}{2}\right)\mathrm{d}x \le \frac{e^{(-a^2/2)}}{a}, \quad a>0. \quad (13)$$

Next, we estimate $E_1(t)$, our estimate is different from [18]. Let

$$G_N(t) = \frac{1}{\sqrt{2\pi}}\int_{\delta-\pi}^{\pi-\delta}\sqrt{\frac{(N-1)M}{\pi-\delta}}e^{-\left((N-1)Mw^2\right)/2(\pi-\delta)}e^{iwt}\mathrm{d}w. \quad (14)$$

Since $G_N(t)\in\mathscr{B}_{\pi-\delta}(\mathbb{R})$, by equation (2), we have

$$f(t)G_N(t-x) = \sum_{n=-\infty}^{\infty}\sum_{m=1}^{M} f\left(\tau_{m,n}\right)\psi_{m,n}(t)G_N\left(\tau_{m,n}-x\right), \quad (15)$$

choosing $x=t$ and G_N is even function, then

$$f(t)G_N(0) = \sum_{n=-\infty}^{\infty}\sum_{m=1}^{M} f\left(\tau_{m,n}\right)\psi_{m,n}(t)G_N\left(t-\tau_{m,n}\right) := V_{f,N}(t). \quad (16)$$

Since

$$g_N(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{(N-1)M}{\pi-\delta}}e^{-\left((N-1)Mw^2\right)/2(\pi-\delta)}e^{iwt}\mathrm{d}w. \quad (17)$$

We have

$$G_N(t)-g_N(t) = -\frac{2}{\sqrt{2\pi}}\int_{\pi-\delta}^{\infty}\sqrt{\frac{(N-1)M}{\pi-\delta}}e^{-\left((N-1)Mw^2\right)/2(\pi-\delta)} \cdot\cos(wt)\mathrm{d}w, \quad (18)$$

$$\left|f(t)G_N(0)-f(t)\right| \le \left(1-G_N(0)\right)\|f\|_{L^2(\mathbb{R})} \lesssim e^{-((N-1)M(\pi-\delta))/2}\cdot\|f\|_{L^2(\mathbb{R})}. \quad (19)$$

We write $A\lesssim B$ if $A\le c_\delta B$ for some positive constant c_δ depends on δ. Note that

$$\left|E_1(t)\right| \le \left|f(t)G_N(0)-f(t)\right| + \left|S_{f,N}(t)-V_{f,N}(t)\right|. \quad (20)$$

To this end, we compute

Table 1: $\delta = 0.6\pi, M = 3, t_{[M]} = (0, 1.101, 1.523)$.

N	$E_{1,\delta,N,t_{[M]}}$	$E_{2,\delta,N,t_{[M]}}$	$E_{\delta,N,M}$
5	$1.1184E-05$	0.002609644	0.006175379
6	$1.1354E-06$	0.001248875	0.001027137
7	$1.246E-07$	$8.4595E-05$	0.000168452
8	$1.4392E-08$	0.000623966	$2.734E-05$
9	$1.7291E-09$	0.000858093	$4.4034E-06$
10	$2.1409E-10$	0.000723839	$7.0477E-07$
11	$2.7075E-11$	0.000387667	$1.1223E-07$
12	$3.4960E-12$	$1.99208E-05$	$1.7798E-08$
13	$4.5764E-13$	0.000255964	$2.8128E-09$
14	$6.2616E-14$	0.000372274	$4.4321E-10$
15	$1.1102E-14$	0.000332194	$6.9657E-11$

Table 2: $\delta = 0.8\pi, M = 4, t_{[M]} = (0, 1.221, 1.505, 2.668)$.

N	$E_{1,\delta,N,t_{[M]}}$	$E_{2,\delta,N,t_{[M]}}$	$E_{\delta,N,M}$
6	$5.499E-05$	0.003396916	0.036594255
7	$1.2193E-05$	0.000724752	0.011249566
8	$2.9539E-06$	0.002792515	0.003422797
9	$6.77669E-07$	0.003210469	0.001033254
10	$1.69413E-07$	0.00248397	0.000309981
11	$4.15186E-08$	0.001187713	$9.25296E-05$
12	$1.02189E-08$	0.00015787	$2.75058E-05$
13	$2.65649E-09$	0.00105366	$8.14808E-06$
14	$6.54824E-10$	0.001367498	$2.40656E-06$
15	$1.72059E-10$	0.001151512	$7.0897E-07$

$$\begin{aligned}
\left|S_{f,N}(t) - V_{f,N}(t)\right| &\le \sqrt{(N-1)M}\left|\sum_{n=-\infty}^{\infty}\sum_{m=1}^{M} f(\tau_{m,n})\psi_{m,n}(t) \times \int_{\pi-\delta}^{\infty} e^{-((N-1)Mw^2)/2(\pi-\delta)}\cos\left(w\left(t-\tau_{m,n}\right)\right)\mathrm{d}w\right| \\
&\le \sqrt{(N-1)M}\left|\sum_{|n|\ge 3}\sum_{m=1}^{M}\frac{f(\tau_{m,n})\psi_{m,n}(t)}{t-\tau_{m,n}} \times \left(\int_{\pi-\delta}^{\infty} -\left(e^{-((N-1)Mw^2)/2(\pi-\delta)}\right)'\sin\left(w\left(t-\tau_{m,n}\right)\right)\mathrm{d}w\right.\right. \\
&\quad \left.\left. + e^{-(N-1)M(\pi-\delta)^2/2}\sin\left((\pi-\delta)\left(t-\tau_{m,n}\right)\right)\right)\right| + 5\sqrt{(N-1)M}M\mu\|f\|_{L^2(\mathbb{R})}\int_{\pi-\delta}^{\infty} e^{(-((N-1)Mw^2)/2)(\pi-\delta)}\mathrm{d}w \\
&\le M\sqrt{(N-1)M}e^{-((N-1)M(\pi-\delta))/2}\mu\|f\|_{L^2(\mathbb{R})},
\end{aligned} \tag{21}$$

where we use the fact $\sum_{|n|\ge 3}\sum_{m=1}^{M}(|\psi_{m,n}(t)|/t-\tau_{m,n}) < \infty$ for $|t| \le M$. Combining (10) and (18)–(20) proves Theorem 1. □

2. Numerical Experiments

The band-limited function under investigation takes the form

$$f_\delta(t) = \frac{2\sin\delta t}{t} + \frac{\sin\delta(t-5)}{t-5}, \quad \delta < \pi. \tag{22}$$

The truncation error of the Gaussian regularized periodic nonuniform sampling series is measured by

$$E_{1,\delta,N,t_{[M]}} := \max_{|j|\le 100M}\left|f_\delta\left(\frac{j}{100}\right) - S_{f_\delta,N}\left(\frac{j}{100}\right)\right|, \tag{23}$$

where $t_{[M]}$ stands for $(t_1, t_2, \ldots, t_M)$ which is defined in (4). The truncation error of the periodic nonuniform sampling series is measured by

$$E_{2,\delta,N,t_{[M]}} := \max_{|j|\le 100M}\left|f_\delta\left(\frac{j}{100}\right) - \sum_{n=-N}^{N}\sum_{m=1}^{M} f_\delta(\tau_{m,n})\psi_{m,n}\left(\frac{j}{100}\right)\right|. \tag{24}$$

The following error is the theoretical estimate in Theorem 1:

$$E_{\delta,N,M} = M\sqrt{NM}\exp\left(\frac{-M(\pi-\delta)(N-1)}{2}\right). \tag{25}$$

We omit C_δ, $\|f\|_{L^2(\mathbb{R})}$ and μ here. The above errors for different choices of δ and $t_{[M]}$ are listed in Tables 1–3. The numerical experiments show that the truncation error accords with our theoretical estimation. Figure 1 shows the truncation error of the entire sampling interval when $N = 15$.

Table 3: $\delta=0.8\pi, M=10, t_{[M]}=(0,1,1.2,1.4,2.1,6,7.1,7.5,8.3,9.1)$.

N	$E_{1,\delta,N,t_{[M]}}$	$E_{2,\delta,N,t_{[M]}}$	$E_{\delta,N,M}$
4	$3.0833E-05$	0.1155	0.0051
5	$8.3643E-07$	0.0621	$2.4659E-04$
6	$2.5212E-08$	0.0372	$1.1673E-05$
7	$8.1356E-10$	0.0241	$5.4487E-07$
8	$2.8218E-11$	0.0165	$2.5172E-08$
9	$1.6206E-12$	0.0117	$1.1537E-09$

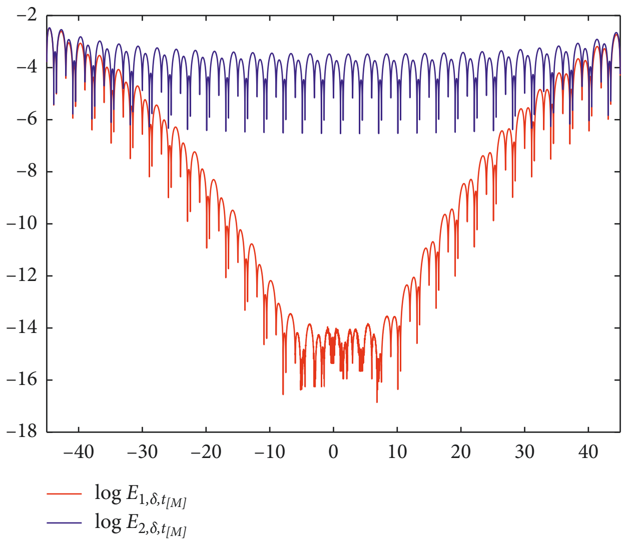

Figure 1: $t_{[M]}=(0,1.101,1.523)$, $j\in[-4500,4500]$, $\delta=0.6\pi$.

3. Conclusion

The convergence order [18, 25–27] of Gaussian regularization of the Shannon sampling series is the best among other known regularization methods [28–32] because of the good time-frequency concentration of the Gaussian function. In this paper, we proposed the Gaussian regularized periodic nonuniform sampling series and proved that this series is strictly exponentially decaying. Thus, its truncation error is superior to [9]. More important, our method is much simpler. The approximation algorithm for some discrete model is discussed in [33]. The distance between its discrete model and Paley–Wiener space is given in [34] (see Corollary 2 and 3), from which we can know that there is no way to compare the results in [33, 34] and ours. Moreover, the maximum distance between sampling points only needs to be less than M (see (4)) for periodic nonuniform sampling, while Theorem 1 in [34] tells us that the maximum sampling distance required for the more general nonuniform sampling they discussed is less than 1.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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