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Research Article

An Efficient Algorithm with Stabilized Finite Element Method for the Stokes Eigenvalue Problem

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This paper provides a two-space stabilized mixed finite element scheme for the Stokes eigenvalue problem based on local Gauss integration. The two-space strategy contains solving one Stokes eigenvalue problem using the $P_1 - P_1$ finite element pair and then solving an additional Stokes problem using the $P_2 - P_2$ finite element pair. The postprocessing technique which increases the order of mixed finite element space by using the same mesh can accelerate the convergence rate of the eigenpair approximations. Moreover, our method can save a large amount of computational time and the corresponding convergence analysis is given. Finally, numerical results are presented to confirm the theoretical analysis.

1. Introduction

The Stokes eigenvalue problem is one of the most important eigenvalue problems and plays an important role in the stability analysis of nonlinear partial differential equations [1]. The eigenvalue problems are used in many application areas: structural mechanics and fluid mechanics. Thus, development of the efficient numerical methods for studying the eigenvalue problems has practical meanings and has been noticed by many researchers. At the time of writing, numerous works are devoted to these problems (see [2–10] and the references cited therein).

Many effective postprocessing strategies that improve the convergence rate for the approximations of the eigenvalue problems by the finite element methods have been well developed. The two-grid method is one of these efficient postprocessing methods. The basic idea of two-grid scheme is first introduced by Xu [11, 12] for the nonsymmetric and nonlinear elliptic problems. Hence, it can be seen as a postprocessing technique and can take less CPU time compared to the one grid methods. To the best of our knowledge, some details of the two-grid scheme can be found in the works of Xu and Zhou [13], Chien and Jeng [14, 15], Chen et al. [7, 16], Hu and Cheng [17], Yang et al. [18, 19], Huang et al. [8], and Weng et al. [20, 21]. The two-space method is actually the iterative Galerkin method, which was first used for solving integral equation eigenvalue problems by Sloan [22] and differential equation eigenvalue problems by Lin and Xie [23]. Particularly, Racheva and Andreev [24] have proposed a postprocessing method for the $2m$-order self-adjoint eigenvalue problems by two-grid method or the two-space method. A similar method has been given for the Stokes eigenvalue problem [7, 25], elliptic eigenvalue problem [16], and the biharmonic eigenvalue problem [26] by mixed finite element methods.

In fact, two-space method [27–29] can be cast in the framework of Xu's work regarding the two-grid method. However, the two-space method is different from the two-grid method. This two-space method consists in solving the original Stokes eigenvalue problem in the k-order mixed finite element space and one additional Stokes source problem in an augmented mixed finite element space by a $k+1$-order mixed finite element space on the same mesh. Besides, the two-space method only needs one mesh size while the two-grid method needs two mesh sizes, a coarse mesh, and a fine mesh. In fact, the two-space method can avoid the discussion on the relation of the coarse and fine meshes. For this reason, in the present paper we establish a two-space discretization scheme for the Stokes eigenvalue problem.

Recently, more attention has been paid to the lowest equal order finite element pairs for simulating the incompressible flow. The lowest equal order finite element pairs offer some computational advances; for example, they are simple and have practical uniform data structure and adequate accuracy, because they show an identical degree distribution for both the velocity and pressure. Moreover, they are of practical importance in scientific computation owing to their very convenient computational cost. However, the lowest equal order mixed finite element pairs do not satisfy the inf-sup condition. Numerical tests show that the violation of the inf-sup condition often brings about unphysical pressure oscillations. In order to avoid the instability problem, the stabilized finite element methods are applied to the incompressible flow. Therefore, a lot of work focuses on stabilization (see [30–37]) of the lowest equal order pairs. Particularly, based on the work of Bochev et al. [30], Li et al. [31, 32] used the projection of the pressure onto the piecewise constant space to add the stabilized term for $P_1 - P_1$ element and Zheng et al. [35] used the projection of the pressure-gradient onto the piecewise constant space to add the stabilized term for $P_2 - P_2$ element.

Influenced by the work mentioned above, the paper focuses on the method, which combines two-space discretization scheme with a stabilized finite element method based on local Gauss integration technique for the Stokes eigenvalue problem. The paper is organized as follows. In Section 2, we introduce the studied Stokes eigenvalue problem and the notations and some well-known results used throughout this paper. Some stabilized finite element strategies based on two local Gauss integrations are recalled in Section 3. In Section 4, a two-space stabilized finite element algorithm is constructed and its error estimates are discussed. In Section 5, numerical experiments are reported for illustrating the theoretical results and the high efficiency of the proposed method. Finally, we will conclude our presentation in Section 6 with a few comments and also possible future research topics.

2. Preliminaries

In this paper, we consider the following Stokes eigenvalue problem:

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \lambda \mathbf{u} \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} \quad \text{on } \Gamma, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded and convex domain with a Lipschitz-continuous boundary Γ, $p(\mathbf{x})$ represents the pressure, $\mathbf{u}(\mathbf{x})$ is the velocity vector, and $\lambda \in \mathbb{R}$ is the eigenvalue.

We shall introduce the following Hilbert spaces:

$$\begin{aligned} \mathbf{V} &= \left[H_0^1(\Omega)\right]^2, \\ Y &= \left[L^2(\Omega)\right]^2, \\ W &= L_0^2(\Omega) = \left\{q \in L^2(\Omega), \int_\Omega q\,dx = 0\right\}. \end{aligned} \tag{2}$$

The spaces $[L^2(\Omega)]^m$, $m = 1, 2$, are equipped with the L^2-scalar product $(\cdot,\cdot)$ and L^2-norm $\|\cdot\|_0$. The norm and seminorm in $[H^k(\Omega)]^2$ are denoted by $\|\cdot\|_k$ and $|\cdot|_k$, respectively. The space $\mathbf{V}$ is equipped with the norm $\|\nabla\cdot\|_0$ or its equivalent norm $\|\cdot\|_1$ due to Poincaré inequality. Spaces consisting of vector-valued functions are denoted in boldface. Furthermore, the norm in the space dual to V is given by

$$\|\mathbf{u}\|_{-1} = \sup_{\mathbf{v}\in\mathbf{V}, \|\mathbf{v}\|_1 = 1} (\mathbf{u},\mathbf{v}). \tag{3}$$

Therefore, we define the following bilinear forms $a(\cdot,\cdot)$, $d(\cdot,\cdot)$, and $b(\cdot,\cdot)$ on $\mathbf{V}\times\mathbf{V}$, $\mathbf{V}\times W$, and $\mathbf{V}\times\mathbf{V}$, respectively, by

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) &= (\nabla\mathbf{u},\nabla\mathbf{v}), \quad \forall \mathbf{u},\mathbf{v}\in\mathbf{V}, \\ d(\mathbf{v},q) &= (\operatorname{div}\mathbf{v},q), \quad \forall \mathbf{v}\in\mathbf{V},\ \forall q\in W, \\ b(\mathbf{u},\mathbf{v}) &= (\mathbf{u},\mathbf{v}),\ \forall \mathbf{u},\mathbf{v}\in\mathbf{V} \end{aligned} \tag{4}$$

and a generalized bilinear form $B((\cdot,\cdot),(\cdot,\cdot))$ on $(\mathbf{V}\times W)\times(\mathbf{V}\times W)$; that is,

$$\begin{aligned} B((\mathbf{u},p),(\mathbf{v},q)) &= a(\mathbf{u},\mathbf{v}) - d(\mathbf{v},p) + d(\mathbf{u},q), \\ &\qquad \forall (\mathbf{u},p),(\mathbf{v},q)\in\mathbf{V}\times W. \end{aligned} \tag{5}$$

With the above notations, the variational formulation of problem (1) reads as follows: Find $(\mathbf{u},p;\lambda)\in(\mathbf{V}\times W)\times\mathbb{R}$ with $\|\mathbf{u}\|_0 = 1$, such that

$$B((\mathbf{u},p),(\mathbf{v},q)) = \lambda b(\mathbf{u},\mathbf{v}), \quad \forall (\mathbf{v},q)\in\mathbf{V}\times W. \tag{6}$$

From [1], we know that eigenvalue problem (6) has an eigenvalue sequence $\{\lambda_j\}$

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \tag{7}$$

and corresponding eigenvectors

$$\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \tag{8}$$

with the orthogonal property $b(\mathbf{u}_i,\mathbf{u}_j) = \delta_{ij}$.

Let

$$\begin{aligned} M(\lambda_i) = \{&\mathbf{u}\in\mathbf{V}, \mathbf{u} \text{ is an eigenvector of (6)} \\ &\text{corresponding to } \lambda_i\}. \end{aligned} \tag{9}$$

Moreover, the bilinear form $d(\cdot,\cdot)$ satisfies the inf-sup condition for all $q\in W$

$$\sup_{\mathbf{v}\in\mathbf{V}} \frac{|d(\mathbf{v},q)|}{\|\mathbf{v}\|_1} \ge \beta\|q\|_0, \tag{10}$$

where $\beta > 0$ is a constant depending only on Ω. Therefore, the generalized bilinear form B satisfies the continuity property and coercive condition

$$\left|B\left((\mathbf{u}, p),(\mathbf{v}, q)\right)\right| \leq C\left(\|\mathbf{u}\|_1 + \|p\|_0\right) \times \left(\|\mathbf{v}\|_1 + \|q\|_0\right),$$
$$\sup_{(\mathbf{v},q)\in(\mathbf{V},W)} \frac{\left|B\left((\mathbf{u}, p),(\mathbf{v}, q)\right)\right|}{\|\mathbf{v}\|_1 + \|q\|_0} \geq \beta_1\left(\|\mathbf{u}\|_1 + \|p\|_0\right), \tag{11}$$

where C and β_1 are the positive constants depending only on Ω. Throughout the paper we use c or C to denote a generic positive constant whose value may change from place to place, which remains independent of the mesh parameter.

3. A Stabilized Mixed Finite Element Method

From now on, h is a real positive parameter tending to 0. The finite element subspaces $\overline{\mathbf{V}}_h \times \overline{W}_h, \mathbf{V}_h \times W_h$ of $\mathbf{V} \times W$ are characterized by T_h, a partitioning of Ω into triangles T with the mesh size h, assumed to be uniformly regular in the usual sense [38]. Then we define them as follows:

$$\begin{aligned}
\overline{\mathbf{V}}_h &= \left\{v_h = (v_1, v_2) \in \left(C^0(\Omega)\right)^2 \cap \mathbf{V} : v_i|_T \right.\\
&\qquad \left. \in P_1(T),\ \forall T \in T_h,\ i = 1,2\right\},\\
\overline{W}_h &= \left\{w \in C^0 \cap W : w|_T \in P_1(T),\ \forall T \in T_h\right\},\\
\mathbf{V}_h &= \left\{v_h = (v_1, v_2) \in \left(C^0(\Omega)\right)^2 \cap \mathbf{V} : v_i|_T \right.\\
&\qquad \left. \in P_2(T),\ \forall T \in T_h,\ i = 1,2\right\},\\
W_h &= \left\{w \in C^0 \cap W : w|_T \in P_2(T),\ \forall T \in T_h\right\},\\
\mathbf{M}_h^1 &= \left\{v_h = (v_1, v_2) \in C_0(\Omega)^2 \cap \mathbf{V} \mid v_i|_T \in P_1(T) \right.\\
&\qquad \left. \oplus B(T),\ \forall T \in \mathscr{T}_h,\ i = 1,2\right\},\\
\mathbf{M}_h^2 &= \left\{v_h = (v_1, v_2) \in C_0(\Omega)^2 \cap \mathbf{V} \mid v_i|_T \in P_2(T) \right.\\
&\qquad \left. \oplus B(T),\ \forall T \in \mathscr{T}_h,\ i = 1,2\right\},
\end{aligned} \tag{12}$$

where $P_k(T)$ represents the set of all polynomials on T of degree less than $k \in N$ and $B(T)$ denotes the space of bubble functions. The bubble functions are defined as follows:

$$B(T) = \left\{v_h \in C(T) \mid v_h \in \operatorname{Span}\left\{\lambda^0\lambda^1\lambda^2\right\}\right\} \qquad \forall T \in \mathscr{T}_h, \tag{13}$$

where λ^i are area coordinates on T, $i = 0, 1, 2$. The area coordinate is also known as a triangle barycentre coordinate, where the three components $(\lambda^0, \lambda^1, \lambda^2)$ are of the ratio between the area of the three triangles and the area of the mother triangle.

It is known that this choice of the approximate spaces $\mathbf{M}_h^1 \times \overline{W}_h$ or $\mathbf{M}_h^2 \times W_h$ satisfies the inf-sup condition in [38], but this choice of the approximate spaces $\overline{\mathbf{V}}_h \times \overline{W}_h$ or $\mathbf{V}_h \times W_h$ does not satisfy the inf-sup condition [30, 32, 35]. As a consequence, we give a stabilized finite element approximation based on local Gauss integration technique (see [32, 35]). The idea is as follows.

Let $\Pi : L^2(\Omega) \to R_0$ be the standard L^2-projection:

$$(p, q)(\Pi p, q), \quad \forall p \in W,\ q \in R_0, \tag{14}$$

where $R_0 = \{q \in W : q|_T \in P_0(T),\ \forall T \in T_h\}$.

The projection operator Π has the following properties:

$$\begin{aligned}
\|\Pi p\|_0 &\leq c\|p\|_0, \quad \forall p \in W,\\
\|p - \Pi p\|_0 &\leq ch\|p\|_1, \quad \forall p \in H^1(\Omega).
\end{aligned} \tag{15}$$

The $P_1 - P_1$ stabilized bilinear terms are used by

$$\overline{G}(\overline{p}_h, q) = (\overline{p}_h - \Pi\overline{p}_h, q - \Pi q), \quad \overline{p}_h, q \in \overline{W}_h, \tag{16}$$

and the $P_2 - P_2$ stabilization term is given by

$$G(p, q) = (\nabla p - \Pi\nabla p, \nabla q - \Pi\nabla q), \quad \forall p, q \in W_h. \tag{17}$$

The stabilized term which is defined by local Gaussian quadrature can be rewritten as

$$G(p, q) = \sum_{T\in T_h}\left(\int_{T,2} \nabla p \cdot \nabla q\, d\mathbf{x} - \int_{T,1} \nabla p \cdot \nabla q\, d\mathbf{x}\right), \qquad \forall p, q \in W_h, \tag{18}$$

where $\int_{T,i} g(x, y) d\mathbf{x}$ denotes a Gaussian quadrature over T which is exact for polynomials of degree i, $i = 1, 2$. In particular, when $i = 1$, the trial function $\nabla p \in W_h$ is projected to the piecewise constant space. Besides, the stabilized term $\overline{G}(\overline{p}_h, q)$ can be rewritten as

$$\overline{G}(\overline{p}_h, q) = \sum_{T\in T_h}\left(\int_{T,2} \overline{p}_h \cdot q\, dx\, dy - \int_{T,1} \overline{p}_h \cdot q\, dx\, dy\right), \qquad \forall \overline{p}_h, q \in \overline{W}_h, \tag{19}$$

where the trial function $\overline{p}_h \in \overline{W}_h$ must be projected to R_0 when $i = 1$ for any $q \in \overline{W}_h$. Indeed, Becker and Hansbo have found [33] that the stabilized methods of [30, 32] are identical from a numerical point of view for these low-order approximations.

By adding the stabilization term into the generalized bilinear form $B((\cdot,\cdot),(\cdot,\cdot))$, we define

$$\begin{aligned}
B_h\left((\mathbf{u}_h, p_h),(\mathbf{v}, q)\right) &= B\left((\mathbf{u}_h, p_h),(\mathbf{v}, q)\right) - G(p_h, q),\\
\overline{B}_h\left((\overline{u}_h, \overline{p}_h);(v, q)\right) &= B\left((\overline{u}_h, \overline{p}_h);(v, q)\right) - \overline{G}(\overline{p}_h, q).
\end{aligned} \tag{20}$$

Then the corresponding discrete variational formulation for the Stokes eigenvalue problem reads as follows: find $(\mathbf{u}_h, p_h; \lambda_h) \in (\mathbf{V}_h \times W_h) \times \mathbb{R}$ with $\|\mathbf{u}_h\|_0 = 1$, such that

$$B_h((\mathbf{u}_h, p_h), (\mathbf{v}, q)) = \lambda_h r(\mathbf{u}_h, \mathbf{v}), \quad \forall (\mathbf{v}, q) \in \mathbf{V}_h \times W_h \tag{21}$$

and find $(\overline{\mathbf{u}}_h, \overline{p}_h; \overline{\lambda}_h) \in (\overline{\mathbf{V}}_h \times \overline{W}_h) \times \mathbb{R}$ with $\|\overline{\mathbf{u}}_h\|_0 = 1$, such that

$$\overline{B}_h((\overline{u}_h, \overline{p}_h); (v, q)) = \overline{\lambda}_h r(\overline{\mathbf{u}}_h, \mathbf{v}), \quad \forall (\mathbf{v}, q) \in \overline{\mathbf{V}}_h \times \overline{W}_h. \tag{22}$$

Remark 1. For the $P_{kb} - P_k$ ($\mathbf{M}_h^k \times W_h, k = 1, 2$) pair which satisfy inf-sup condition, there are points of difference between them. The $P_k - P_k$ stabilized method in this article only adds the stabilized term with respect to the pressure space. However, the $P_{kb} - P_k$ method has the implicit stabilized term in the velocity space.

With $\mathbf{v} = \mathbf{u}_h, q = p_h$ and thanks to the positive definiteness of $a(u_h, u_h)$, we deduce that the discrete eigenvalues λ_{jh} are positive. Let the eigenvalue of (21) be ordered as follows:

$$0 < \lambda_{1h} \le \lambda_{2h} \le \lambda_{3h} \le \cdots \le \lambda_{Nh}, \tag{23}$$

and let us consider the corresponding eigenfunctions

$$(\mathbf{u}_{1h}, p_{1h}), (\mathbf{u}_{2h}, p_{2h}), (\mathbf{u}_{3h}, p_{3h}), \ldots, (\mathbf{u}_{Nh}, p_{Nh}), \tag{24}$$

where $r(\mathbf{u}_{ih}, \mathbf{u}_{jh}) = \delta_{ij}$, $1 \le i, j \le N$, N denotes the dimension of the finite element space.

Similarly, let $M_h(\lambda_{ih})$ be the eigenspace associated with λ_{ih}; that is,

$$M_h(\lambda_{ih}) = \{\mathbf{u}_h \in \mathbf{V}_h, \mathbf{u}_h \text{ is an eigenfunction of (21) corresponding to } \lambda_{ih}\}. \tag{25}$$

For (22) with $P_1 - P_1$ pairs, it can be given similarly. The corresponding nature of the eigenvalues is omitted for the sake of simplicity.

The next theorem shows the continuity property and the weak coercivity property of the bilinear form $B_h((\mathbf{u}_h, p_h), (\mathbf{v}, q))$ for the finite element space $\mathbf{V}_h \times W_h$ in [35] and $\overline{B}_h((\overline{\mathbf{u}}_h, \overline{p}_h); (v, q))$ for the finite element space $\overline{\mathbf{V}}_h \times \overline{W}_h$ in [30, 32].

Theorem 2. *For all $(\mathbf{u}_h, p_h), (\mathbf{v}, q) \in \mathbf{V}_h \times W_h$, there exist positive constants C and β_2, independent of h, such that*

$$\begin{aligned} &|B_h((\mathbf{u}_h, p_h), (\mathbf{v}, q))| \\ &\quad \le C(\|\mathbf{u}_h\|_1 + \|p_h\|_0) \times (\|\mathbf{v}\|_1 + \|q\|_0), \\ &\sup_{(\mathbf{v},q)\in(\mathbf{V}_h, W_h)} \frac{|B_h((\mathbf{u}_h, p_h), (\mathbf{v}, q))|}{\|\mathbf{v}\|_1 + \|q\|_0} \\ &\quad \ge \beta_2 (\|\mathbf{u}_h\|_1 + \|p_h\|_0). \end{aligned} \tag{26}$$

Moreover, for all $(\overline{\mathbf{u}}_h, \overline{p}_h), (\mathbf{v}, q) \in (\overline{\mathbf{V}}_h \times \overline{W}_h)$, there exist positive constants C_1 and β_3, independent of h, such that

$$\begin{aligned} &\overline{B}_h((\overline{\mathbf{u}}_h, \overline{p}_h); (\mathbf{v}, q)) \\ &\quad \le C_1(\|\overline{\mathbf{u}}_h\|_1 + \|\overline{p}_h\|_0) \times (\|\mathbf{v}\|_1 + \|q\|_0), \\ &\sup_{(\mathbf{v},q)\in(\overline{\mathbf{V}}_h, \overline{W}_h)} \frac{|\overline{B}_h((\overline{\mathbf{u}}_h, \overline{p}_h); (\mathbf{v}, q))|}{\|\mathbf{v}\|_1 + \|q\|_0} \\ &\quad \ge \beta_3 (\|\overline{\mathbf{u}}_h\|_1 + \|\overline{p}_h\|_0). \end{aligned} \tag{27}$$

The next theorem contains the convergence result of eigenfunctions and eigenvalues for the Stokes eigenvalue problem in [8, 20].

Theorem 3. *With (u, p, λ) belonging to $(H^3(\Omega)^2 \cap \mathbf{V}) \times (H^2(\Omega) \cap W) \times \mathbb{R}$ and being the exact solution of (6), one deduces that there exists a discrete eigenpair $(\mathbf{u}_h, p_h; \lambda_h)$ of (21) which satisfies the following error estimates:*

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_0 + h(\|\mathbf{u} - \mathbf{u}_h\|_1 + \|p - p_h\|_0) &\le ch^3, \\ |\lambda - \lambda_h| &\le ch^4. \end{aligned} \tag{28}$$

Furthermore, if the exact solution $(\mathbf{u}, p; \lambda) \in (H^2(\Omega)^2 \cap \mathbf{V}) \times (H^1(\Omega) \cap W) \times \mathbb{R}$, then $(\overline{\mathbf{u}}_h, \overline{p}_h; \overline{\lambda}_h) \in \overline{\mathbf{V}}_h \times \overline{W}_h \times \mathbb{R}$ of problem (22) satisfies

$$\begin{aligned} \|\mathbf{u} - \overline{\mathbf{u}}_h\|_0 + h(\|\mathbf{u} - \overline{\mathbf{u}}_h\|_1 + \|p - \overline{p}_h\|_0) &\le ch^2, \\ |\lambda - \overline{\lambda}_h| &\le ch^2. \end{aligned} \tag{29}$$

4. Two-Space Stabilized Finite Element Scheme and Error Estimates

In this section, we shall present a two-space stabilized finite element algorithm to reduce the computational cost. The two-space stabilized finite element approximation consists of three steps.

Step 1. On the mesh size h, solve the following Stokes eigenvalue problem by $P_1 - P_1$ pair and find $(\overline{\mathbf{u}}_h, \overline{p}_h; \overline{\lambda}_h) \in (\overline{\mathbf{V}}_h \times \overline{W}_h) \times \mathbb{R}$ and $\|\overline{\mathbf{u}}_h\|_0 = 1$, such that, for all $(\mathbf{v}, q) \in \overline{\mathbf{V}}_h \times \overline{W}_h$,

$$\overline{B}_h((\overline{\mathbf{u}}_h, \overline{p}_h); (\mathbf{v}, q)) = \overline{\lambda}_h(\overline{\mathbf{u}}_h, \mathbf{v}). \tag{30}$$

Step 2. On the same mesh size h, solve the following Stokes problem by $P_2 - P_2$ pair and find $(\mathbf{u}^h, p^h) \in \mathbf{V}_h \times W_h$ such that for all $(\mathbf{v}^h, q^h) \in \mathbf{V}_h \times W_h$

$$B_h((\mathbf{u}^h, p^h); (\mathbf{v}^h, q^h)) = \overline{\lambda}_h(\overline{\mathbf{u}}_h, \mathbf{v}^h). \tag{31}$$

Step 3. Compute the eigenvalue by the Rayleigh quotient

$$\lambda^h = \frac{B_h((\mathbf{u}^h, p^h); (\mathbf{u}^h, p^h))}{(\mathbf{u}^h, \mathbf{u}^h)}, \tag{32}$$

where $\mathbf{u}^h \in \mathbf{V}_h \setminus \{0\}$.

Next, we will study the convergence of the two-space stabilized finite element solution. To do this, we define the Galerkin projection operator $(R_h, Q_h) : (\mathbf{V}, W) \to (\mathbf{V}_h, W_h)$ by

$$B_h\left(\left(R_h(\mathbf{v},q), Q_h(\mathbf{v},q)\right), (\mathbf{v}_h, q_h)\right) = B\left((\mathbf{v},q), (\mathbf{v}_h, q_h)\right), \quad \forall (\mathbf{v}_h, q_h) \in (\mathbf{V}_h, W_h). \tag{33}$$

By Theorem 2, (R_h, Q_h) is well defined and the following approximation properties are fulfilled in [20].

Lemma 4. *For all* $(\mathbf{u}, p) \in (\mathbf{H}^3(\Omega)^2 \cap \mathbf{V}, H^2(\Omega) \cap W)$*, one has*

$$\left\|\mathbf{u} - R_h(\mathbf{u},p)\right\|_1 + \left\|p - Q_h(\mathbf{u},p)\right\|_0 \le ch^2\left(\|\mathbf{u}\|_3 + \|p\|_2\right). \tag{34}$$

The following identity that relates the errors in the eigenvalue and eigenvector can be found in [3].

Lemma 5. *Let* $(\mathbf{u}, p; \lambda)$ *be an eigenpair of (6); for any* $\mathbf{s} \in \mathbf{V} \setminus \{0\}$ *and* $w \in W$*, one has*

$$\frac{B((\mathbf{s},w),(\mathbf{s},w))}{r(\mathbf{s},\mathbf{s})} - \lambda = \frac{B((\mathbf{s}-\mathbf{u}, w-p),(\mathbf{s}-\mathbf{u}, w-p))}{r(\mathbf{s},\mathbf{s})} - \lambda\frac{r(\mathbf{s}-\mathbf{u},\mathbf{s}-\mathbf{u})}{r(\mathbf{s},\mathbf{s})}. \tag{35}$$

The next theorem provides the error estimates for our two-space scheme.

Theorem 6. *Let* $(\mathbf{u}^h, p^h; \lambda_h)$ *be the ith discrete eigenpair. Then the ith eigenpair* $(\mathbf{u}, p; \lambda)$ *of the Stokes operator is such that*

$$\left\|\mathbf{u} - \mathbf{u}^h\right\|_1 + \left\|p - p^h\right\|_0 \le ch^2\left(\|\mathbf{u}\|_3 + \|p\|_2\right), \tag{36}$$

$$\left|\lambda - \lambda^h\right| \le ch^4\left(\|\mathbf{u}\|_3 + \|p\|_2\right)^2. \tag{37}$$

Proof. Denoted by $(\theta_h, \rho_h) = (R_h(\mathbf{u},p) - \mathbf{u}^h, Q_h(\mathbf{u},p) - p^h)$, subtracting (6) from (31), we derive from (33)

$$B_h\left((\theta_h,\rho_h);(\mathbf{v},q)\right) = \lambda\left(\mathbf{u} - \overline{\mathbf{u}}_h, \mathbf{v}\right) + \left(\lambda - \overline{\lambda}_h\right)\left(\overline{\mathbf{u}}_h, \mathbf{v}\right). \tag{38}$$

Let $(\mathbf{v}, q) = (\theta_h, \rho_h)$ in (38), by using Theorem 2, Sobolev embedding theorem, and Theorems 3, we obtain

$$\begin{aligned} \nu\left\|\nabla\theta_h\right\|_0 + \left\|\rho_h\right\|_0 &\le \beta_2^{-1}\left(\lambda\left\|\mathbf{u} - \overline{\mathbf{u}}_h\right\|_{-1} + \left|\lambda - \overline{\lambda}_h\right|\left\|\overline{\mathbf{u}}_h\right\|_{-1}\right) \\ &\le ch^2\left(\|\mathbf{u}\|_2 + \|p\|_1\right). \end{aligned} \tag{39}$$

Combining the triangle inequality with Lemma 4, we deduce

$$\begin{aligned} &\left\|\left(\mathbf{u} - \mathbf{u}^h\right)\right\|_1 + \left\|p - p^h\right\|_0 \\ &\quad \le \left\|\theta_h\right\|_1 + \left\|\left(\mathbf{u} - R_h(\mathbf{u},p)\right)\right\|_1 + \left\|\rho_h\right\|_0 \\ &\qquad + \left\|p - Q_h(\mathbf{u},p)\right\|_0 \le ch^2\left(\|\mathbf{u}\|_3 + \|p\|_2\right), \end{aligned} \tag{40}$$

and finally we obtain (36).

Moreover, using (32) and Lemma 5, we have

$$\begin{aligned} \lambda^h - \lambda &= \frac{B_h\left(\left(\mathbf{u}^h, p^h\right);\left(\mathbf{u}^h, p^h\right)\right)}{\left(\mathbf{u}^h, \mathbf{u}^h\right)} - \lambda \\ &= \frac{B\left(\left(\mathbf{u}^h - \mathbf{u}, p^h - p\right);\left(\mathbf{u}^h - \mathbf{u}, p^h - p\right)\right) - G\left(p^h, p^h\right)}{\left(\mathbf{u}^h, \mathbf{u}^h\right)} \\ &\quad - \lambda\frac{\left(\mathbf{u}^h - \mathbf{u}, \mathbf{u}^h - \mathbf{u}\right)}{\left(\mathbf{u}^h, \mathbf{u}^h\right)}. \end{aligned} \tag{41}$$

Taking the norm and using (15) and (36), we conclude the proof that is

$$\begin{aligned} \left|\lambda - \lambda^h\right| &\le c\left\|\mathbf{u} - \mathbf{u}^h\right\|_1^2 + h^2\left\|p^h - \Pi p^h\right\|_0^2 + c\left\|p - p^h\right\|_0^2 \\ &\le c\left\|\mathbf{u} - \mathbf{u}^h\right\|_1^2 + h^2\left\|p^h - p + p - \Pi p + \Pi p - \Pi p^h\right\|_0^2 + c\left\|p - p^h\right\|_0^2 \\ &\le c\left\|\mathbf{u} - \mathbf{u}^h\right\|_1^2 + h^2\left\|p - \Pi p\right\|_0^2 + c\left\|p - p^h\right\|_0^2 \\ &\le ch^4\left(\|\mathbf{u}\|_3 + \|p\|_2\right)^2. \end{aligned} \tag{42}$$

□

Remark 7. From Theorem 3, for the usual $P_2 - P_2$ stabilized finite element solution $(\mathbf{u}_h, p_h; \lambda_h)$ which involves solving a Stokes eigenvalue problem with mesh size h, we have the following error estimates:

$$\begin{aligned} \left\|\mathbf{u} - \mathbf{u}_h\right\|_0 + h\left(\left\|\mathbf{u} - \mathbf{u}_h\right\|_1 + \left\|p - p_h\right\|_0\right) &\le ch^3, \\ \left|\lambda - \lambda_h\right| &\le ch^4. \end{aligned} \tag{43}$$

Furthermore, if we use the two-space stabilized finite element method, then we get the convergence rate of the same order as the usual stabilized finite element method from Theorem 6. However, our method is more efficient than the $P_2 - P_2$ stabilized finite element scheme in the same mesh because our method for solving Stokes eigenvalue problem is to compute an initial approximation based on a lower number of nodes, which takes less CPU time.

Table 1: Relative error and convergence rate for $P_2 - P_2$ pair.

$1/h$	λ^h	$\lvert\lambda - \lambda^h\rvert/\lvert\lambda\rvert$	Rate	CPU time
8	52.4269	$1.570E-3$		0.156
16	52.3505	$1.111E-4$	3.821	0.672
32	52.3451	$7.345E-6$	3.919	3.469
64	52.3447	$5.245E-7$	3.808	27.001

Table 2: Relative error and convergence rate of two-space method with $P_2 - P_2$ pair.

$1/h$	λ^h	$\lvert\lambda - \lambda^h\rvert/\lvert\lambda\rvert$	Rate	CPU time
8	52.4594	$2.191E-3$		0.14
16	52.3529	$1.570E-4$	3.803	0.609
32	52.3452	$1.029E-5$	3.930	2.766
64	52.3447	$6.536E-7$	3.977	13.875

Table 3: Relative error and convergence rate of two-space method with $P_{2b} - P_2$ pair.

$1/h$	λ^h	$\lvert\lambda - \lambda^h\rvert/\lvert\lambda\rvert$	Rate	CPU time
8	52.518	$3.311E-3$		0.218
16	52.3594	$2.817E-4$	3.555	0.875
32	52.3458	$2.033E-5$	3.793	3.984
64	52.3448	$1.338E-6$	3.926	22.916

Remark 8. For the two-space algorithm with $P_{2b} - P_2$ ($\mathbf{M}_h^2 \times W_h$) pair which satisfies inf-sup condition, we obtain the same result. The procedure of the two-space method with $P_{2b} - P_2$ pair could be described in the following manner: Firstly, we solve the following Stokes eigenvalue problem by $P_{1b} - P_1$ ($\mathbf{M}_h^1 \times W_h$) pair on the mesh size h; then, we should solve the new Stokes problem by $P_{2b} - P_2$ pair on the same mesh size h. Finally we can compute the eigenvalue by the Rayleigh quotient.

5. Numerical Experiments

In this section we present numerical results to check the theoretical analysis contained in Theorem 6. Our goal is to confirm the theoretical results of the new two-space stabilized finite element method for the two-dimensional Stokes eigenvalue approximated by the equal order finite element pairs based on local Gauss integration.

In our numerical experiments, Ω is the unit square domain $[0,1] \times [0,1]$ in $\mathbb{R}^2$. The domain Ω is uniformly divided by the triangulations of mesh size h. Here, we just consider the first eigenvalue of the Stokes eigenvalue problem for the sake of simplicity. Following [4], we employ the approximation $\lambda_1 = 52.3446911$ as the reference solution for the first eigenvalue. Note that in these computations we set $\nu = 1$.

When solving the Stokes problem with a mesh size h, we need the solutions $\overline{\lambda}_h$ and $\overline{\mathbf{u}}_h$ generated by a lower finite element pair $P_1 - P_1$. To do this we interpolate the solutions $\overline{\lambda}_h$ and $\overline{\mathbf{u}}_h$ onto the grid with the same mesh size h, but increasing the order of the mixed finite element space. In conclusion, the solution of the two-space method is obtained by one simple eigenvalue problem by a lower finite element pair and one time interpolation by a higher finite element pair $P_2 - P_2$.

Our goal in this test is to validate the merit of the two-space method as compared with the $P_2 - P_2$ stabilized method and the two-space method with $P_{2b} - P_2$ pair. The eigenvalue approximation λ_h, the eigenvalue error, the convergence rates, and the CPU time for the stabilized mixed finite element methods for different values of h are tabulated in Tables 1, 2, and 3. From Tables 1, 2, and 3, we can see that the three methods work well and keep the convergence rates just as predicted by the theoretical analysis, but our two-space method can take less CPU time. For the two-space method with $P_{2b} - P_2$ pair, the two-space method with $P_2 - P_2$ pair approximates the velocity variable with a lower number of nodes, so our method can save a lot of time.

Next numerical test is about the second, third, and fourth eigenvalues $\lambda_{2,3,4}^h$. The reference values are computed over a fine mesh $h = 1/64$ and the results are $\lambda_{2,3,4} = 92.1245411, 92.1245843, 128.209971$. Then, in Figure 1, we exhibit the $O(h^4)$ convergence rate as has been predicted in Theorem 6 with the two-space method.

Moreover, we give two plots of numerical solutions of two kinds of two-space schemes at the mesh $1/h = 48$ in Figure 2 for the details. Figure 2 shows the stability of two schemes.

6. Conclusions

In this paper, we presented the two-space algorithm for the Stokes eigenvalue problem discretized by stabilized mixed finite element scheme, based on local Gauss integration technique. The main feature of our method is to combine two equal order stabilized methods, then use the first-order mixed

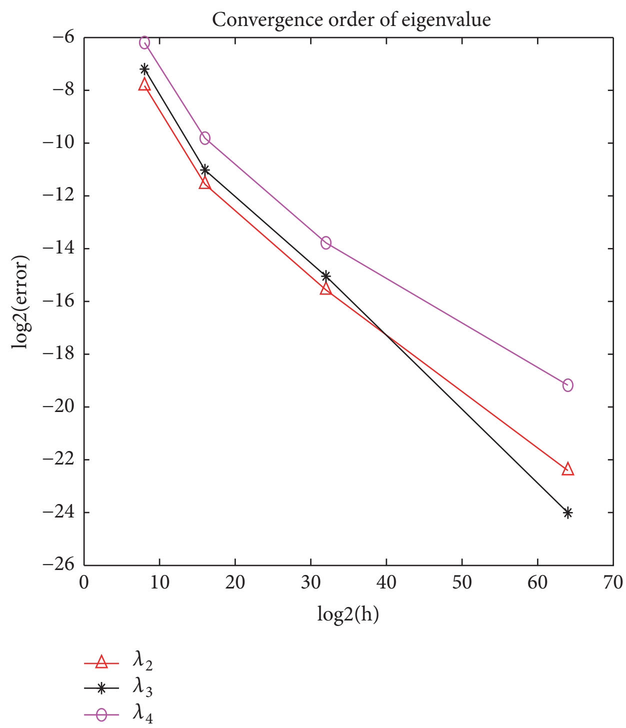

Figure 1: The convergence rate of the eigenvalue for $\lambda_{2,3,4}$ on the unit square with the two-space method.

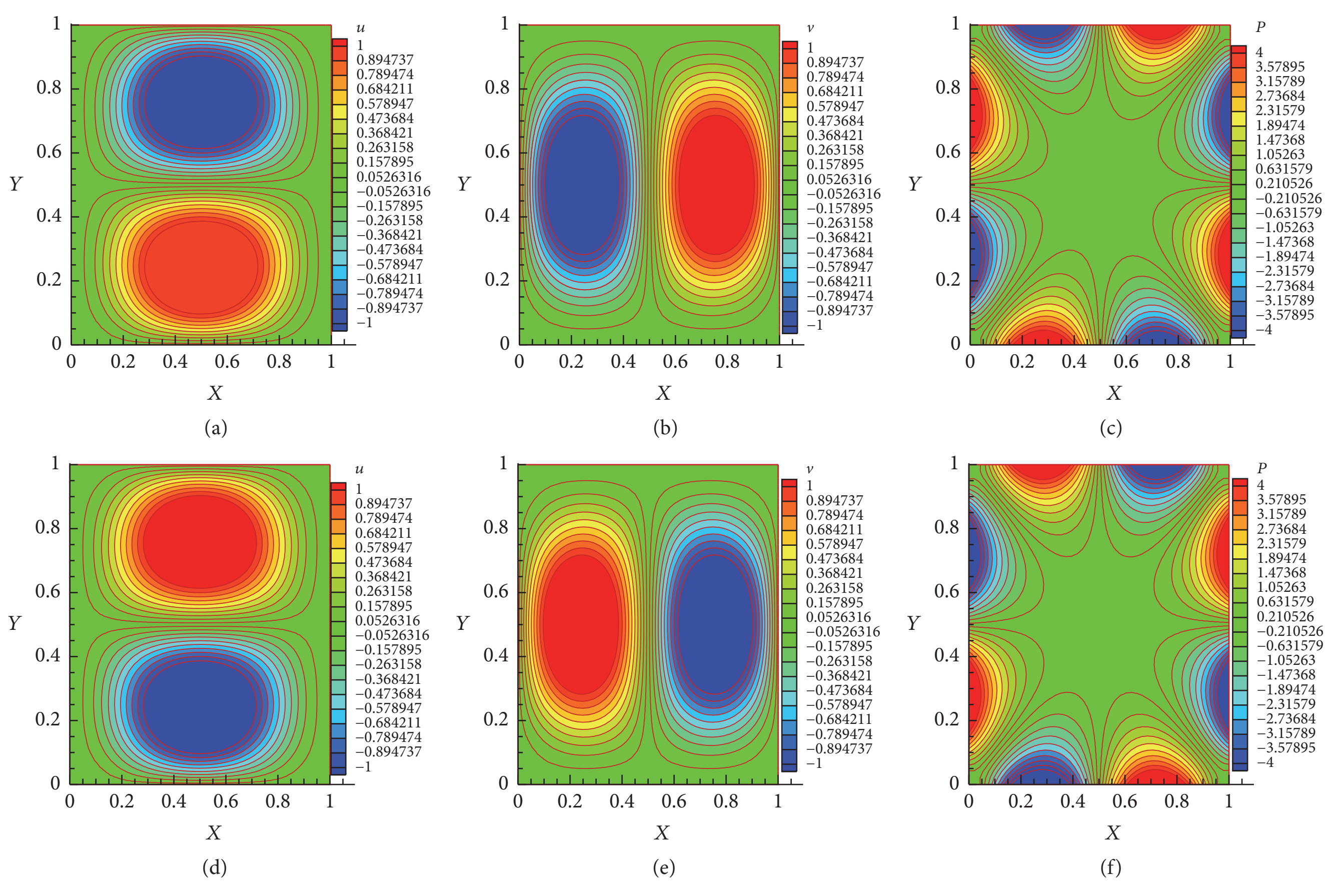

Figure 2: Plots of the velocity and pressure at $h = 1/48$: numerical solution of two-space method with $P_2 - P_2$(a–c) and numerical solution of two-space method with $P_{2b} - P_2$(d–f) for u_{1h}, u_{2h}, p_h.

finite element space to solve the original Stokes eigenvalue problem, and solve the Stokes source problem in the second-order mixed finite element space on the same mesh. Moreover, the related error estimates have been derived. Finally, numerical tests show that the two-space stabilized mixed finite element method is numerically efficient for solving the Stokes eigenvalue problem. The two-space algorithm can achieve the same accuracy as the stabilized finite element solution as the $P_2 - P_2$ stabilized method by taking less CPU time. Obviously, this method can be extended to the case of three dimensions easily. And there are some open questions including the possible extension of the method to other linear and nonlinear eigenvalue problems.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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