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Combinatorial Problem Arising in Chaotic Control

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An optimization problem arising in the analysis of controllability and stabilization of cycles in discrete time chaotic systems is considered.

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1 INTRODUCTION

In recent times, the control *of* chaos has become more and more important. It is often desirable in practice that chaos be eliminated and that a system exhibit periodic or near periodic behaviour [17]. For example, in [7] an experiment was conducted in which a drug was injected at an approximate moment to restore a periodic beat pattern in a rabbit's heart.

We will consider only the discrete time case where the corresponding dynamics of a control system can be described by

$$x_n = f(x_{n-1}, u_{n-1}), \quad x_{n-1}, x_n \in X$$

where X is the phase space of the system and u_{n-1} is a control influence. The set of all possible control influences U contains the element $\mathbf{0}$, which corresponds to trivial control, that is, no control at all. Let $\mathbf{x}$ be a periodic motion of the unperturbed chaotic system $x_n = f(x_{n-1}, \mathbf{0})$ which is considered as an admissible or desired regime of the system. Our aim is to synthesize a control $u_n = u(n, x_n)$ that stabilizes the closed system onto the periodic motion $\mathbf{x}$.

The process of control is naturally divided into two stages: firstly, bringing the system to within close vicinity of the desirable periodic motion, and secondly, local stabilization of this periodic motion. The main methodology of controlling chaos in the second stage has so far been by means of linear control theory [17, 20]. The mathematical model of chaotic dynamics

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is linearized around the desired fixed point or periodic orbit to enable application of the linear control theory [18].

The first stage, that is approaching a region where a linearized model works, is a different problem. At least two factors should be taken into consideration: (A) The control influences should be rather small, in the sense that u_n should be close to zero control $\mathbf{0}$. This restriction is categorical in many ecological and biological applications. (B) Some areas of the state space are less desirable to visit than others and so time spent visiting these areas should be minimized.

A trivial strategy is to just wait for a trajectory of the unperturbed system $x_n = f(x_{n-1}, \mathbf{0})$ to be within a suitable vicinity of the desirable motion. The principal success of this strategy for some systems is guaranteed w.p.1 by ergodic theorems. However the approach is not very practical, firstly, because the waiting time could be rather long and, secondly, because it does not take into account the aforementioned restrictions of type (B).

Another natural approach is to consider a rough discretization of a system and try to bring the discretization to within a neighbourhood of the desired cycle. Technically it means the following. A reasonably fine lattice Ξ, is chosen in the phase space. Then the multivalued dynamics of the set Ξ, when the transition from one point to another is allowed if it can be realized by means of sufficiently small (satisfying restriction (A)) control influences. Finally the optimal path satisfying the multivalued dynamics and the corresponding control influences should be chosen. Note that there is a tradeoff when taking into account both factors (A) and (B). Therefore a kind of multi-criteria Pareto optimization [15] should be used.

This paper is devoted to the analysis of this last approach. The main results are described in Section 2. The first Subsection 2.1 is devoted to the description of the discretization procedures. We prove that the procedure is correct and give a two-sided estimate of its accuracy. In Subsection 2.2 we describe an algorithm to solve the corresponding combinatorial problems. Theoretical questions concerning the effectiveness of the algorithm are considered in Subsection 2.3. We show that the algorithm is quasi-polynomial and that there does not exist a truly polynomial algorithm for the corresponding combinatorial problem. In Section 3 the approach is applied to some classical systems, such as the tent map and the Anosov toroidal automorphisms. Results of experimental calculations demonstrate that the suggested algorithm works quite well. Indeed, discretizations where the number of lattice points is of the order 10^5 can be considered in less than one minute of CPU time.

2 MAIN RESULTS

2.1 Discretization Procedure

Suppose that X and U are compact metric spaces with the metrics ρ_X and ρ_U. Let $f: X \times U \mapsto X$ be a continuous function which describes the dynamics of a control system under consideration. Let $p(u)$ and $q(x)$ be scalar-valued nonnegative functions which characterize the price of control and the quality of a phase point respectively. Infinity is admitted for the function $p(u)$; this implies that the corresponding controls are prohibited. The functions $p(u)$ and $q(x)$ are *upper semi-continuous* which means that the relations

$$p(u) = \lim_{\varepsilon \mapsto 0} \sup_{\rho_U(u,v)<\varepsilon} p(v), \qquad q(x) = \lim_{\varepsilon \mapsto 0} \sup_{\rho_X(x,y)<\varepsilon} q(y)$$

hold. Suppose that the price of trivial control $\mathbf{0}$ is equal to zero. Consider the question as to whether we can reach a particular "target" set S_ω commencing from a given set S_0 with reasonable accumulated price and, simultaneously, with reasonable accumulated quality. In the previous setting this area S_ω can be interpreted as a neighbourhood of periodic motion that allows the use of local control. Suppose there is a path $\pi_x = x_0, \ldots, x_n$ from a given point $x_0 \in S_0$ to $x_n \in S_\omega$ with the corresponding control sequence $\pi_u = u_0, \ldots, u_{n-1}$, satisfying $p(u_i) < \infty$. That is, $x_i = f(x_{i-1}, u_{i-1})$, $i = 1, \ldots, n$. The sequence of pairs $\pi = (x_0, u_0), \ldots, (x_{n-1}, u_{n-1})$ will also be called a path. The accumulated price along a path π is $p(\pi) = \sum_{i=0}^{\eta-1} p(u_i)$ and the accumulated quality is $q(\pi) = \sum_{i=0}^{n-1} q(x_i)$. The general quality of this path can be characterized by the pair $(p(\pi), q(\pi))$ and we denote by $\Pi = \Pi(f, S_0, S_\omega, p, q)$ the totality of all possible paths from S_0 to S_ω which have a finite accumulated price.

Consider the set

$$K(f, S_0, S_\omega, p, q) = \overline{\bigcup_{\pi\in\Pi} K(p(\pi), q(\pi))} \tag{1}$$

where the overline denotes closure and $K(a, b) = \{(a_1, b_1): a_1 \geq a \text{ and } b_1 \geq b\}$. If the set $\Pi(f, S_0, S_\omega, p, q)$ is empty then $K(f, S_0, S_\omega, p, q)$ is defined to also be the empty set. The set (1) characterizes the possible outcomes in the following sense: if (a, b) is an interior point of $K(f, S_0, S_\omega, p, q)$ then there exists a path π satisfying both inequalities $p(\pi) < a$ and $q(\pi) < b$; if on the other hand $(a, b) \in IR^2 \backslash K(f, S_0, S_\omega, p, q)$ then for each path $\pi = \Pi(f, S_0, S_\omega, p, q)$ either $p(\pi) > a$ or $q(w) > b$ holds. These considerations explain why the following two problems are of interest.

Problem 1 Find the set $K(f, S_0, S_\omega, p, q)$.

Problem 2 For a given (a, b) belonging to the interior of $K(f, S_0, S_\omega, p, q)$ find a path $\pi \in \Pi(f, S_0, S_\omega, p, q)$ satisfying both $p(\pi) < a$ and $q(\pi) < b$.

Recall that an element $(a, b) \in M \subset IR^2$, is a *Pareto minimal* element [15] of the closed set M if for each $(a_1, b_1) \in M$, either $a_1 > a$ or $b_1 > b$. The set (1) can also be defined by the formula $K(f, S_0, S_\omega, p, q) = \cup_{(a,b)\in\mathcal{P}} K(a, b)$ where $\mathcal{P}$ is the totality of the Pareto minimal elements of the closure of the set $\{(p(\pi), q(\pi)): \pi \in \Pi(f, S_0, S_\omega, p, q)\}$. The set $K(f, S_0, S_\omega, p, q)$ is technically more convenient than the set $\mathcal{P}$ itself as it is known to be more robust to perturbations, in particular, to the discretization procedures considered below.

We now describe how the above problems can be correctly discretized. Consider the case when the mapping $f(x, u)$ is a local surjection in u, in the sense that *there exists a continuous function* $\sigma(\varepsilon)$, $\varepsilon > 0$ *which satisfies* $\sigma(0) = 0$ *such that for any* (x, u) *the set* $\{f(x, v): \rho_U(u, v) \leq \sigma(\varepsilon)\}$ *contains the ball* $\rho_X(x, y) \leq \varepsilon$. This condition can be interpreted as local one-step controllability; see Subsection 2.4 for further discussion of this restriction.

Let Ξ be a certain finite lattice in X which intersects with the sets S_0 and S_ω. Denote $\Xi_0 = \Xi \cap S_0$ and $\Xi_\omega = \Xi \cap S_\omega$ and let $G = G_\Xi(f, S_0, S_\omega, p, q)$ be a directed graph with the set of nodes Ξ and set of arcs defined as follows. If $\xi \in \Xi \backslash \Xi_\omega$, $\eta \in \Xi \backslash \Xi_0$ and the set $U(\xi, \eta)$ of controls satisfying $p(f(\xi, u) = \eta)$, $p(u) < \infty$ is not empty, then we include the arc $\alpha: \xi \mapsto \eta$ in the directed graph $G_\Xi(f, S_0, S_\omega, p, q)$ with length and weight given by

$$\ell(\alpha) = \inf\{p(u): u \in U(\xi, \eta)\}, \qquad w(\alpha) = q(\xi). \tag{2}$$

Note that 'inf' in the definition (2) cannot be swapped with 'min' because the price function $p(\cdot)$ is only supposed to be *semi*-continuous. The constructed directed graph $G_\Xi(f, S_0, S_\omega, p, q)$ with corresponding lengths and weights of arcs is a straightforward Ξ-discretization of the control system under consideration. Denote by $\Pi_\Xi = \Pi_\Xi(f, S_0, S_\omega, p, q)$ the set of all paths in this graph from Ξ_0 to Ξ_ω.

Denote by $\mathcal{P}_\Xi(f, S_0, S_\omega, p, q)$ the totality of Pareto minimal elements in the finite set $\{(\ell(\pi), w(\pi)) \colon \pi \in \Pi_\Xi\}$.

Problem 1a Find the set $P_\Xi(f, S_0, S_\omega, p, q)$.

Problem 2a For a given $(a, b) \in P_\Xi(f, S_0, S_\omega, p, q)$ find a path

$$\pi \in \Pi_\Xi(f, S_0, S_\omega, p, q) \quad \text{with } \ell(\pi) = a \text{ and } w(\pi) = b.$$

Problems 1a and 2a represent correct discretizations of Problems 1 and 2 in the sense that under natural technical restrictions the set

$$K_\Xi(f, S_0, S_\omega, p, q) = \bigcup_{(a,b)\in\mathcal{P}_\Xi(f,S_0,S_\omega,p,q)} K(a, b)$$

converges to $K(f, S_0, S_\omega, p, q)$ as the density of the set Ξ increases. For each (a, b) belonging to the interior of $K(f, S_0, S_\omega, p, q)$ and each sufficiently dense Ξ there exists an element $(a_1, b_1) \in \mathcal{P}_\Xi(f, S_0, S_\omega, p, q)$ satisfying both $a_1 < a$ and $b_1 < b$. Let us formulate a simple accurate statement in this direction.

Denote by $h(\Xi) = \sup_{x\in X} \inf_{\xi\in\Xi} \rho_X(\xi, x)$ the *step size* of the lattice Ξ. Recall also that the Hausdorff separation of the set $A \in \mathbb{R}^2$ from $B \subset A$ is the value $\operatorname{sep}(A, B) = \sup_{a\in A} \inf_{b\in B} |a - b|$.

PROPOSITION 1 *Suppose that the sets S_0 and S_ω are finite and $S_0, S_\omega \subset \Xi$. Then*

$$K_\Xi(f, S_0, S_\omega, p, q) \subseteq K(f, S_0, S_\omega, p, q). \tag{3}$$

For each $\varepsilon > 0$ there exists $h(\varepsilon) > 0$ such that the relation $h(\Xi) < h(\varepsilon)$ implies that

$$\operatorname{sep}[K(f, S_0, S_\omega, p, q), K_\Xi(f, S_0, S_\omega, p, q)] \le \varepsilon. \tag{4}$$

For each (a, b) belonging to the interior of $K(f, S_0, S_\omega, p, q)$ there exists $h(a, b) > 0$ such that the set $P_\Xi(f, S_0, S_\omega, p, q)$ contains an element (a_1, b_1) with $a_1 < a, b_1 < b$ when $h(\Xi) < h(a, b)$.

Proof To prove (3), suppose that there exists a path $\pi_\xi = \xi_0, \xi_1, \ldots, \xi_{k-1}, \xi_k, \xi_i \in \Xi$ with $\xi_0 \in \Xi_0$, $\xi_k \in \Xi$ along the sequence of arcs $\alpha_1, \ldots, \alpha_k$ with total length $\ell(p) = \sum_{i=1}^k \ell(\alpha_i)$ and total weight $w(p) = \sum_{i=1}^k w(\alpha_i)$. We need to show that for each $\varepsilon > 0$ there exists a path $\pi \in \Pi(f, S_0, S_\omega, p, q)$ with $p(\pi) = \ell(\pi)$ and $q(\pi) = w(\pi_\xi) + \varepsilon$. It suffices to define $\pi = (\xi_0, u_0), (\xi_1, u_1), \ldots, (\xi_k, u_k)$ where u_i are such that $p(u_i) < \ell(\alpha_{i-1}) + \varepsilon/k$. The last definition is correct by virtue of the first equality of (2). To prove (4), first choose a finite set P of paths in $\Pi(f, S_0, S_\omega, p, q)$ such that

$$\operatorname{sep}\left[K(f, S_0, S_\omega, p, q), \bigcup_{\pi\in P} K(p(\pi), q(\pi))\right] \le \frac{\varepsilon}{2}.$$

This is possible because $K(f, S_0, S_\omega, p, q)$ belongs to the positive quadrant of the real plane. It remains to show that for each particular path $\pi \in P$ there exists a path π^Ξ in the directed graph

G_Ξ with $p(\pi^\Xi) \le p(\pi) + \varepsilon/2$, $q(\pi^\Xi) \le q(\pi) + \varepsilon/2$, for all sufficiently small $h(\Xi)$. Note that the x-component of a particular path π can be approximated with a precision of $h(\Xi)$ by an auxiliary path π_x^Ξ which goes through the lattice Ξ. If h is small enough, then this path π_x^Ξ can be treated as the x-component of a path π^Ξ with a price $p(\pi^\Xi) \le p(\pi) + \varepsilon/2$ because $f(x, v)$ is a local surjection in v, and p is upper semicontinuous. On the other hand, for sufficiently small h, $q(\pi^\Xi) \le q(\pi) + \varepsilon/2$ as q is upper semicontinuous. That is, the path π^Ξ has the necessary properties to be a path in the directed graph and the proposition is proved. Finally, the existence of the function $h(a, b)$ follows from (4) which has just been proved. ■

The standard examples show that under the assumptions of the above proposition, the sets $\mathcal{P}_\Xi(f, S_0, S_\omega, p, q)$ do not necessarily converge to the totality of the Pareto minimal elements of the closure of the set $\{(p(\pi), q(\pi)) : \pi \in \Pi(f, S_0, S_\omega, p, q)\}$.

It is convenient to be able to estimate the distance

$$\operatorname{sep}[K(f, S_0, S_\omega, p, q), K_\Xi(f, S_0, S_\omega, p, q)]$$

explicitly. Let us formulate a simple assertion which is useful in this context. Denote by $c(\varepsilon)$, $\varepsilon \ge 0$ a continuity module of the mapping $x \mapsto f(x, u)$, $x \in X$, $u \in U$, that is, $c(\cdot)$ is continuous, satisfies $c(0) = 0$ and the inclusions $x, y \in X$, $u \in U$ imply $\rho_X(f(x, u), f(y, u)) \le c(\rho_X(x, y))$; $\sigma(\cdot)$ is the function from the definition of the local surjection property.

LEMMA 1 *Suppose that the sets S_0 and S_ω are finite and S_0, $S_\omega \subset \Xi$. Then the inclusions*

$$K_\Xi(f, S_0, S_\omega, p, q) \subseteq K(f, S_0, S_\omega, p, q) \subseteq K_\Xi(f, S_0, S_\omega, p_*, q_*) \quad (5)$$

$$K(f, S_0, S_\omega, p^*, q^*) \subseteq K_\Xi(f, S_0, S_\omega, p, q) \subseteq K(f, S_0, S_\omega, p, q) \quad (6)$$

$$K(f, S_0, S_\omega, p, q) \subseteq K_\Xi(f, S_0, S_\omega, p_*, q_*) \subseteq K(f, S_0, S_\omega, p_*, q_*) \quad (7)$$

are valid where

$$p_*(u) = \inf_{\rho_U(u,v) < \sigma(h(\Xi) + c(h(\Xi)))} p(v), \quad q_*(x) = \inf_{\rho_X(x,y) < h(\Xi)} q(y),$$

$$p^*(u) = \sup_{\rho_U(u,v) < \sigma(h(\Xi) + c(h(\Xi)))} p(v), \quad q^*(x) = \sup_{\rho_X(x,y) < h(\Xi)} q(y).$$

The **proof** is straightforward and so is omitted.

The inclusion (5) provides two-sided estimates for the set $K(\mathcal{P}(f, S_0, S_\omega, p, q))$ in terms of the corresponding discretizations. The estimates (6) and (7) and the semi-continuity of functions p and q guarantee that the aforementioned two-sided estimates converge as the step size $h(\Xi)$ approaches zero.

The constructions given in this subsection are of some general interest. Any straightforward computer model of a dynamical system is a discretization due to finite machine arithmetic [3]. Some dangerous pathologies ([5] and references therein) can occur in the course of such implementations. In this context, Lemma 1 shows how such dangerous effects could be controlled by means of modifying the price and quality functions.

2.2 Algorithm

Let us describe an algorithm for Problem 1a, which we denote by $\mathcal{A}1$.

INPUT:

The directed graph G with the set Ξ of nodes and the set A of arcs;

The disjoint subsets Ξ_0 and Ξ_ω of Ξ;
Non-negative lengths $\{\ell(\alpha), \alpha \in A\}$ and weights $\{w(\alpha), \alpha \in A\}$ of arcs.

OUTPUT:

The totality $\mathcal{P}$ of Pareto minimal elements in the set $(\ell(\pi), w(\pi))$ corresponding to all paths from Ξ_0 to Ξ_ω in the directed graph G.

During the work of the algorithm, *labels* will be assigned to the nodes of the graph G and at each step the set $\mathcal{P}$ will also contain labels. The character of the labels and the procedure of their assignment varies a little from its classical variant [6]. Essentially, the differences are as follows: firstly, a *label* is an ordered pair (ℓ, w) with both elements being either a non-negative number or ∞; secondly, every node during the work of the algorithm can have more than one label at the same time. The meaning of the label (ℓ, w) assigned to some node $\xi \in \Xi$ after k steps of the algorithm is as follows:

(a) there exists a path π from the node ξ_0 to the node ξ with length $\ell(\pi)$ and weight $w(\pi)$;
(b) for each k-step path π from Ξ_0 to Ξ_ω there exists a label from $\mathcal{P}$ which is coordinate-wise less then or equal to $(\ell(\pi), w(\pi))$;
(c) for each k-step path π from Ξ_0 to another node $\eta \in \Xi \backslash \Xi_\omega$ there exists either a label of η or a label from $\mathcal{P}$ which is coordinate-wise less than or equal to $(\ell(\pi), w(\pi))$.

Before starting the algorithm, the label (0, 0) is assigned to the nodes $\xi \in \Xi_0$, the label (∞, ∞) is assigned to all other nodes, the auxiliary set Ξ_{new} is defined to be equal to Ξ_0 and the set $\mathcal{P}$ is defined as the empty set.

WORK OF THE ALGORITHM: At each stage the following four actions are performed.

Action 1

Subaction 1.1 The set Ξ_{ends} of nodes which are endpoints of arcs commencing at nodes belonging to $\Xi_{\text{new}} \backslash \Xi_0$ are determined.

Subaction 1.2 The set of arcs A_ω which end at $\xi \in (\Xi \bigcap \Xi_{\text{ends}})$ and start at Ξ_{new}.

Subaction 1.3 For each $\xi \in (\Xi_{\text{ends}} \backslash \Xi)$ the set of arcs $A(\xi)$ which end at ξ and start at Ξ_{new} are derived.

Action 2 The set Ξ_{new} is redefined as the empty set.

Action 3

Subaction 3.1 The set Λ is arranged to include: (i) elements of the set $\mathcal{P}$; (ii) the set of pairs $(\ell(\alpha) + \ell, w(\alpha) + w)$, where $\alpha \in A$ and (ℓ, w) belongs to the set of labels of the start point of α.

Subaction 3.2 The set Λ is searched for the set $\mathcal{P}_{\text{new}}$ of all Pareto minimal elements.

Subaction 3.3 The set $\mathcal{P}$ is redefined as $\mathcal{P}_{\text{new}}$.

Action 4 The set $\Xi_{\text{ends}} \backslash \Xi$ is ordered in an arbitrary way and for successive elements of this set the following subactions are performed.

Subaction 4.1 The set Λ is arranged to include: (i) all labels of the node ξ; (ii) the set of pairs $(\ell(\alpha) + \ell, w(\alpha) + w)$, where $\alpha \in A(\xi)$ and (ℓ, w) belongs to the set of labels of the start point of α.

Subaction 4.2 The set Λ is searched for the set of all Pareto minimal elements. Those minimal elements which are *coordinate-wise less than at least one element of* $\mathcal{P}$ become the only labels of the node ξ. If the node ξ has at least one new label, then it is included in the set Ξ_{new}.

Stopping Rule The process stops if the set Ξ_{new} becomes empty. After stopping, the set $\mathcal{P}$ is declared to be the output of the algorithm.

THEOREM 1 *Algorithm* $\mathcal{A}1$ *correctly solves Problem* 1*a*.

Proof The algorithm stops in no more than $\nu + 1$ steps where ν denotes the cardinality of Ξ. Correctness of the algorithm follows from the properties (a), (b) and (c) of the labels which were given in the description of the algorithm. These properties can be easily proved by induction on the number of steps. ■

$\mathcal{A}1$ is similar, but different, to the algorithm suggested in [10], which allows negative weights but is not as efficient as $\mathcal{A}1$ in the case of non-negative weights. Other algorithms to solve Problem 1a were also considered. The particular variant given above was chosen as most suitable for analysis of graphs arising in discretizations of control systems described in the previous subsection. Algorithm $\mathcal{A}1$ can be modified to solve simultaneously Problem 2a (see Appendix).

2.3 Complexity Questions

We commence with an analysis of the complexity of Problem 1a in the Cook-Karp sense. Recall [9] that combinatorial recognition problems are divided into two quite different types: problems which are $\mathcal{NP}$-complete and those which are not. Basically, $\mathcal{NP}$-complete problems are defined as problems which are at least as hard to solve in polynomial time as any other problem of combinatorial optimization [2, 9]. A recognition problem P is said to be $\mathcal{NP}$-complete if another recognition problem P_1, which has been proved to be $\mathcal{NP}$-complete, can be polynomially reduced to problem P. Informally, the class of $\mathcal{NP}$-complete problems consists of "computationally hard" problems for which an effective algorithm has not been found that solves the corresponding problem in polynomial time with respect to the minimal length of the complete description of the problem. Note, that in general the class of $\mathcal{NP}$-complete problems are more common than polynomially solvable problems.

A special place in the totality of all combinatorial problems is occupied by the subclass of *problems with numerical parameters*; in the case of Problem 1a these are the corresponding lengths and weights of arcs. For $\mathcal{NP}$-complete problems of this sort, there are *pseudo-polynomial algorithms* ($\mathcal{PPT}$-algorithms) for which the time of its work is bound by the sum of absolute values of the numerical parameters characterizing the individual problem. (Here as usual in complexity analysis the numerical parameters ℓ and w are non-negative integers.) Pseudo-polynomial algorithms are sufficient in all reasonable situations. Additional specifics and discussion on this matter can be found in [9].

THEOREM 2 *Algorithm A1 is a PPT-algorithm. Problem 1a is $\mathcal{NP}$-complete.*

Proof The first assertion – that $\mathcal{A}1$ is a $\mathcal{PPT}$-algorithm. Without loss of generality we can assume that the directed graph G is complete. Then each of the numerical parameters appearing in work of the algorithm does not exceed the sum S over all lengths and weights of arcs of the directed graph G. Therefore the quantity of the labels of each node also does not exceed S. This implies that the number of operations at each step of the algorithm is bounded from above by S^3. It remains to note that the maximum quantity of steps also does not exceed S.

To prove the second assertion it suffices to polynomially reduce the well-known $\mathcal{NP}$-complete problem PARTITION to the following modification of Problem 1a.

Problem 1b Let G be a directed graph with the set $\Xi = \{\xi_0, \dots, \xi_n\}$ of nodes and with the set A of arcs. Let ℓ and w be two integer valued functions defined on A. Let L and W be integers. Then, does there exist a path connecting the node v_0 to the node v_n such that the total length of arcs is less than or equal to L and the total weight of arcs is less than or equal to W?

The problem PARTITION (see [9]) is as follows. Let $P = \{p_i, i \in \{1, 2, \dots, s\} = N_s\}$ be a set of natural numbers. We wish to find whether there is the subset I of the set N_s such that

$$\sum_{i \in I} p_i = \sum_{i \in N_s \setminus I} p_i. \tag{8}$$

Consider the PARTITION problem with the individual set P. We construct the individual Problem 1b to which the problem PARTITION can be reduced. For the directed graph G suppose $n = 3s$,

$$A = \bigcup_{i=1}^{s} \{(\xi_{3i-3} \mapsto \xi_{3i-2}), (\xi_{3i-2} \mapsto \xi_{3i}), (\xi_{3i-3} \mapsto \xi_{3i-l})\}$$

and

$$\ell(\xi_{3i-3} \mapsto \xi_{3i-2}) = w(\xi_{3i-3} \mapsto \xi_{3i-1}) = p_i, \quad i \in N_s.$$

All other $\ell(a)$ and $w(a)$, $a \in A$ are zero.

We now prove that the answer to the individual Problem 1b with the above graph G and $L = W = 1/2 \sum_{i=1}^{s} p_i$ is positive if and only if there exists a partition of the set $P = \{p_1, \dots, p_s\}$ into two subsets satisfying (8). For every $i \in N_s$ any path from ξ_0 to ξ_{3s} includes either the pair of arcs $\{(\xi_{3i-3} \mapsto \xi_{3i-2}), (\xi_{3i-2} \mapsto \xi_{3i})\}$ or the pair $\{(\xi_{3i-3} \mapsto \xi_{3i-1}), (\xi_{3i-1} \mapsto \xi_{3i})\}$. Consider some path from ξ_0 to ξ_{3s}. Denote by I a subset of such i from $N - s$ for which this path contains the first pair of arcs. Then the length and weight of this path are $\sum_{i \in I} p_i$ and $\sum_{i \in N_s \setminus I} p_i$ respectively. These values are not more than $1/2 \sum_{i=1}^{s} p_i$ if and only if equality (8) holds. That is, the individual PARTITION problem under consideration is reduced to the constructed individual problem Problem 1. The theorem is proved. ■

The assertion about $\mathcal{NP}$-completeness of Problem 1a can be found without proof in [9]. Authors refer to the unpublished private statement of Megiddo. The existence of a $\mathcal{PPT}$-algorithm for solving the PARTITION problem based on the idea of dynamical programming was also given in [9]. It was this that prompted us to construct a $\mathcal{PPT}$-algorithm for Problem 1a. Lastly, note that the construction of a polynomial algorithm to solve Problem 1a is impossible for the following reason: for each positive integer n there exists graphs with n nodes such that the quantity of Pareto minimal paths connecting the given nodes η and ξ is equal to the overall quantity of all paths connecting these nodes (see [13]).

2.4 Remarks

2.4.1 Discontinuous Systems

Many important systems with chaotic behaviour have points of discontinuity. For instance the β-mapping, Belykh mapping [1], etc. All considerations above, including Proposition 1 remain intact provided that the system is generated by a mapping f with some simple discontinuity points in the variable x. In particular, it is sufficient to assume that for any open set $Y \subset X$ the preimage $f^{-1}(Y)$ is contained in the closure of its interior.

2.4.2 Generalized Prices

Let each pair $(x, u) \in X \times U$ has a generalized price $p_g(x, u)$ which is an element of a cone K of non-negative elements in a semi-ordered linear space L supplemented by an infinity element ∞. We suppose that the function p_g is upper semi-continuous. The generalized price of a finite path $\pi = x_0, x_1, \dots, x_k$ is defined in an additive way: $p_g(\pi) = \sum_0^{n-1} p_g(x_i, u_i)$. Problems 1 and 2 are modified accordingly. For a lattice Ξ which intersects S_0 and S we can introduce the corresponding directed graph G_Ξ with generalized length $\ell_g(\alpha)$ of its arcs and denote by Π_Ξ the set of possible paths. We arrive at the following modifications of Problems 1a and 2a.

Problem 1g Find the Pareto minimal elements in the set $\{\ell_g(\pi)\colon \pi \in \Pi_\Xi\}$.

Problem 2g For a given Pareto minimal pair find a corresponding path π.

Algorithm $\mathcal{A}1$ works for Problem 1g with the following modification: the labels should be understand to be elements from $K \bigcup \infty$, which imply the subsequent evident corrections. For instance the pairs $(\ell(a), w(a))$ should be swapped to the generalized lengths $\ell_g(a)$.

2.4.3 Locally Controllable Systems

The set U can be of a lesser dimension than X in which case the mapping $f(x, u)$ itself is not a local surjection in u. To use the methods described above, it is convenient to consider a certain iteration of the mapping f. To avoid cumbersome formulas we will illustrate this idea by a simple example. Let Tor^d be the standard d-dimensional torus, that is, the factorization of $\mathbb{R}^d$ by the integer lattice. Each matrix A with integer entries $a_{ij}, i, j = 1, \dots, d$ defines an algebraic endomorphism f_A of Tor^d. If the eigenvalues of this matrix are irrational, then the mapping is chaotic and each hyperbolic diffeomorphism of Tor^d can be in a natural sense reduced to such a mapping.

Consider a control system of the form $x_n = f_A(x_{n-1}) + bu_{n-1}$ where u is a scalar, $|u| \le 1$, and b is an element of $\mathbb{R}^d$ treated as an element of the group of toroidal shifts. Consider reaching a periodic orbit S from a certain set S_0 with given price and quality functions $p(u)$ and $q(x)$. Suppose that the vectors $b, Ab, \dots, A^{d-1}b$ are linearly independent. An auxiliary control system $x_n = F(x_{n-1}, \bar{u}_{n-1})$ is defined in the following way: for $\bar{u} = (u^1, \dots, u^d)$, $|u^i| \le 1$, the set $F(x, \bar{u}) = y_d$ where $y_0 = x$ and $y_i = f(y_{i-1}) + bu^i, i = 1, \dots, d$. This system has the same periodic points as the original system and we consider reaching the same set S as before. Finally, define the generalized price $p_g(x, \bar{u})$ by $p_g(x, \bar{u}) = \sum_{i=1}^d q_*(y_{i-1}) + p(u^i)$ where $y_i, i = 0, \dots, d-1$ are the same as above and $q_*(x) = q(x)$ for $x \in \mathrm{Tor}^d \backslash S$ whereas $q_*(x) \equiv 0$ for $x \in S$. The constructed auxiliary problem is equivalent to the original and the mapping

$F(x, \bar{u})$ is a local surjection: that is, there exists a continuous function $\sigma(\varepsilon)$, $\varepsilon \geq 0$, $\sigma(0) = 0$, such that for any x and $\bar{u}$ the set $\{F(x, \bar{v})$: $\rho_U(\bar{u}, \bar{v}) \leq (\varepsilon)\}$ covers the ball $\rho_X(x, y) \leq \varepsilon$ due to the Kalman controllability criterion [19].

2.4.4 Existence of Paths with Finite Prices

As was mentioned previously, the set of possible controls $u \in U$, constitutes a reasonably small neighbourhood around the trivial control. For this reason it is convenient to characterize couples $(f(x, u), S)$ such that the set S can be achieved for each $p(u)$. That is, the set $\Pi(f, S_0, S_\omega, p, q)$ and therefore the set $K(f, S_0, S_\omega, p, q)$ are non-empty for any upper semi-continuous $p(u)$ with $p(\mathbf{0}) = \mathbf{0}$. Let X be a smooth manifold and μ_0 be an absolutely continuous measure. *I.e.* $\mu_0(X) = 0$ whenever X has zero Lebesgue measure. Define the sequence of measures $\{\mu_n\}$ by $\mu_{n+1}(S) = \mu_n(F^{-1}(S))$, $n = 0, 1, 2, \ldots$ where $F(x) = f(x,)$. A probability measure μ_* is said to be *attractive* if all such sequences $\{\mu_n\}$ converge Cesàro weakly to μ_*. Recall that Cesàro means $(\mu_0 + \mu_1 + \cdots + \mu_{n-1})/n$ converge weakly to μ_* for all choices of absolutely continuous μ_0 on X. Existence of an attracting measure is proved or supposed to be true for many classes of chaotic systems.

Let μ_* be an attractive measure whose support intersects S. It is easy to show that the set $\Pi(f, S_0, S_\omega, p, q)$ is non-empty for each $p(u)$. Moreover, if the lattice Ξ intersects S_0 and S and the set where $p(u)$ takes finite values contains the ball $\rho_U(u, u^0) \leq \sigma[h(\Xi) + c(h(\Xi))]$ where $\sigma(\cdot)$ and $c(\cdot)$ are the same as in Lemma 1, then the set $\mathcal{P}_\Xi(f, S_0, S_\omega, p, q)$ is non-empty. See also Theorem 5, [4].

3 NUMERICAL EXPERIMENTS

3.1 Tent Map

The tent map, or the baker map, is defined (p. 179 [8]) by the formula

$$T(x) = 1 - |2x - 1|, \quad 0 \leq x \leq 1.$$

This mapping is the simplest continuous chaotic dynamical system and occupies a special place in the modern dynamical systems theory (see [16] and references therein). Consider the case when the corresponding control system is described by the relation $x_n = f_T(x_{n-1}, u_{n-1})$, $0 \leq x \leq 1$, $|u| \leq 1$ where

$$f_T(x, u) = \begin{cases} T(x) + u, & \text{if } 0 \leq T(x) + u \leq 1; \\ 0, & \text{if } T(x) + u < 0; \\ 1, & \text{if } T(x) + u > 1. \end{cases}$$

$f_T(x, u)$ is the point nearest to $T(x) + u$ belonging to $[0, 1]$. Suppose that the price of control is defined by the formula

$$p(u; \varepsilon) = \begin{cases} |u|, & \text{if } |u| < \varepsilon, \\ \infty, & \text{if } |u| \geq \varepsilon \end{cases}$$

with $\varepsilon = 0.001$, *i.e.* controls with $|u| \geq \varepsilon$ are prohibited. Suppose further, that all points of $[0, 1]$ have the same quality $q(x) \equiv 1$. As such, $q(\pi)$ for a certain path is just the corresponding number of iterations.

The tent map has two equilibriums: 0 and 2/3. We will consider the question of the "cheapest" transition from the first to the second. That is, we are interested in reaching the set

$S = \{2/3\}$ from the set $S_0 = \{0\}$. We will use uniform lattices $\Xi_\nu = \{0, 1/\nu, \ldots, (\nu-1)/\nu, 1\}$. The lattice Ξ_ν contains the point 2/3 if 3 is a divisor of ν and we will only consider such lattices in this subsection. Algorithm $\mathcal{A}1$ was used to find the Pareto sets $\mathcal{P}_{\Xi_\nu}(T, S_0, S_\omega, p, q)$ for different ν. The experiments were carried out on a SUN workstation with 2 UltraSPARC 167 MHz processors with 256 MB memory. Typical results are presented in Table I.

The "upper" elements of the Pareto sets are converging to a limit as ν increases, which is in line with Proposition 1. The corresponding sets $\bigcup_{(a,b)\in\mathcal{P}(\nu)} K(a,b)$ should approximate the set $K(f_T, S_0, S_\omega, p, q)$. To estimate rigorously the precision we apply Lemma 1. In this case $h(\Xi_\nu) = 1/2\nu$, $\sigma(\varepsilon) = \varepsilon$ and $c(\varepsilon) = 2\varepsilon$. That is,

$$\bigcup_{(a,b)\in\mathcal{P}(\nu)} K(a,b) \subseteq K(f_T, S_0, S_\omega, p, q) \subseteq \bigcup_{(a,b)\in\mathcal{P}(f_T, S_0, S_\omega, p_\nu, q)} K(a,b)$$

where

$$p_\nu(u) = \inf_{\rho_U(u,\nu)<3/(2\nu)} p(\nu) = \max\{0, p(u) - 3/(2\nu)\}.$$

The set $\mathcal{P}_{\Xi_\nu}(f_T, S_0, S_\omega, p_\nu, q)$ was calculated analogously to $\mathcal{P}_{\Xi_\nu}(f_T, S_0, S_\omega, p, q)$ for the case $\nu = 60{,}000$.

Let us discuss in brief some statistics of the algorithm for the case $\nu = 24{,}000$. The CPU time was 31 seconds and the overall number of steps was 103. The maximal number of elements in the set Ξ_{ends} was 11,864 and the maximal number of arcs with the same end-node was 8. Table II presents information about the dynamics of the size $\aleph(\Xi_{\text{new}})$ of the set Ξ_{new}.

Two particular things about this table are: (i) the number $\aleph(\Xi_{\text{new}})$ increased geometrically at the first 10 steps, then it decreased drastically in the next two steps to the value 27; (ii) then it "unexpectedly" became 1 at step 54 and remained equal to 1 for the 50 last steps. Qualitatively such behaviour is quite typical and can be explained as follows. The first reduction at steps 10–11 was due to the fact that at step 10 the destination point 2/3 was reached and from this moment onwards the italicized part of the subaction 4.2 of the algorithm takes over. The second reduction at step 54 occurs because all of the Pareto elements have been found and it remains only to check whether the trajectory commecing at the point $1/\nu$ would eventually hit the fixed point at 2/3.

Let us discuss finally which maximal number of steps can be expected. For each positive integer i denote by $\div_2(i)$ the exponent in the maximal divisor of i of the form 2^k and denote $\text{odd}(i) = i/2^k$. From [14] it follows that the number of steps in the algorithm is always less than or equal to $k + m$ where $k = \div_2(i)$ and m denotes the period of the point $2/\text{odd}(\nu)$ under iterations of the tent map; if, additionally, the Riemann hypothesis is true, then the proportion

TABLE I Pareto Sets for the Tent Map with Various ν and $=0.001$.

$\nu = 12{,}000$	$\nu = 24{,}000$	$\nu = 36{,}000$	$\nu = 48{,}000$	$\nu = 60{,}000$
(10, 1.7E−3)	(10, 1.7E−3)	(10, 1.7E−3)	(10, 1.6E−3)	(10, 1.6E−3)
(11, 8.3E−4)	(11, 7.1E−4)	(11, 7.0E−4)	(11, 6.7E−4)	(11, 6.7E−4)
(12, 5.0E−4)	(12, 4.2E−4)	(12, 3.9E−4)	(12, 3.5E−4)	(12, 3.5E−4)
(13, 3.3E−4)	(13, 2.5E−4)	(13, 2.2E−4)	(13, 2.1E−4)	(13, 2.0E−4)
(14, 2.5E−4)	(14, 1.7E−4)	(14, 1.4E−4)	(14, 1.3E−4)	(14, 1.2E−4)
(15, 1.7E−4)	(15, 1.2E−4)	(15, 1.1E−4)	(15, 0.8E−4)	(15, 0.8E−4)
	(58, 8.8E−5)	(30, 8.8E−5)	(16, 6.3E−5)	(16, 6.7E−5)
		(107, 5.6E−5)	(59, 4.2E−5)	(25, 4.5E−5)
				(257, 3.3E−5)

TABLE II The Cardinality of the Set Ξ Agains Step Number for $v = 24{,}000, = 0.001$.

Step	$\aleph(\Xi)$	*Step*	$\aleph(\Xi)$	*Step*	$\aleph(\Xi)$	*Step*	$\aleph(\Xi)$
1	1	15	27	29	16	43	15
2	23	16	24	30	16	44	14
3	68	17	26	31	16	45	15
4	115	18	22	32	15	46	15
5	206	19	21	33	14	47	15
6	390	20	22	34	15	48	15
7	758	21	22	35	14	49	14
8	1494	22	21	36	15	50	15
9	2966	23	19	37	14	51	14
10	5910	24	20	38	15	52	14
11	3623	25	19	39	14	53	15
12	27	26	17	40	15	54	1
13	26	27	15	41	14	55	1
14	28	28	16	42	15	56	1
Steps from 57 up to 103: $\aleph(\Xi) = 1$							

of those $v = 3k$ for which the number of steps in the algorithm is equal to $2v/3$ is of magnitude $\ln(k)^{-1}$.

3.2 Anosov Algebraic Automorphism

Let Tor^2 be the standard 2-dimensional torus, that is, the factorization of $\mathbb{R}^2$ by the integer lattice. Each matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with integer entries and $\det A = \pm 1$ defines an algebraic automorphism $f_A(x)$ on Tor^2. If $b, c \neq 0$ then this mapping is chaotic and each hyperbolic diffeomorphism on Tor^2 can be in a natural sense reduced to such mapping (see p. 114 [8]). We consider the corresponding 2-dimensional control system described by

$$x_n = f_A(x_{n-1}, u_{n-1}) = f_A(x_{n-1}) + u_{n-1}$$

where $x = (x^1, x^2) \in \mathrm{Tor}^2$ and $u = (u^1, u^2)$, $|u^1|, |u^2| \leq 1$ is a two-dimensional vector to be understood as an operator on Tor^2. We suppose that the price of control is defined by the formula

$$p(u; \varepsilon) = \begin{cases} |u^1| + |u^1|, & \text{if } |u^1|, |u^2| < \varepsilon, \\ \infty, & \text{otherwise} \end{cases}$$

with $\varepsilon = 0.01$ and all points of [0, 1] have the same quality $q(x) \equiv 1$. Let, in particular,

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

The mapping $f_A(x)$ has a fixed point at (0, 0) and a 2-cycle with elements (1/5, 2/5) and (3/5, 4.5). We are interested in transition from the set $S_0 = (0, 0)$ to the aforementioned 2-cycle, S. We use the uniform lattice

$$\Xi_v^2 = \{(i/v, j/v): i, j = 0, 1, \ldots, v - 1\}$$

where v is divisible by 5 (so as the elements of the two-cycle belong to the lattice).

TABLE III Pareto Sets for the Anosov Mapping for Different v with = 0.01.

$v = 120$	$v = 140$	$v = 160$	$v = 180$	$v = 200$
(8, 0.041)	(8, 0.050)	(8, 0.050)	(7, 0.066)	(7, 0.084)
(10, 0.033)	(9, 0.043)	(9, 0.037)	(8, 0.039)	(8, 0.035)
(12, 0.025)	(10, 0.035)	(10, 0.031)	(9, 0.031)	(10, 0.025)
(15, 0.017)	(12, 0.021)	(11, 0.025)	(11, 0.028)	(14, 0.020)
			(14, 0.022)	(19, 0.015)
			(18, 0.017)	(27, 0.010)

Table III is analogous to Table I. For $v = 200$ the algorithm took 67 steps and the maximal number of elements in Ξ_{new} was 9920 at step 9. In this case, Lemma 1 was too "coarse" to get good estimates (moreover it shows that the set Ξ cannot be reached in less than 7 iterations). Nevertheless, using the particular structure of the mapping shows that the precision of the algorithm is satisfactory.

Analogous experiments were carried out for other chaotic mappings, such as the twisted horse-shoe mapping, the Belych mapping, etc. [1, 8].

A Few Personal Words

I worked at the Institute for Control Problems (IPU) for around 20 years (firstly in the Tsypkin Laboratory No 7 and then in the Krasnoselskii Laboratory No 61). The IPU was distinguished among Soviet academic institutions in many aspects, such as:

- a concentration of great scientists and great personalities (certainly, these two sets intersect);
- flow of principal new ideas and approaches, often fusing different scientific strands.

These points may be convincingly justified. The influence on the world science can be tracked through the citation index. The number of specialists of my and younger generations who made quite remarkable careers, some times in science, some times in quite different areas (a time of changes in Russia!) is phenomenal.

I am glad to acknowledge that the itemized above anomalies are nowadays in place in IPU. I have visited IPU in 2000, when celebrating its 40th anniversary. It was a great conference!

A couple of words about this paper. The first author, Vladimir Bondarenko is, like me, a pupil of Mark Krasnosel'skii. He has never formally worked in IPU, however all his life was closely connected with our beloved institute Jamie Mustard is a young Australian mathematician (this paper was to a large extent written when I and Vladimir Bondarenko visited Australia). Anyway, Jamie was influenced strongly by IPU-people. This is a (probably incomplete) list of those who visited us in those months: Nikolay Kuznetsov, Boris Polyak, Alexander Krasnosel'skii, Boris Miller, Victor Kozyakin, Nikolay Bobylev.

We all have very pleasant, longtime memories of our collegues and teachers who have died. Greetings to all IPU-people who remember me and/or whom I remember. Special thanks to organizers of this volume, A. Poznyak and V. Utkin.

Alexei Pokrovskii

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References

[1] Belykh, V. N. (1982) Models of discrete systems of phase synchronization, In: Shakhgildyan and Belyustina (Eds.), *Systems of Phase Synchronization*, Chapter 10. Moscow: Radio and Communication, pp. 161–176 (in Russian).
[2] Bondarenko, V. A. (1995) *Polyhedral Graphs and Complexity in Combinatorial Optimization.* Yaroslavl: Yaroslavl University Press (in Russian).
[3] Cavanagh, J. J. F. (1984) *Digital Computer Arithmetic. Design and Implementation*, McGraw-Hill Book Company.
[4] Diamond, P., Kloeden, P. E. and Pokrovskii, A. V. (1995) Cycles of spatial discretizations of shadowing dynamical systems, *Mathematische Nachrichten*, **171**, 415–422.
[5] Diamond, P. and Pokrovskii, A. V. (1996) Statistical laws for computational collapse of discretized chaotic mappings, *J. Bifurcation & Chaos*, **6**, 2389–2399.
[6] Dijcstra, E. W. (1959) A note on two problems in connection with graphs, *Numerische Mathematik*, **1**, 269–271.
[7] Ditto, W. L. and Spano, M. L. (1994) Control and anti-control of chaos in hearts and brains, In: Stonier, R. J. and Yu, X. H. (Eds.), *Complex Systems: Mechanism of Adaptation.* Amsterdam: IOS Press, p. 359.
[8] Sinai, Ya. G. (Ed.) (1989) *Dynamical Systems. Ergodic Theory with Applications to Dynamical Systems and Statistical Mechanics*, Series: Encylopedia of Mathematical Sciences, Vol. 2. Berlin; New York: Springer-Verlag.
[9] Garey, M. R. and Johnson, D. S. (1979) *Computers and Intractability: A Guide to the Theory of $\mathcal{NP}$-Completeness*, San Francisco: Freeman.
[10] Garraway, R. L. and Morin, T. L. (1990) Generalized dynamic programming for multi-criteria optomization. *European J. Oper. Res.*, **44**(1), 55–104.
[11] Grötschel, M., Lovász, L. and Schrijver, A. (1988) *Geometric Algorithms and Combinatorial Optimization.* Heidelberg, Berlin: Springer-Verlag.
[12] Guckenheimer, J., Oster, G. and Ipaktachi, A. (1977) The dynamics of density dependent population models, *J. Math. Biol.*, **4**, 101–147.
[13] Jemelitschev, V. A. and Perepelitsa, V. A. (1988) Multicriteria problems on graph spanning tree, *Soviet. Math. Dokl.*, **37**, (1), 114–117.
[14] Kloeden, P. E., Mustard, J. and Pokrovskii, A. V. (1999) Statistical properties of some spatially discretized dynamical systems, *ZAMP*, **50**, 638–660.
[15] Stadler, W. (1988) Multicriteria optimization in engineering and sciences, In: Stadler, W.(Ed.), *Mathematics Concepts and Methods in Science and Engineering*, Vol. 37. New York; London: Plenum Press.
[16] Nusse, H. E. and Yorke, J. A. (1988) Is every approximate trajectory of some process near an exact trajectory of a nearby process?, *Commun. Math. Phys.*, **114**, 363–379.
[17] Ott, E. (1993) *Chaos in Dynamical Systems*, Cambridge University Press.
[18] Ott, E., Grebogi, C. and Yorke, J. A. (1990) Controlling chaos, *Physical Review Letters*, **64**, 1196–1199.
[19] Padulo, L. and Arbib, M. A. (1974) *System Theory; A Unified State-Space Approach to Continuous and Discrete Systems.* Philadelphia: Saunders.
[20] Romeiras, F. J., Gregori, C., Ott, E. and Dayawansa, W. P. (1992) Controlling chaotic dynamical systems, *Physica D*, **58**, 165–492.

APPENDIX

We now describe an algorithm for Problem 2a, which we denote by $\mathcal{A}2$.

INPUT:

Same as the input for Algorithm $\mathcal{A}1$ plus an element $(a, b) \in \mathcal{P}_{\Xi}(f, S_0, S, p, q)$.

OUTPUT:

The path π_ξ satisfying $\ell(\pi_\xi) = a$, $w(\pi_\xi) = b$.

The work of the algorithm consists of two stages.

Stage 1 At this stage, the labels are assigned to the nodes of the graph G which are either the symbol $\emptyset$ or an ordered sequence of the form $= \{(0, 0), (\ell_1, w_1), \ldots, (\ell_n, w_n)\}$ where n is a positive integer. The pair (ℓ_n, w_n) is referred to as the final element of the label. The set of labels assigned to a node ξ is denoted by $\Lambda(\xi)$. Before starting the algorithm the label $(0, 0)$ is assigned to all nodes $\xi \in \Xi_0$ and the label $\emptyset$ is assigned to other nodes; the auxiliary set Ξ_{new} is defined to be equal to Ξ_0.

At each step the following three actions are performed (which are analogous to Actions 1, 2 and 4 in Algorithm $\mathcal{A}1$).

Action 1
Subaction 1.1 The set Ξ_{ends} of nodes which are endpoints of arcs commencing at nodes belonging to $\Xi_{\text{new}} \setminus \Xi_0$ are determined.

Subaction 1.2 For each $\xi \in \Xi_{\text{ends}}$ the set of arcs $A(\xi)$ which end at ξ and start at Ξ_{new} are derived.

Action 2 The set Ξ_{new} is redefined as the empty set.

Action 3 The set Ξ_{ends} is ordered in an arbitrary way and for successive elements of this set the following subactions are performed.

Subaction 3.1 The set M is arranged which includes: (i) all of the final elements of the labels of node ξ which are not $\emptyset$; (ii) the totality of sums $(\ell(\alpha) + \ell, w(\alpha) + w)$, where $\alpha \in A(\xi)$ and (ℓ, w) belongs to the set consisting of all the final elements of the labels of the starting point of α. Denote by M_{old} the subset of M containing only the elements from (i).

Subaction 3.2 The set M is searched for all Pareto minimal elements. If a Pareto minimal element (ℓ, w) belongs to the set M_{old} then the corresponding sequence is included in the set $\Lambda(\xi)$. If (ℓ, w) does not belong to the set M_{old} and both $\ell \leq a$ and $w \leq b$ hold, then it can be written as $(\ell, w) = (\ell_n + \ell(\alpha), w_n + w(\alpha))$ for some α and (ℓ_n, w_n) to be the final element of the label $\{(0,0), (\ell_1, w_1), \ldots, (\ell_n, w_n)\} \in \Lambda(\eta)$ where α goes from η to ξ. In this case we add to the set of labels of ξ the sequence $\{(0,0), \ldots, (\ell_n, w_n), (\ell, w)\}$. If the node ξ has at least one new label, then it is included in the set Ξ_{new}.

Stopping Rule for the First Stage The process stops when at least one node $\xi \in \Xi$ has a non-empty label.

Stage 2 After the first stage is finished we have a directed graph G with labels. Now we define a path π_ξ in the directed graph G. Choose the node $\xi \in \Xi$ which has a non-empty label. This can be written as $\{(0,0), \ldots, (\ell_{n-1}, w_{n-1}), (\ell_n, w_n)\}$ where $\ell_n = a$ and $w_n = b$. Define a path $\pi_\xi = \xi_0, \xi_1, \ldots, \xi_n$ by induction in the following manner. Let $\xi_n = \xi$. Suppose that ξ_k has been defined. Then choose ξ_{k-1} such that it has a label $\{(0,0), \ldots, (\ell_{k-1}, w_{k-1})\}$ and there exists an arc α from ξ_{k-1} to ξ_k satisfying $\ell(\alpha) = \ell_k - \ell_{k-1}$ and $w(\alpha) = w_k - w_{k-1}$. The path π_ξ is the output of the algorithm.

THEOREM 3 *Algorithm $\mathcal{A}2$ correctly solves Problem 2a.*

Proof is straightforward, so is omitted.