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Observability Conditions of Linear Time-Varying Systems and Its Computational Complexity Aspects

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We propose necessary and sufficient observability conditions for linear time-varying systems with coefficients being time polynomials. These conditions are deduced from the Gabrielov–Khovansky theorem on multiplicity of a zero of a Noetherian function and the Wei–Norman formula for the representation of a solution of a linear time-varying system as a product of matrix exponentials. We define a Noetherian chain consisted of some finite number of usual exponentials corresponding to this system. Our results are formulated in terms of a Noetherian chain generated by these exponential functions and an upper bound of multiplicity of zero of one locally analytic function which is defined with help of the Wei–Norman formula. Relations with observability conditions of bilinear systems are discussed. The case of two-dimensional systems is examined.

Key words: Linear system, Time-varying system, Observability, Matrix exponential, Rank of a matrix, Multiplicity of zero, Computational complexity

AMS Classification Codes: 3B07, 93B27, 93B40

1 INTRODUCTION

Observability is one of principal qualitative properties characterizing a control system. It has an significant theoretical impact in elaborating of control and systems theory. Besides, there are various important applications of observability theory in engineering sciences due to related concepts of an observer, a feedback etc. In this paper we shall deal with linear systems of a type

$$\dot{x}(t) = A(t)x = \sum_{s=1}^{m} u_s(t)A_s x(t), \quad y = Cx(t) \tag{1}$$

where $A_1, \dots, A_m$ are linear independent constant $(n \times n)$-matrices; C is a constant $(p \times n)$-matrix; $x \in \mathbf{R}^n; y \in \mathbf{R}^p$. Below a vector-function $u(t) = (u_1(t), \dots, u_m(t))$ is assumed to be analytic in some neighborhood of zero and is referred to as an input. We shall consider the system (1) as a bilinear system or as a linear time-varying system with specified functions $u_s(t); s = 1, \dots, m$. In accordance with this point of view we remind one classical definition.

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DEFINITION 1 *The system* (1) *is called observable if for any* $x \neq 0$ *one can find an input* $u(t)$ *such that for the corresponding solution* $\Phi(u,t)x$ *of* (1) *we have that* $C\Phi(u,t)x$ *is a nonzero time function. The system* (1) *with a specified input* $u(t) = u^*(t)$ *is called* u^**-observable if for any* $x \neq 0$ *we have that* $C\Phi(u^*, t)\, x$ *is a nonzero time function.*

Observability conditions for linear time varying systems and bilinear systems and related controllability conditions for linear time-varying systems were described in many papers: Bittanti, Colaneri and Guardabassi [1], D'Alessandro, Isidori and Ruberti [3], Funahashi, Adachi and Inagaki [4], Kratz and Liebscher, [6], Leiva and Lehman [7], Leiva and Zambrano [8], Silverman and Meadows [9], Szigeti [12].

Computational complexity of observability algorithms for linear time-varying systems is not an well-studied topic up to nowadays, see Starkov and Garnica [10]. Another related reference is Starkov [11] where complexity questions were considered for polynomial control systems. Following to Szigeti [12] we apply the Wei–Norman formula, see Wei and Norman [13], for the solution of the system (1). The differential-algebraic treatment of the Szigeti's paper leads to observability conditions given in terms of differential-algebraic independence of some finite number of functions arisen as a result of the application of the Wei–Norman formula. Results obtained on this way seem difficult for practical computations. Instead of this, we use the pure algebraic approach. The principal theorem of this paper is deduced from the Gabrielov theorem on multiplicities of zeros of polynomials on trajectories of polynomial vector fields, see the refined version due to Gabrielov and Khovansky [5]. Our main contribution is to present necessary and sufficient u^*-observability conditions which are formulated in terms of observability property of a corresponding bilinear system and, besides, are described with help of the computation of a multiplicity bound of a zero for some Noetherian function denoted below by ${}^c w$ at 0. Ways for computing this multiplicity bound concern computational complexity of observability tests and are discussed in this paper. As compared with the short conference paper Starkov and Garnica [10], the case of a n-dimensional system (1) is examined in this paper. We provide the refined multiplicity bound of a zero of ${}^c\omega$ expressed only in terms of the dimension of the matrix Lie algebra generated by $A(t)$, the dimension of the system and degrees of functions $u_s(t), s = 1, \dots, m$. This bound is obtained without any additional assumptions respecting to the system (1).

2 PRELIMINARIES

By $\sigma(B)$ we denote the spectrum of the constant $(n \times n)$-matrix B. Let $Lie(A_1, \dots, A_m)$ be the matrix Lie algebra generated by matrices $A_1, \dots, A_m$. Let $A_1, \dots, A_m, \dots, A_\ell$, $m \le \ell \le n^2$, be a basis of $Lie(A_1, \dots, A_m)$. Observability analysis in case $\ell = 1$ is clear. Indeed, let $A(t) = u(t)A$ for some constant $(n \times n)$-matrix A; u is a nonzero input. Then we have that $(A(t), C)$ is u-observable $\Leftrightarrow$ (A, C) is observable $\Leftrightarrow$ $(A(t), C)$ is observable. Here the first equivalence is followed from the use of the appropriate time transformation combined with a small time shift. The second equivalence is due to results of D'Alessandro, Isidori and Ruberti [3]. We remark here that controllability property for this class of systems was studied in Leiva and Zambrano [8], see also Szigeti [12]. Below everywhere in this paper we suppose that $\ell > 1$. We introduce the notation

$$\sum_{s=1}^{m} u_s A_s = (A_1, \dots, A_\ell)\bar{u},$$

with $\bar{u} = (u_1, \dots, u_m, 0, \dots, 0)^T \in \mathbf{R}^\ell$. Let $[X, Y]$ be the Lie bracket of $(n \times n)$-matrices X and Y. Suppose that the multiplication table for $(adX)Y = [X, Y]$ is given by formulae

$[A_i, A_j] = \sum_{k=1}^{\ell} \gamma_{ij}^k A_k$. Now by use of results of Wei and Norman [13], the fundamental matrix $\Phi(u, t)$ for (1) can be given locally in time around zero as

$$\Phi(u, t) = \exp(A_1 g_1(t)) \cdots \exp(A_\ell g_\ell(t)) \tag{2}$$

where $g = (g_1, \ldots, g_\ell)^T$ is satisfied the equation

$$\dot{g} = \Xi(g)\bar{u}, \quad g(0) = 0. \tag{3}$$

Here in (3) Ξ (0) is nonsingular;

$$\prod_{j=1}^{i-1} \exp(g_j adA_j)A_i = \sum_{k=1}^{\ell} s_{ki}(t)A_k, \quad i = 1, \ldots, \ell \tag{4}$$

and $\Xi(g) = (S(g))^{-1}, S(g) = \|s_{ij}(g)\|; i, j = 1, \ldots, \ell$. Also, let $\rho(g) = \det \Xi(g)$.

3 ONE FORMULA FOR THE OUTPUT OF THE LINEAR TIME-VARYING SYSTEM

In this section we describe one formula for the output of the system (1). It is deduced from formulae for matrix exponentials borrowed from the paper Cheng and Yau [2]. Assume that A_s has distinct eigenvalues $\lambda_{1s}, \ldots, \lambda_{r(s)s}, s = 1, \ldots, \ell$, and its minimal polynomial has the form

$$mp_{A_s}(x) = (x - \lambda_{1s})^{m_{1s}} \ldots (x - \lambda_{r(s)s})^{m_{r(s)s}};$$
$$m_{1s} + \cdots + m_{r(s)s} = m(s) \leq n.$$

We also can write

$$\exp(g_s(t)A_s) = \sum_{k_s=0}^{m(s)-1} f_{k_s}(g_s(t))A_s^{k_s},$$

with

$$f_{k_s}(t) = \sum_{j_s=1}^{r(s)} \sum_{i_s=0}^{m_{j_s}-1} P_{i_s j_s k_s} \frac{t^{i_s}}{i_s!} \cdot \exp(\lambda_{j_s} t).$$

Numbers $P_{i_s j_s k_s}$ are calculated according with formulae from Cheng and Yau [2]. By substitution of these formulae into (2) we have:

$$\begin{aligned} C\Phi(u, t)x &= C \prod_{s=1}^{\ell} \sum_{k_s=0}^{m(s)-1} A_s^{k_s} \sum_{j_s=1}^{r(s)} \sum_{i_s=0}^{m_{j_s}-1} P_{i_s j_s k_s} \frac{g_s^{i_s}(t)}{i_s!} \cdot \exp(\lambda_{j_s} g_s(t))x \\ &= \sum_{\Theta_1} CA_1^{k_1} \cdots A_\ell^{k_\ell} \sum_{\Theta_2} (P_{i_1 j_1 k_1} \cdots P_{i_\ell j_\ell k_\ell} / i_1! \cdots i_\ell!) \\ &\quad \times g_1^{i_1}(t) \cdots g_\ell^{i_\ell}(t) \exp\left(\sum_{s=1}^{\ell} \lambda_{j_s} g_s(t)\right)x; \end{aligned} \tag{5}$$

$$\Theta_1 = \{k_s = 0, \ldots, m(s) - 1; s = 1, \ldots, \ell\};$$
$$\Theta_2 = \{i_s = 0, \ldots, m_{j_s} - 1; j_s = 1, \ldots, r(s); s = 1, \ldots, \ell\}$$

The formula (5) is valid in some neighborhood of the point $t = 0$.

4 u^*-OBSERVABILITY *VERSUS* OBSERVABILITY

By Ω_1 we denote a set of vector-rows of matrices $CA_1^{k_1}\cdots A_\ell^{k_\ell}$ in the formula (5). Let $\xi_1,\ldots,\xi_\eta\in\Omega_1$ be a basis in a vector space spanned by Ω_1. Then (5) is reduced up to

$$C\Phi(u,t)x = \|G_{ij}(g(\bar u;t))\|\cdot\|\xi_{sq}\|x \tag{6}$$

$i=1,\ldots,p; j=1,\ldots,\eta; s=1,\ldots,\eta; q=1,\ldots,n$. In the formula (6) $\|\xi_{sq}\|=\|\xi_1^T\cdots\xi_\eta^T\|^T$; G_{ij} are analytic functions defined in some neighborhood of $t=0$. Now we come to

PROPOSITION 2 *The system* (1) *is u^*-observable if and only if*

$$\eta=n;\ rank\|\xi_{sq}\|=n$$

and there is $\varepsilon>0$ such that

$$Ker\, G(g(\bar u^*;t))=\{0\}\quad for\ 0<|t|<\varepsilon, \tag{7}$$

with $G(g(\bar u;t)) := \|G_{ij}(g(\bar u;t))\|$.

If we have a single-output system then we take the last condition in the form of the linear independence of functions

$$G_j(g(\bar u;t)) := G_{1j}(g(\bar u;t)),\quad j=1,\ldots,n.$$

Proof The proof is followed from one of inequalities for a rank of a multiplication of two rectangular matrices. ■

We introduce the set Ω_2 of vector-rows of matrices

$$\left\{\begin{array}{c} CA_{a(1)}^{k(1)}\cdots A_{a(n-1)}^{k(n-1)};\\ k(1);\ldots;k(n-1)\in\{0;1\};a(1);\ldots;a(n-1)\in\{1;\ldots;m\}.\end{array}\right\}$$

PROPOSITION 3 *If $rank\,\Omega_1=n$ then* 1) *$rank\,\Omega_2=n$*; 2) *the system* (1) *is observable.*

Proof. Assume that there is a nonzero vector $x\in\mathbf{R}^n$ such that $\Omega_2x=0$. Then $CA_sx=0; s=1,\ldots,m; CA_iA_jx=0; i,j=1,\ldots,m$. As a result, $C[A_i,A_j]x=0;\ i,j=1,\ldots,m$. By iteration of this argument, we get that $CLie(A_1,\ldots,A_m)x=0$ and we come to the contradiction with our condition. So (1) is proved. It remains to remind that the system (1) is observable if and only if $rank\,\Omega_2=n$. It follows from Funahashi, Adachi and Inagaki [4]. ■

Remark 1 The set $\{u$ is analytic locally around zero $|Ker\,G(g(\bar u;t))=\{0\}\}$ defines a class of analytic inputs u for which the system (1) is u-observable provided rank $\|\xi_{sq}\|=n$.

Let $d \in \mathbf{N}$. We introduce the linear space *Inp* (d) of polynomial vector functions $u : \mathbf{R}^1 \to \mathbf{R}^m$, with deg $u_s(t) \leq d$; $s = 1, \dots, m$. Let $j^d u(0)$ be the jet of the order d of $u(t)$ at zero. Consider the following $(pn \times n)$-matrix

$$M = \left[\frac{d^i}{dt^i} |_{t=0} \, G(g(\bar{u}; t)) \right]; \quad i = 0, 1, \dots, n-1. \tag{8}$$

Iterated derivatives in (8) are calculated with help of the vector field in (3), but for arbitrary initial condition $g(0) = g$ in some neighborhood of $g = 0$. Taking into account of (3) with the initial condition $g(0) = 0$ we obtain that $\frac{d^i}{dt^i}|_{t=0} \, G(g(\bar{u}; t))$ depends on $j^d u(0)$ in a polynomial way.

5 NOETHERIAN CHAINS, NOETHERIAN FUNCTIONS AND ONE THEOREM RELATED TO THESE CONCEPTS

We note that Proposition 2 does not provide any effective algorithm for checking the condition (7). We discuss this circumstance on an example of single-output systems (1). Since functions $G_j, j = 1, \dots, n$, are locally analytic we check their linear independence property with help of considering an overdetermined system of linear homogeneous equations arisen from the application of the method of indefinite coefficients. Coefficients in this linear system of equations are multiple time derivatives of $G_j, j = 1, \dots, n$, taken at $t = 0$. *A priori* it is not clear up to what order we have to compute these time derivatives in order to establish linear independence of functions $G_j, j = 1, \dots, n$. That is why computational complexity aspect is important in the process of the application of Proposition 2. In Sections 7, 8 and 9 we answer this question in an explicit way and present results of computations respecting to the condition (7).

In what follows in this section we remind some necessary matter taken from Gabrielov and Khovansky [5]. We formulate

DEFINITION 4 *A Noetherian chain of order* m *and degree* α *is a system* $f = (f_1(x), \dots, f_m(x))$ *of germs of analytic functions at the origin* 0 *of* $\mathbf{C}^n$ *or* $\mathbf{R}^n$, *satisfying Pfaffian equations*

$$\frac{\partial f_i}{\partial x_j} = g_{ij}(x, f_1(x), \dots, f_m(x)), \quad i = 1, \dots, m; \; j = 1, \dots, n$$

where g_{ij} *are polynomials with respect to* x; f *of degree* $\leq \alpha$; $\alpha \geq 1$. *A function* $\delta(x) = P(x, f_1(x), \dots, f_m(x))$, *where* P *is a polynomial in* x *and* f *of degree* $\leq \beta$ *is called a Noetherian function of degree* β *with the Noetherian chain* $f_1, \dots, f_m$.

As it was noted in Gabrielov and Khovansky [5], these definitions can be formulated in terms of a ring $\mathbf{K}$ of Noetherian functions defined in an open domain $\mathbf{U} \subset \mathbf{C}^n$ or $\mathbf{R}^n$. Recall here that $\mathbf{K}$ consists of analytic functions on $\mathbf{U}$ such that (1) $\mathbf{K}$ contains the polynomial ring $\mathbf{C}[x_1, \dots, x_n]$ ($\mathbf{R}[x_1, \dots, x_n]$ appropriately) and is finitely generated over the ring $\mathbf{C}[x_1, \dots, x_n]$ ($\mathbf{R}[x_1, \dots, x_n]$). (2) $\mathbf{K}$ is closed under differentiation. Our main tool is the following theorem presented in Gabrielov and Khovansky [5].

THEOREM 5 *Let ξ be a time-varying vector field on $\mathbf{C}^n \times \mathbf{C}(\mathbf{R}^n \times \mathbf{R})$, with coefficients ξ_j depending on time and Noetherian of degree η for the Noetherian chain $f = (f_1(x), \dots, f_m(x))$ of degree α. Let ψ be a Noetherian function of degree β, with the same Noetherian chain. Suppose that $\xi(0) \neq 0$ and if γ is a solution of ξ through the origin then the restriction $\psi|_\gamma \neq 0$. Under these conditions the multiplicity of the zero of $\psi|_\gamma$ at 0 does not exceed*

$$F(m, n, \alpha, \eta, \beta) = \frac{1}{2}\sum_{k=0}^{m+n}[2\beta + 2k(\eta + \alpha - 1)]^{2m+2n+2}$$

6 THE NOETHERIAN CHAIN OF A LINEAR TIME-VARYING SYSTEM

The goal of this section is to show that there is a natural Noetherian chain corresponding to the system (1).

Let $\kappa = (3\ell - 6)/2 + 2$ provided ℓ is even and $\kappa = (3\ell - 3)/2$ provided ℓ is odd. We formulate

PROPOSITION 6 *We state that either (1) $S(g) = P(g)$, where elements P_{ij} of the matrix P are polynomials of degree $\leq \kappa$ with respect to g, or (2) there are $r \leq 2\ell$ exponential functions $v_1(g), \dots, v_r(g)$ such that (2a) $S(g) = P(g, v_1(g), \dots, v_r(g))$, where elements P_{ij} of the matrix P are polynomials of degree $\leq \kappa$ with respect to g, $v_1, \dots, v_r$; (2b) functions $(v_1(g), \dots, v_r(g))$ form a Noetherian chain of degree 1 and order r.*

Proof Firstly, we remark that for different choices of the basis in the Lie algebra $Lie(A_1, \dots, A_m)$ we obtain expressions (2) with different matrices $S(g)$ and vectors $\bar{u}$. However, the germ at zero of a solution of the system (3) with the initial condition $g(0) = 0$, generated by any of these expressions is the same. It follows from the uniqueness theorem of the solution of the differential equation. Therefore we can take the basis with the simplest multiplication table. It is some subset of $\{I_{ij}; i, j = 1, \dots, n\}$, where I_{ij} is a matrix containing the only one unit on the (i, j)-position; other elements are zeroes. Now we write a list of useful formulae for matrices $\{I_{ij}; i, j = 1, \dots, n\}$:

$$\begin{aligned}
\exp(g\,ad\,I_{ij})I_{ji} &= I_{ji} + g(I_{ii} - I_{jj}) - g^2 I_{ij}, \quad i \neq j;\\
\exp(g\,ad\,I_{ij})I_{jk} &= I_{jk} + gI_{ik}, \quad i \neq j; \ i \neq k;\\
\exp(g\,ad\,I_{ii})I_{ik} &= I_{ik}\exp g, \quad i \neq k;\\
\exp(g\,ad\,I_{ij})I_{ki} &= I_{ki} - gI_{kj}, \quad i \neq j; \ j \neq k;\\
\exp(g\,ad\,I_{ii})I_{ki} &= I_{ki}\exp(-g), \quad i \neq k;\\
\exp(g\,ad\,I_{ii})q(g)I_{ik} &= I_{ik}q(g)\exp g, \quad i \neq k;\\
\exp(g\,ad\,I_{ii})q(g)I_{ki} &= I_{ki}q(g)\exp(-g), \quad i \neq k;
\end{aligned} \tag{9}$$

for any analytic function $q(g)$. By substitution of these formulae into (4) we obtain that each function $s_{ki}(g)$ is expressed in the form

$$\sum_\alpha b_\alpha(g)\exp\left(\sum_{s=1}^{l}\alpha_s g_s\right); \tag{10}$$

here $\alpha = (\alpha_1, \ldots, \alpha_l)$; each $\alpha_s \in \{\pm 1; 0\}$; all b_α are polynomials with respect to g. Since $\{\exp(\pm g_s); s = 1, \ldots, \ell\}$ forms a Noetherian chain of degree 1 we deduce that either (1) there is an integer $1 \le r \le 2\ell$ and $v_1, \ldots, v_r \in \{\exp(\pm g_s); s = 1, \ldots, \ell\}$ with desirable properties, or (2) a quasiexponential in the formula (10) is reduced to a polynomial for any $s_{ki}(g)$. It remains to estimate the degree of P_{ij} with respect to $\{g, \exp(\pm g_s); s = 1, \ldots, \ell\}$. We note that if $s_{ki}(g)$ has a maximal possible degree with respect to $\{g, \exp(\pm g_s); s = 1, \ldots, \ell\}$ then it is contained in (4) with $i = l$. Now we rename $\{g_s; s = 1, \ldots, \ell\}$ such that I_{ij} corresponds to g_{ij} for each (i, j). Then it follows from (9) that the expression

$$\exp(g_{jk} ad\, I_{jk}) \exp(g_{ki} ad\, I_{ki}) \exp(g_{ij} ad\, I_{ij}) I_{ji}, \tag{11}$$

with $i \ne j$; $k \ne i$ and $j \ne k$, contains the term $g_{jk}^2 g_{ki} g_{ij}^2 I_{jk}$ of maximal degree among all possible products (11) with 3 exp operators. Now we can see that if we separate successively the right-side of (4) with $i = \ell$ by groups with 2 exp operators then we derive from (9) that $\deg s_{ij}(g) \le \kappa$. ∎

7 MAIN RESULTS: THE CASE OF SINGLE-OUTPUT SYSTEMS

In this section we examine the case of single-output systems and concern the relation between u-observability and observability described in terms of the multiplicity of a zero of some Noetherian function at 0. Below we shall consider that matrices A_s in (1) are chosen from the set $\{I_{ij}; i, j = 1, \ldots, n\}$. By fixing some input $u^*(t) \in Inp(d)$ we shall write $M(g, j^d u^*(0))$ as $M_*(g)$. We define a function $\omega(g) = \det M_*(g)$ being analytic in some neighborhood of zero.

Remark 2 We note that columns of the matrix $M_*(g)$ are linear independent if and only if the vector field in (3) is tangent with the finite order of tangency to the surface $\omega = 0$ at $g = 0$. This remark allows us to apply Theorem 5 to the solution of our problem.

We introduce the following objects:

(1) the set $\Gamma = \{\exp(\lambda_{sj} g_j) \mid \lambda_{sj} \in \sigma(A_j) - \{0\}; j = 1, \ldots, \ell\}$;
(2) if we have the second case of Proposition 6 then by $\mathcal{R}_1$ we denote a ring of Noetherian functions generated by $v_1, \ldots, v_r$ over the ring $\mathbf{C}[g_1, \ldots, g_\ell]$;
(3) the ring $\mathcal{R}_2$ which is a ring of Noetherian functions generated by some subset $\Gamma_0 \subset \Gamma$ over the ring $\mathbf{C}[g_1, \ldots, g_\ell]$ such that $\Gamma \subset \mathcal{R}_2$;
(4) if we have the first case of Proposition 6 then let $\mathcal{R} = \mathcal{R}_2$ and $q = card\, \Gamma_0$, otherwise let $\mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2$ be a ring of Noetherian functions generated by some set $\{h_1, \ldots, h_q\} \subset \{\exp(\pm g_j) \mid j = 1, \ldots, \ell\}$ and q is the minimal integer with this property.

The main result of this paper is the following.

THEOREM 7 *Let ζ be a complexified solution of the system* (3) *with $u = u^*$ passing at $t = 0$ through* $\mathbf{0}$. *We establish that the system* (1) *is u^*-observable if and only if* (1) *is observable and the multiplicity of the zero of the complexified polynomial ${}^c\omega$ restricted on the solution ζ at zero does not exceed the number*

$$F(q + 1, \ell, \ell_\kappa + 2, \tau_1, \tau_2), \tag{12}$$

with $q \le 2\ell$; $\tau_1 = (\ell - 1)\kappa + d + 1$; $\tau_2 = n^2\ell + n(n-1)((\ell - 1)\kappa + d + 1)/2$.

Proof Necessity. Let (1) be u^*-observable. Then by Proposition 2, $rank\,\Omega_1 = n$. By Proposition 3, (1) is observable. By Remark 2, it is sufficient to demonstrate that the number (12) gives the upper bound for the multiplicity of the zero of the complexified polynomial ${}^c\omega$ restricted on the solution ζ at zero.

Since the set $\Gamma \subset \{\exp(\pm g_j) \mid j = 1, \dots, \ell\}$ we obtain that $q \leq 2\ell$. We note that there is an analytic function h_{q+1} of ℓ variables defined in some neighborhood of zero such that (1) $\Xi(g) = Q(g, h_1(g), \dots, h_{q+1}(g))$; (2) $h_1, \dots, h_{q+1}$ form a Noetherian chain of degree $\ell_\kappa + 2$ and order $q+1$; (3) $Q = \|Q_{ij}\|$; Q_{ij} are polynomials of degree which does not exceed $(\ell - 1)\kappa + 1$.

Indeed, let $h_{q+1}(g) = \rho(g)$. Then $h_{q+1}(g)$ is analytic in some neighborhood of zero and, also, the matrix Q in the prescribed form is existed because of the nonsingularity of the matrix $S(0)$. Also, we have

$$\frac{\delta}{\delta g_i} h_{q+1} = -h_{q+1}^2 \frac{\delta}{\delta g_i} \det S, \quad i = 1, \dots, \ell. \tag{13}$$

Therefore by use of Proposition 6 the degree of the right-side in (13) is less or equal to

$$2 + \ell \max \deg s_{ij} \leq \ell\kappa + 2.$$

Then again by use of Proposition 6 we obtain the estimate

$$\deg Q_{ij} \leq (\ell - 1) \max \deg s_{ij} + 1 \leq (\ell - 1)\kappa + 1.$$

Because of its definition, ω is a Noetherian function with respect to the Noetherian chain $h_1, \dots, h_{q+1}$. We estimate its degree: $\deg \omega \leq \max \deg G_{i_1} \dot{G}_{i_2} \cdots G_{i_n}^{(n-1)}$ where the maximum is taken with respect to all permutations $(i_1, \dots, i_n)$ of the vector $(1, \dots, n)$. Since $\deg G_{i_1} \leq n\ell$, $i_1 \in \{1, \dots, n\}$ we have that for $j \geq 1$

$$\deg G_{i_j}^{(j-1)} \leq n\ell + (j-1)((\ell - 1)\kappa + 1 + d).$$

Hence,

$$\deg \omega \leq n^2\ell + n(n-1)((\ell - 1)\kappa + 1 + d)/2.$$

Sufficiency. As it was reminded in Proposition 3, $rank\,\Omega_2 = n$. We define the set

$$\Omega_3 = \left\{ \begin{array}{c} CA_{a(1)}^{k(1)} \cdots A_{a(\ell)}^{k(\ell)}; \\ k(1); \dots; k(\ell) \in \{0; 1\}; a(1); \dots; a(\ell) \in \{1; \dots; \ell\}. \end{array} \right\}$$

Now we will show that $span\,\Omega_1 = span\,\Omega_3$. Indeed, for some integer k a vector-row

$$CI_{i_1 j_1}^{n(i_1, j_1)} \cdots I_{i_k j_k}^{n(i_k, j_k)}$$

is equal to $CI_{i_1 i_k}$ for $n(i_s, j_s) = 1$, $s = 1, \dots, k-1$; $n(i_k, j_k) > 1$ and $j_1 = i_k = j_k$; $j_1 = i_2, j_2 = i_3, \dots, j_{k-1} = i_k$, or it is equal to $CI_{i_1 j_k}$ for $n(i_s, j_s) = 1, s = 1, \dots, k$, and $j_1 = i_2, j_2 = i_3, \dots, j_{k-1} = i_k$, otherwise it is equal to the zero vector-row.

Since $\Omega_2 \subset \Omega_3$ we get that $rank\,\Omega_1 = n$. Now by Remark 2, the bound (12) computed for the input u^* and Proposition 2, the system (1) is u^*-observable. ∎

Below we discuss two improvements of the bound (12).

1. Firstly, we note that the reason to use the basis chosen from the set $\{I_{ij}; i, j = 1, \dots, n\}$, is due to availability of estimates of Proposition 6 and a simple relation between Ω_1 and Ω_2.

However, sometimes other bases can lead to better results. In order to see this, we assume that the basis $A_1, \ldots, A_\ell$ can be taken in the form: I (the identity matrix) and some subset $\mathcal{J}$ of $\{I_{ij}; i,j = 1, \ldots, n\}$. Then, by use of considerations of Theorem 3 from Wei and Norman [13], we give the better bound. Namely, we have

COROLLARY 8 *The assertion of Theorem* 7 *is valid if* $q+1$ *is replaced by* q *and* ℓ *is replaced by* $\ell - 1$ *everywhere in the bound* (12).

Proof Let $A(t) = D(t) + a(t)I$ in (1) where $A_\ell := I$; $D(t)$ is generated by $\mathcal{J}$. Then the corresponding fundamental matrix $\Phi(t) = V(t)\exp(\int_0^t a(\vartheta)d\vartheta I)$, with

$$\dot{V} = D(t)V. \tag{14}$$

We can show exactly in the same way as above that $rank\,\Omega_1 = n$ if and only if $rank\,\Omega_2 = n$. Since zeroes of $C\Phi(t)x$ and $CV(t)x$ are the same the result of this Corollary proceeds from Theorem 7 applied to the vector analog of the system (14) with $y = Cx$. ■

Another important case is occured when $\rho \in \mathcal{R}$. Here we have

COROLLARY 9 *Let* $\deg \rho \leq \theta + 1$ *with respect to the Noetherian chain* $(h_1, \ldots, h_q)$. *Then the bound* (12) *can be replaced by* $F(q, \ell, 1, \tau_3, \tau_4)$, *with* $\tau_3 = (\ell - 1)\kappa + d + \theta + 1$ *and* $\tau_4 = n^2\ell + n(n-1)((\ell-1)\kappa + d + \theta)/2$.

Proof The proof is followed from the estimate $\deg Q_{ij} \leq (\ell - 1)\deg s_{ij} + \deg \rho$. ■

In Section 9 we demonstrate that it is possible to obtain a better bound for two-dimensional systems.

8 THE CASE OF MULTIOUTPUT SYSTEMS

It follows from computations of Section 6 that elements of the matrix $\Xi(g)$ and the function ω are real functions. Let $\{\omega_s, s = 1, \ldots, \binom{pn}{n}\}$ be a set of minors corresponding to all $(n \cdot n)$-submatrices of the matrix (8). Let

$$\omega = \sum_{s=1}^{\binom{pn}{n}} \omega_s^2.$$

Since the solution of the complexified system (3) with the initial condition $g(0) = 0$ is real we come to the following assertion. We have that there is $\varepsilon > 0$ such that $Ker\,G(g(\bar{u}^*, t)) = \{0\}$ for $0 < |t| < \varepsilon$ if and only if the vector field (3) is tangent with the finite order of tangency to the surface $\omega = 0$ at the point 0. After this we repeat arguments of Section 7 and obtain the bound (12) with τ_2 replaced by $2\tau_2$.

Also, results of Corollary 8 and Corollary 9 remain valid for the multioutput case. Here τ_2 from the bound of Corollary 8 is replaced by $2\tau_2$ and τ_4 from the bound of Corollary 9 is replaced by $2\tau_4$.

9 EXAMPLE: THE TWO-DIMENSIONAL CASE

We present results of computations for two-dimensional systems, with $y = Cx = x_1$, borrowed from the paper Starkov and Garnica [10]. We shall not use the bound (12). Instead of this, we fulfill direct computations for the basis $A_1 = I_{12} - I_{21}$; $A_2 = I_{11} - I_{22}$; $A_3 = I_{12}$; $A_4 = I_{11} + I_{22}$ of the matrix Lie algebra generated by (1). This basis is taken from Theorem 3, see the paper Wei and Norman [13]. Then our problem is reduced to the same problem for

$$\dot{x}(t) = \sum_{s=1}^{3} u_s(t) A_s x(t).$$

By computation of the matrix $\Xi(g)$ for the new system, we obtain that $\Xi(g)$ has the form

$$\begin{pmatrix} 1 & -\sin(2g_1) & (1-\cos(2g_1))/2 \\ 0 & \cos(2g_1) & -\sin(2g_1)/2 \\ 0 & 2\exp(-2g_2)\sin(2g_1) & \exp(-2g_2)\cos(2g_1) \end{pmatrix}.$$

In this matrix g_s is corresponded to A_s, $s = 1, 2, 3$. We note that

$$\exp(ig_1), \exp(ig_1), \exp g_2, \exp(-g_2)$$

form the Noetherian chain which generates the ring $\mathcal{R}$ in notations of Section 7. By direct calculations, ω is a Noetherian function of degree $d + 7$. Hence, we have the following bound: $F(4, 3, 1, d + 4, d + 7)$. Since this bound is better than general bounds given above we can obtain better results for some special classes of systems (1), *e.g.*, for block-triangular matrices $A(t)$ with only one- or two-dimensional blocks on the main diagonal.

Results on observability of three-dimensional systems (1) are presented briefly in Starkov and Garnica [10]. Details of cumbersome computations for this class of systems can be found in the master thesis of Garnica (CITEDI-IPN, Tijuana, Mexico, 2000).

10 CONCLUSIONS AND FUTURE DIRECTIONS

The main contribution of this paper consists in necessary and sufficient u^*-observability conditions of linear time-varying systems with a polynomial vector function u^*. Conditions obtained are connected with observability property of bilinear systems and also they are formulated in terms of the estimate of multiplicity of a zero of one analytic function which is arisen from the application of the Wei–Norman formula concerning the representation of a solution of a linear time-varying system as a product of matrix exponentials. This multiplicity is computed with help of the Gabrielov–Khovansky theorem. Then the main result is improved in two special cases and, also, we concern the multioutput system case. By use of direct computations, we give the better multiplicity bound for two-dimensional systems. As it can be seen from the matter of Sections 7;9, sharpness of bounds computed is a difficult problem opened for future investigations. In essence, this important topic is depended on sharpness of the Cabrielov–Khovansky bound and, to the best of the author's knowledge, this problem has not been examined yet. A possible continuation of this work consists in considering of the system (1) with the analytic time-varying observation law $y = C(t)x$ and, also, in efforts to improve general multiplicity bounds for more specific classes of (1) of small dimensions. It would be also interesting to apply one of formal algebra packages for a continuation of this work towards applications. Finally, we remark that due to duality between controllability and observability the similar multiplicity bound is elaborated in case of studies of controllability property of (1).

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