Mathematical Problems in Engineering, 2002, Vol. 8(4–5), pp. 389–411

Robust Stochastic Maximum Principle: Complete Proof and Discussions

ALEX S. POZNYAK*

CINVESTAV-IPN, Departimento de Control Automatico, AP 14-740, CP 07300, Mexico D.F., Mexico

(Received 7 February 2002)

This paper develops a version of Robust Stochastic Maximum Principle (RSMP) applied to the Minimax Mayer Problem formulated for stochastic differential equations with the control-dependent diffusion term. The parametric families of first and second order adjoint stochastic processes are introduced to construct the corresponding Hamiltonian formalism. The Hamiltonian function used for the construction of the robust optimal control is shown to be equal to the Lebesque integral over a parametric set of the standard stochastic Hamiltonians corresponding to a fixed value of the uncertain parameter. The paper deals with a cost function given at finite horizon and containing the mathematical expectation of a terminal term. A terminal condition, covered by a vector function, is also considered. The optimal control strategies, adapted for available information, for the wide class of uncertain systems given by an stochastic differential equation with unknown parameters from a given compact set, are constructed. This problem belongs to the class of minimax stochastic optimization problems. The proof is based on the recent results obtained for Minimax Mayer Problem with a finite uncertainty set [14, 43–45] as well as on the variation results of [53] derived for Stochastic Maximum Principle for nonlinear stochastic systems under complete information. The corresponding discussion of the obtain results concludes this study.

Key words: Robust control; Maximum principle; Minimax problem; Stochastic processes

AMS (MOS) Subject Classification: 05C38, 15A15, 05A15, 15A18

1 INTRODUCTION

During the last decades, the minimax control problem, dealing with different classes of nonlinear systems, has received much attention from many researchers because of its theoretical and practical importance. Basically, the results of this area are based on two classical approaches: *Maximum Principle* (MP) [41] and *Dynamic Programming method* (DP) [3]. In the case of a complete model description, both of them can be directly applied to construct the optimal control.

Various forms of the *Stochastic Maximum Principle* have been published in the literature [8, 9, 28, 30, 37]. All of these publications have usually dealt with the systems whose diffusion coefficients did not contain control variables and the control region was assumed to be convex. In [4] the case of the diffusion coefficients that depend smoothly on a control variable, was considered. Later this approach was extended to the class of partially observable systems [5, 31], where the optimal control consists of two basic components: state estimation and control via the obtained estimates. In the nonlinear case, the so called, “the innovation based technique” [32] and the Duncan–Mortensen–Zakai approach [25, 57] where the stochastic

* E-mail: apoznyak@ctrl.cinvestav.mx

ISSN 1024-123X print; ISSN 1563-5147 online

partial differential equation for the normalized conditional density function of the state to be estimated were used. The most advanced results concerning the maximum principle for nonlinear stochastic differential equations with controlled diffusion term were obtained by the Fudan University group, led by X. Li (see Ref. [53, 56] and bibliography within).

Since MP, mostly considered in this paper, and DP are believed to be two "equivalent" approaches to study the optimal control problems, several publications dealing with DP should be mentioned. There has been significant development due to the notion of "viscosity solution" introduced by Lions in [20] (see also Ref. [40]). Beside this, some different approaches to DP are known. Among those we can cite the elegant work of Krylov [35] (stochastic case) and of Clarke–Vinter [18, 19, 51] (deterministic case) within the framework of "generalized gradient".

Faced with some *uncertainties* (parametric type, unmodelled dynamics, external perturbations etc.) these results cannot be applied. There are two ways to overcome the uncertainty problems:

- the first is to apply the adaptive approach [26] to identify the uncertainty on-line and then use these estimates to construct a control [27];
- the second one, which will be considered in this paper, is to obtain a solution suitable for a class of given models by formulating a corresponding *minimax control problem*, where the maximization is taken over a set of possible uncertainties and the minimization is taken over all of the control strategies within a given set.

Several approaches for deterministic systems are effective in this situation. One of the important components of Minimax Control Theory is the *game-theoretic approach* [1]. In terms of game theory, the control and the model uncertainty are strategies employed by opposing players in a game: control is chosen to minimize a cost function and the uncertainty is chosen to maximize it. In such an interpretation, the uncertainty should be time varying to present the worst situation for the controller. To the best of our knowledge, the earliest papers in this direction are [23] and [34]. Later, in [36], the *Lagrange Multipliers Approach* was applied to the problems of control and observations with incomplete information. They were formulated as the corresponding minimax problems. This technique, as it is mentioned above, effectively works only for the systems where the uncertainties can be varied in time, and, consequently, can "play" against an applied control strategy. Starting from the pioneering work of [55], which dealt with *frequency domain methods* to minimize the norm of the transfer function between the disturbance inputs and the performance output, the minimax controller design is formulated as an H^{∞}*-optimization problem*. As it was shown in [1] this specific problem can be successfully solved in the time domain, leading to a reapproachment with dynamic game theory and to the establishment of a relation with *risk-sensitive quadratic* (*stochastic*) *control* [24, 29, 33, 39]. The paper [38] presents a control design method for continuous-time plants whose uncertain parameters in the output matrix are known to lie within an ellipsoidal set only. An algorithm for minimax control which at every iteration minimizes approximately the defined Hamiltonian is described in [46]. In [22] "*the cost-to-come*" *method* is used. The authors show the equivalence between the original problem with the incomplete information and the problem with the complete information but of a higher dimension. Recently, robust MP was derived in [14] for a deterministic Mayer problem for systems that contain an unknown parameter from a given finite set and it was generalized in [15] for Bolza and Lagrange problems. A comprehensive survey of different parameter space methods for robust control design oriented to deterministic systems can be found in [47].

For *stochastic uncertain systems*, a minimax control of a class of dynamic systems with mixed uncertainties was investigated in [2]. A continuous deterministic uncertainty which affects the system dynamics, and a discrete stochastic uncertainty leading to jumps in the system structure at random times were studied. The solution presents a finite dimensional compensator using two finite sets of partial differential equations. Robust (non optimal) controller for linear time-varying systems given by stochastic differential equation was studied in [42, 48] where the solution is based on the stochastic *Lyapunov analysis* with the martingale technique implementation. Other problems dealing with discrete time models of deterministic and/or simplest stochastic nature and their corresponding solutions are discussed in [6, 7, 17, 21, 52]. In [50] finite horizon Minimax Optimal Control problem of nonlinear continuous time systems with stochastic uncertainty is considered. The original problem was converted into an unconstrained stochastic game problem and a stochastic version of the S-procedure has been designed to obtained a solution.

The purpose of this paper is to explore the possibilities of the MP approach for a class of minimax control problems for uncertain systems given by a system of stochastic differential equations with a controlled diffusion term and unknown parameters within a given measured compact set. First, for simplicity, the minimax problem belongs to the class of optimization problems on a fixed finite horizon where the cost function contains only a terminal term (without an integral part). The proof is based on the recent results obtained for Minimax Mayer Problem with a finite uncertainty set [14, 43–45] as well as on the results of [53] derived for Stochastic Maximum Principle for nonlinear stochastic systems under complete information. The *Tent Method* [10–13] is used to formulate the necessary conditions of optimality in Hamiltonian form.

2 PROBLEM SETTING

2.1 Stochastic Uncertain System

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbf{P})$ be a given filtered probability space, that is,

- the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is complete,
- the sigma-algebra $\mathcal{F}_0$ contains all the $\mathbf{P}$-null sets in $\mathcal{F}$,
- the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is right continuous: $\mathcal{F}_{t+} := \cap_{s>t}\mathcal{F}_s = \mathcal{F}_t$.1

On this probability space an m-dimensional standard Brownian motion is defined, *i.e.*, $(W(t), t \geq 0)$ (with $W(0) = 0$) is an $\{\mathcal{F}_t\}_{t\geq 0}$-adapted $\mathbb{R}^m$-valued process such that

$$\begin{gathered} E\{W(t) - W(s) \mid \mathcal{F}_s\} = 0 \ (\mathbf{P} - a.s.) \\ E\{[W(t) - W(s)][W(t) - W(s)]^{\mathrm{T}} \mid \mathcal{F}_s\} = (t-s)I \ (\mathbf{P} - a.s.) \\ \mathbf{P}\{\omega \in \Omega\colon W(0) = 0\} = 1 \end{gathered}$$

Consider the stochastic nonlinear controlled continuous-time system with the dynamics $x(t)$ given by

$$x(t) = x(0) + \int_{s=0}^{t} b^{\alpha}(s, x(s), u(s))\, \mathrm{d}t + \int_{s=0}^{t} \sigma^{\alpha}(s, x(s), u(s))\, \mathrm{d}W(s) \tag{1}$$

or, in the abstract (symbolic) form,

$$\begin{cases} \mathrm{d}x(t) = b^{\alpha}(t, x(t), u(t))\, \mathrm{d}t + \sigma^{\alpha}(t, x(t), u(t))\, \mathrm{d}W(t) \\ x(0) = x_0, \quad t \in [0, T] (T > 0) \end{cases} \tag{2}$$

The first integral in (1) is an stochastic ordinary integral and the second one is an Itô integral. In the above $u(t) \in U$ is a control at time t and

$$b^{\alpha}: [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$$
$$\sigma^{\alpha}: [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times m}$$

The parameter α is supposed to be *a priori* unknown and running a given parametric set $\mathcal{A}$ from a space with a countable additive measure m.

For any $\alpha \in \mathcal{A}$ denote

$$b^{\alpha}(t,x,u) := (b_1^{\alpha}(t,x,u), \dots, b_n^{\alpha}(t,x,u))^{\top}$$
$$\sigma^{\alpha}(t,x,u) := (\sigma^{1\alpha}(t,x,u), \dots, \sigma^{n\alpha}(t,x,u))$$
$$\sigma^{j\alpha}(t,x,u) := (\sigma_1^{j\alpha}(t,x,u), \dots, \sigma_m^{j\alpha}(t,x,u))^{\top}$$

It is assumed that

A1: $\{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration generated by $(W(t), t \geq 0)$ and augmented by the **P**-null sets from $\mathcal{F}$.

A2: (U, d) is a separable metric space with a metric d.

The following definition is used subsequently.

DEFINITION 1 *The function* $f: [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^{n\times m}$ *is said to be an* $L_{\phi}(C^2)$*-mapping if*

1. *It is Borel measurable;*
2. *It is* C^2 *in* x *for any* $t \in [0, T]$ *and any* $u \in U$
3. *There exist a constant* L *and a modulus of continuity* $\phi: [0, \infty) \to [0, \infty)$ *such that for any* $t \in [0, T]$ *and for any* $x, u, \hat{x}, u \in \mathbb{R}^n \times U \times \mathbb{R}^n \times U$

$$\|f(t,x,u) - f(t,\hat{x},\hat{u})\| \leq L\|x-\hat{x}\| + \phi(d(u,\hat{u}))$$
$$\|f(t,0,u)\| \leq L$$
$$\|f_x(t,x,u) - f_x(t,\hat{x},\hat{u})\| \leq L\|x-\hat{x}\| + \phi(d(u,\hat{u}))$$
$$\|f_{xx}(t,x,u) - f_{xx}(t,\hat{x},\hat{u})\| \leq \phi(\|x-\hat{x}\| + d(u,\hat{u}))$$

(here $f_x(\cdot, x, \cdot)$ *and* $f_{xx}(\cdot, x, \cdot)$ *are the partial derivatives of the first and the second order).*

In view of this definition, it is also assumed that

A3: for any $\alpha \in \mathcal{A}$ both $b^{\alpha}(t,x,u)$ and $\sigma^{\alpha}(t,x,u)$ are $L_{\phi}(C^2)$-mappings.

Let $A_0 \subset \mathcal{A}$ be measurable subsets with a finite measure, that is, $m(\mathcal{A}_0) < \infty$.

The following assumption concerning the right-hand side of (2) will be in force throughout:

A4: All components $b^{\alpha}(t,x,u)$, $\sigma^{\alpha}(t,x,u)$ are measurable with respect to α, that is, for any $i = 1, \dots, n, j = 1, \dots, m$, $c \in R^1$, $x \in R^n$, $u \in U$ and $t \in [0, T]$

$$\{\alpha: b_i^{\alpha}(t,x,u) \leq c\} \in \mathcal{A}$$
$$\{\alpha: \sigma_j^{i\alpha}(t,x,u) \leq c\} \in \mathcal{A}$$

Moreover, every considered function of α is assume to be measurable with respect to α.

The only sources of uncertainty in this system description are

- the system random noise $W(t)$,
- the *priori* unknown parameter $\alpha \in \mathcal{A}$.

It is assumed that the *past information is available* for the controller.

To emphasize the dependence of the random trajectories on the parameter $\alpha \in \mathcal{A}$ the Eq. (2) is rewritten as

$$\begin{cases} \mathrm{d}x^{\alpha}(t) = b^{\alpha}(t, x^{\alpha}(t), u(t))\,\mathrm{d}t + \sigma^{\alpha}(t, x^{\alpha}(t), u(t))\,\mathrm{d}W(t) \\ x^{\alpha}(0) = x_0, \quad t \in [0, T](T > 0) \end{cases} \tag{3}$$

2.2 A Terminal Condition, A Feasible and Admissible Control

The following definitions will be used throughout this paper.

DEFINITION 2 *A stochastic control $u(\cdot)$ is called a feasible in the stochastic sense (or, s-feasible) for the system* (3) *if*

1. $u(\cdot) \in \mathcal{U}[0, T] := \{u: [0, T] \times \Omega \to U \mid u(\cdot)$ *is* $\{\mathcal{F}_t\}_{t \geq 0}$*-adapted*$\}$
2. $x^{\alpha}(t)$ *is the unique solution of* (3) *in the sense that for any* $x^{\alpha}(t)$ *and* $\hat{x}^{\alpha}(t)$, *satisfying* (3),

$$\mathbf{P}\{\omega \in \Omega: x^{\alpha}(t) = \hat{x}^{\alpha}(t)\} = 1$$

The set of all s-feasible controls is denoted by $\mathcal{U}^{s}_{\text{feas}}[0, T]$. *The pair* $(x^{\alpha}(t); u(\cdot))$, *where* $x^{\alpha}(t)$ *is the solution of* (3) *corresponding to this* $u(\cdot)$, *is called an s-feasible pair.*

The assumptions A1–A4 guarantee that any $u(\cdot)$ from $\mathcal{U}[0, T]$ is s-feasible.
In addition, it is required that the following terminal state constraints are satisfied:

$$E\{h^{j}(x^{\alpha}(T))\} \geq 0 \quad (j = 1, \ldots, l) \tag{4}$$

where $h^{j}: \mathbb{R}^{n} \to \mathbb{R}$ are given functions.
A5: For $j = 1, \ldots, l$ the functions h^{j} are $L_{\phi}(C^{2})$-mappings.

DEFINITION 3 *The control $u(\cdot)$ and the pair $(x^{\alpha}(t); u(\cdot))$ are called an s-admissible control or realizing the terminal condition* (4) *and an s-admissible pair, respectively, if*

1. $u(\cdot) \in \mathcal{U}^{s}_{\text{feas}}[0, T]$
2. $x^{\alpha}(t)$ *is the solution of* (3), *corresponding to this* $u(\cdot)$, *such that the inequalities* (4) *are satisfied.*

The set of all s-admissible controls is denoted by $\mathcal{U}^{s}_{\text{adm}}[0, T]$.

2.3 Highest Cost Function and Robust Optimal Control

DEFINITION 4 *For any scalar-valued function $\varphi(\alpha)$ bounded on $\mathcal{A}$ define the m-truth (or m-essential) maximum of $\varphi(\alpha)$ on $\mathcal{A}$ as follows:*

$$m\text{-}\operatorname{vrai\,max}_{\alpha \in \mathcal{A}} \varphi(\alpha) := \max \varphi^{+}$$

such that

$$m\{\alpha \in \mathcal{A}: \varphi(\alpha) > \varphi^{+}\} = 0$$

It can be easily shown (see, for example, [54]) that the following *integral presentation* for the truth maximum holds:

$$m\text{-}\underset{\alpha\in\mathcal{A}}{vrai\,max}\,\varphi(\alpha) := \sup_{\mathcal{A}_0\subset\mathcal{A}:\ m(\mathcal{A}_0)>0} \frac{1}{m(\mathcal{A}_0)}\int_{\mathcal{A}_0}\varphi(\alpha)\,\mathrm{d}m \tag{5}$$

where the Lebesgue–Stilties integral is taken over all subsets $\mathcal{A}_0 \subset \mathcal{A}$ with positive measure $m(\mathcal{A}_0)$.

Consider the cost function h^α containing a terminal term, that is,

$$h^\alpha := E\{h^0(x^\alpha(T))\} \tag{6}$$

Here $h_0(x)$ is a positive, bounded and smooth *cost function* defined on $\mathbb{R}^n$. The end time-point T is assumed to be finite and $x^\alpha(t) \in \mathbb{R}^n$.

If an admissible control is applied, for every $\alpha \in \mathcal{A}$ we deal with the cost value $h^\alpha = E\{h_0(x^\alpha(T))\}$ calculated at the terminal point $x^\alpha(T) \in \mathbb{R}^n$. Since the realized value of α is *a priori* unknown, define the *worst* (*highest*) *cost*

$$F = \sup_{\mathcal{A}_0\subset\mathcal{A}:\ m(\mathcal{A}_0)>0} \frac{1}{m(\mathcal{A}_0)}\int_{\mathcal{A}_0} E\{h^0(x^\alpha(T))\}\,\mathrm{d}m = m\text{-}\underset{\alpha\in\mathcal{A}}{vrai\,max}\,h^\alpha \tag{7}$$

The function F depends only on the considered admissible control $u(t)$, $t_0 \le t \le t_1$.

DEFINITION 5 *The control $\bar{u}(t)$, $0 \le t \le T$ is said to be robust optimal if*

(i) *it satisfies the terminal condition, that is, it is admissible*;
(ii) *it achieves the minimal worst* (*highest*) *cost F^0* (*among all admissible controls satisfying the terminal condition*).

If the dynamics $\bar{x}^\alpha(t)$ corresponds to this robust optimal control $\bar{u}(t)$ then $(\bar{x}^\alpha(\cdot), \bar{u}(\cdot))$ is called an α-robust optimal pair which does not exist.

Thus the *Robust Optimization Problem* consists of finding an admissible control action $u(t)$, $0 \le t \le T$ which provides

$$\begin{aligned} F^0 := F &= \min_{u(t)} m\text{-}\underset{\alpha\in\mathcal{A}}{vrai\,max}\,h^\alpha \\ &= \min_{u(t)}\max_{\lambda\in\Lambda}\int_{\lambda\in\Lambda}\lambda(\alpha)E\{h^0(x^\alpha(T))\}\,\mathrm{d}m(\alpha) \end{aligned} \tag{8}$$

where the minimum is taken over all admissible control strategies and the maximum over all functions $\lambda(\alpha)$ within, so-called, the set of "*distribution densities*" Λ defined by

$$\Lambda := \left\{\lambda = \lambda(\alpha) = \mu(\alpha)\left(\int_{\alpha\in\mathcal{A}}\mu(\alpha)\,\mathrm{d}m(\alpha)\right)^{-1} \ge 0,\ \int_{\alpha\in\mathcal{A}}\lambda(\alpha)\,\mathrm{d}m(\alpha) = 1\right\} \tag{9}$$

This is the *Stochastic Minimax Bolza Problem.*

3 ROBUST STOCHASTIC MAXIMUM PRINCIPLE

3.1 First and Second Order Adjoint Processes

The adjoint equations and the associated Hamiltonian function are introduced in this section to present the *necessary conditions* of the robust optimality for the considered class of partially unknown stochastic systems which is called the *Robust Stochastic Maximum Principle* (RSMP). If in the deterministic case [14] the adjoint equations are backward ordinary differential equations and represent, in some sense, the same forward equation but in reverse time, in the stochastic case such interpretation is not applicable because any time reversal may destroy the non-anticipativeness of the stochastic solutions, that is, any obtained robust control should not depend on the future. To avoid these problems the approach given in [56] is used that takes into account the adjoint equations introduced for any fixed value of the parameter α and, hence, some of the results from [56] may be applied directly without any changes. So, following [56], for any $\alpha \in \mathcal{A}$ and any admissible control $u(\cdot) \in \mathcal{U}^s_{\mathrm{adm}}[0,T]$ consider

– *the 1st order vector adjoint equations*:

$$\begin{cases} d\psi^\alpha(t) = -\left[b^\alpha_x(t, x^\alpha(t), u(t))^{\mathrm{T}} \psi^\alpha(t) + \sum_{j=1}^m \sigma^{\alpha j}_x(t, x^\alpha(t), u(t))^{\mathrm{T}} q^\alpha_j(t) \right] dt \\ \qquad + q^\alpha(t)\, dW(t) \\ \psi^\alpha(T) = c^\alpha, \quad t \in [0,T] \end{cases} \tag{10}$$

– *the 2nd order matrix adjoint equations*:

$$\begin{cases} d\Psi^\alpha(t) = -\Big[b^\alpha_x(t, x^\alpha(t), u(t))^{\mathrm{T}} \Psi^\alpha(t) + \Psi^\alpha(t) b^\alpha_x(t, x^\alpha(t), u(t)) \\ \qquad + \sum_{j=1}^m \sigma^{\alpha j}_x(t, x^\alpha(t), u(t))^{\mathrm{T}} \Psi^\alpha(t) \sigma^{\alpha j}_x(t, x^\alpha(t), u(t)) \\ \qquad + \sum_{j=1}^m (\sigma^{\alpha j}_x(t, x^\alpha(t), u(t))^{\mathrm{T}} Q^\alpha_j(t) + Q^\alpha_j(t) \sigma^{\alpha j}_x(t, x^\alpha(t), u(t))) \\ \qquad + H^\alpha_{xx}(t, x^\alpha(t), u(t), \psi^\alpha(t), q^\alpha(t)) \Big] dt \\ \qquad + \sum_{j=1}^m Q^\alpha_j(t)\, dW^j(t) \\ \Psi^\alpha(T) = C^\alpha, \quad t \in [0,T] \end{cases} \tag{11}$$

Here $c^\alpha \in L^2_{\mathcal{F}_T}(\Omega, \mathbb{R}^n)$ is a square integrable $\mathcal{F}_T$-measurable $\mathbb{R}^n$-valued random vector, $\psi^\alpha(t) \in L^2_{\mathcal{F}_t}(\Omega, \mathbb{R}^n)$ is a square integrable $\{\mathcal{F}_t\}_{t\geq 0}$-adapted $\mathcal{R}^n$-valued vector random process and $q^\alpha(t) \in L^2_{\mathcal{F}_t}(\Omega, \mathbb{R}^{n\times m})$ is a square integrable $\{\mathcal{F}_t\}_{t\geq 0}$-adapted $\mathbb{R}^{n\times m}$-valued matrix random process. Similarly, $C^\alpha \in L^2_{\mathcal{F}_T}(\Omega, \mathbb{R}^{n\times n})$ is a square integrable $\mathcal{F}_T$-measurable $\mathbb{R}^{n\times n}$-valued random matrix, $\Psi^\alpha(t) \in L^2_{\mathcal{F}_t}(\Omega, \mathbb{R}^{n\times n})$ is a square integrable $\{\mathcal{F}_t\}_{t\geq 0}$-adapted $\mathcal{R}^{n\times n}$-valued matrix random process $Q^\alpha_j(t) \in L^2_{\mathcal{F}_t}(\Omega, \mathbb{R}^{n\times m})$ is a square integrable $\{\mathcal{F}_t\}_{t\geq 0}$-adapted $\mathbb{R}^{n\times n}$-valued matrix random process. $b^\alpha_x(t, x^\alpha, u)$ and $H^\alpha_{xx}(t, x^\alpha, u, \psi^\alpha, q^\alpha)$ is the first and, correspondingly, the second derivatives of these functions by x^α. The function $H^\alpha(t, x, u, \psi, q)$ is defined as

$$H^\alpha(t, x, u, \psi, q) := b^\alpha(t, x, u)^{\mathrm{T}} \psi + \mathrm{tr}[q^{\mathrm{T}} \sigma^\alpha] \tag{12}$$

As it is seen from (11), if $C^\alpha = C^{\alpha \mathrm{T}}$ then for any $t \in [0,T]$ the random matrix $\Psi^\alpha(t)$ is symmetric (but not necessarily positive or negative definite). In (10) and (11), which are the backward stochastic differential equations with the $\{\mathcal{F}_t\}_{t\geq 0}$-adapted solutions, the unknown variables to be selected are the pair of terminal conditions c^α, C^α and the collection $(q^\alpha, Q_j^\alpha\ (j=1,\dots,l))$ of $\{\mathcal{F}_t\}_{t\geq 0}$-adapted stochastic matrices. Note that the Eqs. (3) and (10) can be rewritten in Hamiltonian form as

$$\begin{cases} dx^\alpha(t) = H_\psi^\alpha(t, x^\alpha(t), u(t))^\mathrm{T}\psi^\alpha(t), q^\alpha(t))\,dt + \sigma^\alpha(t, x^\alpha(t), u(t))\,dW(t) \\ x^\alpha(0) = x_0, \quad t \in [0, T] \end{cases} \tag{13}$$

$$\begin{cases} d\psi^\alpha(t) = -H_x^\alpha(t, x^\alpha(t), u(t))^\mathrm{T}\psi^\alpha(t), q^\alpha(t))\,dt + q^\alpha(t)\,dW(t) \\ \psi^\alpha(T) = c^\alpha, \quad t \in [0, T] \end{cases} \tag{14}$$

3.2 Main Result

Now the main result of this paper can be formulated.

THEOREM 6 *(Robust Stochastic Maximum Principle) Let A1–A5 be fulfilled and $(\bar{x}^\alpha(\cdot), \bar{u}(\cdot))$ be the α-robust optimal pairs ($\alpha \in \mathcal{A}$). The parametric uncertainty set $\mathcal{A}$ is a space with countable additive measure $m(\alpha)$ which assumed to be given. Then for every $\varepsilon > 0$ there exist collections of terminal conditions $c^{\alpha,(\varepsilon)}$, $C^{\alpha,(\varepsilon)}$, $\{\mathcal{F}_t\}_{t\geq 0}$-adapted stochastic matrices*

$$(q^{\alpha,(\varepsilon)}, Q_j^{\alpha,(\varepsilon)}(j=1,\dots,l))$$

in (10) and (11), and nonnegative constants $\mu_\alpha^{(\varepsilon)}$ and $\nu_{\alpha j}^{(\varepsilon)}(j=1,\dots,l)$ such that the following conditions are fulfilled:

1 *(Complementary slackness condition): For any $\alpha \in \mathcal{A}$*

(i) *the inequality $|E\{h^0(\bar{x}^\alpha(T))\} - \max_{\alpha\in\mathcal{A}} E\{h^0(\bar{x}^\alpha(T))\}| < \varepsilon$ holds or $\mu_\alpha^{(\varepsilon)} = 0$;*
(ii) *moreover, either the inequality $|E\{h^j(\bar{x}^\alpha(T))\}| < \varepsilon$ holds or $\nu_{\alpha j}^{(\varepsilon)} = 0 (j=1,\dots,l)$;*

2 *(Transversality condition): For any $\alpha \in \mathcal{A}$ the inequality*

$$\left\| c^{\alpha,(\varepsilon)} + \mu_\alpha^{(\varepsilon)} h_x^0(\bar{x}^\alpha(T)) + \sum_{j=1}^{l} \nu_{\alpha j}^{(\varepsilon)} h_x^j(\bar{x}^\alpha(T)) \right\| < \varepsilon \quad (\mathbf{P} - a.s.) \tag{16}$$

$$\left\| C^{\alpha,(\varepsilon)} + \mu_\alpha^{(\varepsilon)} h_{xx}^0(\bar{x}^\alpha(T)) + \sum_{j=1}^{l} \nu_{\alpha j}^{(\varepsilon)} h_{xx}^j(\bar{x}^\alpha(T)) \right\| < \varepsilon \quad (\mathbf{P} - a.s.) \tag{17}$$

hold;

3 *(Nontriviality condition): There exists a set $\mathcal{A}_0 \subset \mathcal{A}$ with positive measure $m(\mathcal{A}_0) > 0$ such that for every $\alpha \in \mathcal{A}_0$ either $c^{\alpha,(\varepsilon)} \neq 0$ or, at least, one of the numbers $\mu_\alpha^{(\varepsilon)}, \nu_{\alpha j}^{(\varepsilon)}$ $(j=1,\dots,l)$ is distinct from 0, that is, with probability one*

$$\forall \alpha \in \mathcal{A}_0 \in \mathcal{A}: \quad |c^{\alpha,(\varepsilon)}| + \mu_\alpha^{(\varepsilon)} + \sum_{j=1}^{l} \nu_{\alpha j}^{(\varepsilon)} > 0 \tag{18}$$

4 *(Maximality condition): the robust optimal control $\bar{u}(\cdot)$ for almost all $t \in [0, T]$ maximizes the generalized Hamiltonian function*

$$\begin{aligned} &\mathcal{H}(t, \bar{x}^\diamond(t), u, \psi^{\diamond,(\varepsilon)}(t), \Psi^{\diamond,(\varepsilon)}(t), q^{\diamond,(\varepsilon)}(t)) \\ &\quad := \int_{\mathcal{A}} \mathcal{H}(t, \bar{x}^\alpha(t), u, \psi^{\alpha,(\varepsilon)}(t), \Psi^{\alpha,(\varepsilon)}(t), q^{\alpha,(\varepsilon)}(t))\,dm(\alpha) \end{aligned} \tag{19}$$

where

$$\begin{aligned}
&\mathcal{H}^\alpha(t, \bar{x}^\alpha(t), u, \psi^{\alpha,(\varepsilon)}(t), \Psi^{\alpha,(\varepsilon)}(t), q^{\alpha,(\varepsilon)}(t)) \\
&\quad := H^\alpha(t, \bar{x}^\alpha(t), u, \psi^{\alpha,(\varepsilon)}(t), q^{\alpha,(\varepsilon)}(t)) - \frac{1}{2}\operatorname{tr}[\bar{\sigma}^{\alpha\mathsf{T}}\Psi^{\alpha,(\varepsilon)}(t)\bar{\sigma}^\alpha] \\
&\quad\quad + \frac{1}{2}\operatorname{tr}[(\sigma^\alpha(t, x^\alpha(t), u) - \bar{\sigma}^\alpha)^\mathsf{T}\Psi^{\alpha,(\varepsilon)}(t)(\sigma^\alpha(t, \bar{x}^\alpha(t), u) - \bar{\sigma}^\alpha)]
\end{aligned} \tag{20}$$

the function $H^\alpha(t, \bar{x}^\alpha(t), u, \psi^{\alpha,(\varepsilon)}(t), q^{\alpha,(\varepsilon)}(t))$ *is given by* (12),

$$\bar{\sigma}^\alpha := \sigma^\alpha(t, \bar{x}^\alpha(t), \bar{u}(t)) \tag{21}$$

$$\begin{aligned}
\bar{x}^\diamond(t) &:= (\bar{x}^{1\mathsf{T}}(t), \ldots, \bar{x}^{N\mathsf{T}}(t))^\mathsf{T}, \quad \psi^{\diamond,(\varepsilon)}(t) := (\psi^{1,(\varepsilon)\mathsf{T}}(t), \ldots, \psi^{N,(\varepsilon)\mathsf{T}}(t))^\mathsf{T} \\
q^{\diamond,(\varepsilon)}(t) &:= (q^{1,(\varepsilon)}(t), \ldots, q^{N,(\varepsilon)}(t)), \quad \Psi^{\diamond,(\varepsilon)}(t) := (\Psi^{1,(\varepsilon)}(t), \ldots, \Psi^{N,(\varepsilon)}(t))
\end{aligned}$$

and $\psi^{i,(\varepsilon)\mathsf{T}}(t), \Psi^{i,(\varepsilon)}(t)$ *verify* (10) *and* (11) *with the terminal conditions* $c^{\alpha,(\varepsilon)}$ *and* $C^{\alpha,(\varepsilon)}$, *respectively, i.e., for almost all* $t \in [0, T]$

$$\bar{u}(t) = \arg\max_{u \in U} \mathcal{H}(t, \bar{x}^\diamond(t), u, \psi^{\diamond,(\varepsilon)}(t), \Psi^{\diamond,(\varepsilon)}(t), q^{\diamond,(\varepsilon)}(t)) \tag{22}$$

4 PROOF OF THEOREM 1 (RSMP)

4.1 Formalism

Consider the random vector space $\mathbb{R}^\diamond$ with the coordinates $x^{\alpha,i} \in L^2_{\mathcal{F}_T}(\Omega, \mathbb{R})(\alpha \in \mathcal{A}$, $i = 1, \ldots, n)$. For each fixed $\alpha \in \mathcal{A}$ we may consider

$$x^\alpha := (x^{\alpha,1}, \ldots, x^{\alpha,ni})^\mathsf{T}$$

as an element of a Hilbert (and, hence, self-conjugate) space $\mathbb{R}^\alpha$ with the usual scalar product given by

$$\langle x^\alpha, \tilde{x}^\alpha \rangle := \sqrt{\sum_{i=1}^n E\{x^{\alpha,i}\tilde{x}^{\alpha,i}\}}, \quad \|\tilde{x}^\alpha\| := \sqrt{\langle x^\alpha, x^\alpha \rangle}$$

However, in the whole space $\mathbb{R}^\diamond$ introduce the norm of the element $x^\diamond = (x^{\alpha,i})$ in another way:

$$\|x^\diamond\| := m\text{-}vraimax_{\alpha \in \mathcal{A}} \sqrt{\sum_{i=1}^n E\{(x^{\alpha,i})^2\}} = \sup_{\mathcal{A}_0 \subset \mathcal{A}:\, m(\mathcal{A}_0) > 0} \frac{1}{m(\mathcal{A}_0)} \int_P \sqrt{\sum_{i=1}^n E\{(x^{\alpha,i})^2\}}\, \mathrm{d}m \tag{23}$$

Consider the set $\mathbb{R}^\diamond$ of all functions from $L^2_{\mathcal{F}_T}(\Omega, \mathbb{R})$ for any fixed $\alpha \in \mathcal{A}$, measurable on $\mathcal{A}$ and with values in $\mathbb{R}^n$, identifying every two functions which coincide almost everywhere. With the norm (23), $\mathbb{R}^\diamond$ is a Banach space. Now we describe its conjugate space $\mathbb{R}_\diamond$. Consider the set of all measurable functions $a(\alpha) \in L^2_{\mathcal{F}_T}(\Omega, \mathbb{R})$ defined on $\mathcal{A}$ with values in $\mathbb{R}^n$. It consists of all covariant random vectors $a_\diamond = (a_{\alpha,i})(\alpha \in \mathcal{A}, i = 1, \ldots, n)$ with the norm

$$\|a_\diamond\| := m\text{-}vraimax_{\alpha \in \mathcal{A}} \sqrt{\sum_{i=1}^n E\{(a_{\alpha,i})^2\}} \tag{24}$$

The set of all such functions $a(\alpha)$ is a linear normal space. In general, this normed space is not complete. The following example illustrates this fact.

Example 7 Consider the case when A is the segment $[0, 1] \subset \mathbb{R}$ with usual Lebesgue measure. Let $\varphi_k(\alpha)$ is the function on $[0, 1]$ that it is equal to 0 for $\alpha > (1/k)$ and is equal to k for $0 \leq \alpha \leq (1/k)$. Then $\int_A \varphi_k(\alpha)\,\mathrm{d}\alpha = 1$, and the sequence $\varphi_k(\alpha)k = 1, 2, \ldots$ is a fundamental one in the norm (24). But their limit function $\lim_{k\to\infty} \varphi_k(\alpha)$ does not exists among measurable and summable functions. Such a limit is the Dirak function $\varphi^{(0)}(\alpha)$ that is equal to 0 for every $\alpha > 0$ and is equal to infinity at $\alpha = 0$ (with the normalization agreement that $\int_A \varphi^{(0)}(\alpha)\,\mathrm{d}\alpha = 1$).

This example shows that the linear normed space of all measurable, summable functions with the norm (24) is, in general, incomplete. The complementation of this space is a Banach space, and we denote it by $\mathbb{R}_\diamond$. This is the conjugate space for $\mathbb{R}^\diamond$. The scalar product of $x^\diamond \in \mathbb{R}$ and $a_\diamond \in \mathbb{R}_\diamond$ can be defined as

$$\langle a_\diamond, x^\diamond \rangle_E := \int_{\mathcal{A}} \sum_{i=1}^{n} E\{a_{\alpha,i} x^{\alpha,i}\}\,\mathrm{d}m$$

for which the Cauchy–Bounyakovski–Schwartz inequality evidently holds.

$$\langle a_\diamond, x^\diamond \rangle_E \leq \|a_\diamond\| \cdot \|x^\diamond\|$$

4.2 Proof of Properties 1–3

In this subsection consider the vector $x^\diamond(T)$ only.

The index $\alpha \in \mathcal{A}$ is said to be $\varepsilon \wedge h^0$-active if the given $\varepsilon > 0$

$$E\{h^0(\bar{x}^\alpha(T))\} > \max_{\alpha\in\mathcal{A}} E\{h^0(\bar{x}^\alpha(T))\} - \varepsilon \tag{25}$$

and, it is $\varepsilon \wedge h^j$-active if

$$E\{h^j(\bar{x}^\alpha(T))\} > -\varepsilon \tag{26}$$

First, assume that there exists a set of a positive measure $G \subset \mathcal{A}$ and a set $\bar{\Omega} \subseteq \Omega(\mathbf{P}\{\omega \in \bar{\Omega}\} > 0)$ such that for all $\varepsilon \wedge h^0$-active indices $\alpha \in \mathcal{A}$ we have that $\|h_x^0(\bar{x}^\alpha(T))\| < \varepsilon$ for all $\omega \in \bar{\Omega} \subseteq \Omega$ and almost everywhere on G. Then selecting (without violation of the transversality and nontriviality conditions)

$$\mu_\alpha^{(\varepsilon)} \neq 0, \quad \mu_{\tilde{\alpha}\neq\alpha}^{(\varepsilon)} = 0, \quad \nu_{\alpha j}^{(\varepsilon)} = 0 \quad (\forall \alpha \in \mathcal{A}, j = 1, \ldots, l)$$

it follows that $c^{\alpha,(\varepsilon)} = \psi^{\alpha,(\varepsilon)}(T) = 0$, $C^{\alpha,(\varepsilon)} = \Psi^{\alpha,(\varepsilon)}(T) = 0$ for almost all $\omega \in \bar{\Omega}$ and almost everywhere on G. In this situation, the only nonanticipative matrices $q^{\alpha,(\varepsilon)}(t) = 0$ and $Q_j^{\alpha,(\varepsilon)}(t) = 0$ are admissible, and for all $t \in [0, T]$, as a result, $H^\alpha(t, x, u, \psi, q) = 0$, $\psi^{\alpha,(\varepsilon)}(t) = 0$ and $\Psi^{\alpha,(\varepsilon)}(t) = 0$ for almost all $\omega \in \bar{\Omega}$ and almost everywhere on G. Thus, all conditions 1–4 of Theorem are satisfied automatically whether or not the control is robust optimal or not. So, it can be assumed that $\|h_x^0(\bar{x}^\alpha(T))\| \geq \varepsilon(\mathbf{P} - a.s.)$ for all $\varepsilon \wedge h^0$-active indices $\alpha \in \mathcal{A}$. Similarly, it can be assumed that $\|h_x^0(\bar{x}^\alpha(T))\| \geq \varepsilon(\mathbf{P} - a.s.)$ for all $\varepsilon \wedge h^j$-active indices $\alpha \in \mathcal{A}$.

Denote by $\Omega_1 \subseteq \mathbb{R}^\diamond$ the *controllability region*, that is, the set of all points $z^\diamond \in \mathbb{R}^\diamond$ such that there exists a feasible control $u(t) \in \mathcal{U}^s_{\text{feas}}[0,T]$ for which the trajectories $x^\diamond(t) = (x^{\alpha,i}(t))$, corresponding to (3), satisfy $x^\diamond(T) = z^\diamond$ with probability one:

$$\Omega_1 := \{z^\diamond \in \mathbb{R}^\diamond : x^\diamond(T) \stackrel{a.s.}{=} z^\diamond,\, u(t) \in \mathcal{U}^s_{\text{feas}}[0,T],\, x^\alpha(0) = x_0\} \tag{27}$$

Let $\Omega_{2,j} \subseteq \mathbb{R}^\diamond$ denote the set of all points $z^\diamond \in \mathbb{R}^\diamond$ satisfying the terminal condition (4) for some fixed index j and any $\alpha \in \mathcal{A}$, that is,

$$\Omega_{2j} := \{z^\diamond \in \mathbb{R}^\diamond : E\{h^j(z^\alpha)\} \geq 0 \forall \alpha \in \mathcal{A}\} \tag{28}$$

Finally, denote by $\Omega_0^{(\varepsilon)} \subseteq \mathbb{R}^\diamond$ the set, containing the optimal point $\bar{x}^\diamond(T)$ (corresponding to the given robust optimal control $\bar{u}(\cdot)$) as well as all points $z^\diamond \in \mathbb{R}^\diamond$ satisfying for all $\alpha \in \mathcal{A}$

$$E\{h^0(z^\alpha)\} \leq \max_{\alpha \in \mathcal{A}} E\{h^0(\bar{x}^\alpha(T))\} - \varepsilon$$

that is, $\forall \alpha \in \mathcal{A}$

$$\Omega_0^{(\varepsilon)} := \{x^\diamond(T) \cup z^\diamond \in \mathbb{R}^\diamond : E\{h^0(z^\alpha)\} \leq \max_{\alpha \in \mathcal{A}} E\{h^0(\bar{x}^\alpha(T))\} - \varepsilon\} \tag{29}$$

In view of these definitions, if only the control $\bar{u}(\cdot)$ is robust optimal (locally), then

$$\Omega_0^{(\varepsilon)} \cap \Omega_1 \cap \Omega_{21} \cap \cdots \cap \Omega_{2l} = \{\bar{x}^\diamond(T)\} \quad (\mathbf{P} - a.s.) \tag{30}$$

Hence, if $K_0^\diamond, K_1^\diamond, K_{21}^\diamond, \ldots, K_{2l}$ are the cones (the local tents) of the sets $\Omega_0^{(\varepsilon)}, \Omega_1, \Omega_{21}, \ldots, \Omega_{2l}$ at their common point $\bar{x}^\diamond(T)$, then these cones are *separable* (see Ref. [12, 13] and the Neustad Theorem 1 in [37]), that is, for any point $z^\diamond \in \mathbb{R}^\diamond$ there exist linear independent functionals $\mathbf{l}_s(\bar{x}^\diamond(T), z^\diamond)(s = 0, 1, 2j; j = 1, \ldots, l)$ satisfying

$$\mathbf{l}_0(\bar{x}^\diamond(T), z^\diamond) + \mathbf{l}_1(\bar{x}^\diamond(T), z^\diamond) + \sum_{j=1}^{l} \mathbf{l}_{2\varepsilon}(\bar{x}^\diamond(T), z^\diamond) \geq 0 \tag{31}$$

The implementation of the Riesz representation theorem for linear functionals ([54]) implies the existence of the covariant random vectors $v^s_\diamond(z^\diamond)(s = 0, 1, 2j; j = 1, \ldots, l)$ belonging to the polar cones $K_{s\diamond}$, respectively, not equal to zero simultaneously and satisfying

$$\mathbf{l}_s(\bar{x}^\diamond(T), z^\diamond) = \langle v^s_\diamond(z^\diamond), z^\diamond - \bar{x}^\diamond(T)\rangle_E \tag{32}$$

The relations (31) and (32), and taking into account that they holds for any $z^\diamond \in \mathbb{R}^\diamond$, imply the property

$$v^0_\diamond(\bar{x}^\diamond(T)) + v^1_\diamond(\bar{x}^\diamond(T)) + \sum_{j=1}^{l} v^{sj}_\diamond(\bar{x}^\diamond(T)) = 0 \quad (\mathbf{P} - a.s.) \tag{33}$$

More details about this construction are in [10, 12, 13].

Consider then the possible structures of these vectors.

(a) Denote

$$\Omega_0^\alpha := \{z^\alpha \in \mathbb{R}^\alpha : \{E\{h^0(z^\alpha)\} > \max_{\alpha \in \mathcal{A}} E\{h^0(\bar{x}^\alpha(T))\} - \varepsilon\} \cup \{\bar{x}^\alpha(T)\}\}$$

Taking into account that $h^0(z^\alpha)$ is $L_\phi(C^2)$-mapping and in view of the identity

$$h(x) - h(\bar{x}) = h_x(\bar{x})^\top (x - \bar{x}) + \int_{\theta=0}^{1} \mathrm{tr}[\theta h_{xx}(\theta \bar{x} + (1-\theta)x)(x-\bar{x})(x-\bar{x})^\top]\, d\theta \tag{34}$$

which is valid for any twice differentiable function $h: \mathbb{R}^n \to \mathbb{R}$ and $x, \bar{x} \in \mathbb{R}^n$, it follows that

$$E\{h^0(\bar{x}^\alpha(T))\} = E\{h^0(z^\alpha)\} + \langle h_x^0(z^\alpha), (\bar{x}^\alpha(T) - z^\alpha)\rangle_E + E\{O(\|z^\alpha - \bar{x}^\alpha(T)\|^2)\} \tag{35}$$

So, the corresponding cone K_0^α at the point $\bar{x}^\diamond(T)$ may be described as

$$K_0^\alpha := \begin{cases} \{z^\alpha \in \mathbb{R}^\alpha : \langle h_x^0(z^\alpha), (\bar{x}^\alpha(T) - z^\alpha)\rangle_E \geq 0\} \text{ if } \alpha \text{ is } \varepsilon \wedge h^0\text{-active} \\ \mathbb{R}^\alpha \text{ if } \alpha \text{ is } \varepsilon \wedge h^0\text{-inactive} \end{cases}$$

Then the direct sum $K_0^\diamond := \oplus_{\alpha\in\mathcal{A}} K_0^\alpha$ is a convex cone with apex point $\bar{x}^\alpha(T)$ and, at the same time, it is the tent $\Omega_0^{(\varepsilon)}$ at the same apex point. The polar cone $K_{0\diamond}$ can be presented as

$$K_{0\diamond} = \mathrm{conv}\left(\bigcup_{\alpha\in\mathcal{A}} K_{0\alpha}\right)$$

(here $K_{0\alpha}$ is a the polar cone to $K_0^\alpha \subseteq \mathbb{R}^\alpha$). Since, $v_\diamond^0(z^\diamond) = (v_\alpha^0(z^\alpha)) \in K_{0\diamond}$, then $K_{0\alpha}$ should have the form

$$v_\alpha^0(z^\diamond) = \mu_\alpha^{(\varepsilon)} h_x^0(z^\diamond) \tag{36}$$

where $\mu_\alpha^{(\varepsilon)} \geq 0$ and $\mu_\alpha^{(\varepsilon)} = 0$ if α is $\varepsilon \wedge h^0$-inactive. So, the statement (1(i)) (*Complementary slackness*) is proven.

(b) Now consider the set Ω_{2j}, containing all random vectors $z^\diamond$ admissible by the terminal condition (4) for some fixed index j and any $\alpha \in \mathcal{A}$. Defining for any α and the fixed index j the set

$$\Omega_{2j}^\alpha := \{z^\alpha \in \mathbb{R}^\alpha : E\{h^j(z^\alpha)\} \geq -\varepsilon\}$$

in view of (35) applied for the function h^j, it follows that

$$K_{2j}^\alpha := \begin{cases} \{z^\alpha \in \mathbb{R}^\alpha : \langle h_x^j(z^\alpha)^\top (z^\alpha - \bar{x}^\alpha(T))\rangle_E \geq 0\} \text{ if } \alpha \text{ is } \varepsilon \wedge h^j\text{-active} \\ \mathbb{R}^\alpha \text{ if } \alpha \text{ is } \varepsilon \wedge h^j\text{-inactive} \end{cases}$$

Let $\Omega_{2j} = \oplus_{\alpha\in\mathcal{A}} \Omega_{2j}^\alpha$ and $K_{2j}^\diamond = \oplus_{\alpha\in\mathcal{A}} K_{2j}^\alpha$. By analogy with the above,

$$K_{2j\diamond} = \mathrm{conv}\left(\bigcup_{\alpha\in\mathcal{A}} K_{2j\alpha}\right)$$

is the polar cone, and hence, $K_{2j\alpha}$ should consist of all

$$v_\alpha^{2j}(z^\alpha) = v_{\alpha j}^{(\varepsilon)} h_x^j(z^\alpha) \tag{37}$$

where $v_{\alpha j}^{(\varepsilon)} \geq 0$ and $v_{\alpha j}^{(\varepsilon)} = 0$ if α is $\varepsilon \wedge h^j$-inactive. So, the statement (1(ii)) (*Complementary slackness*) is also proven.

(c) Consider the polar cone $K_{1\diamond}$. Let us introduce the so-called *needle-shape* (*or, spike*) variation $u^\delta(t)(\delta > 0)$ of the robust optimal control $\bar{u}(t)$ at the time region $[0, T]$ as follows:

$$u^\delta(t) := \begin{cases} \bar{u}(t) & \text{if } [0, T+\delta] \backslash T_{\delta_n} \\ u(t) \in \mathcal{U}_{\mathrm{feas}}^s[0, T] & \text{if } t \in T_{\delta_n} \end{cases} \tag{38}$$

where $T_\delta \subseteq [0, T]$ is a measurable set with Lebesgue measure $|T_\delta| = \delta$, $u(t)$ is any s-feasible control. Here it is assumed that $\bar{u}(t) = \bar{u}(T)$ for any $t \in [T, T+\delta]$. It is clear from this construction that $u^\delta(t) \in \mathcal{U}^s_{\text{feas}}[0, T]$ and, hence, the corresponding trajectories $x^\diamond(t) = (x^{\alpha,i}(t))$, given by (3), also make sense. Denote by $\Delta^\alpha := \lim_{\delta\to 0} \delta^{-1}[x^\alpha(T) - \bar{x}^\alpha(T)]$ the corresponding *displacement vector* (here the limit exists because of the differentiability of the vector $x^\alpha(t)$ at the point $t = T$). By the definition, Δ^α is a tangent vector of the controllability region Ω_1. Moreover, the vector

$$g^\diamond(\beta)|_{\beta=\pm 1} := \lim_{\delta\to 0} \delta^{-1}\left[\int_{s=T}^{T+\beta_\delta} b^\diamond(s, x(s), u(s))\,\mathrm{d}t + \int_{s=T}^{T+\beta_\delta} \sigma^\diamond(s, x(s), u(s))\,\mathrm{d}W(s)\right]$$

is also the tangent vector for Ω_1, since

$$x^\diamond(T+\beta_\delta) = x^\diamond(T) + \int_{s=T}^{T+\beta_\delta} b^\alpha(s, x(s), u(s))\,\mathrm{d}t + \int_{s=T}^{T+\beta_\delta} \sigma^\alpha(s, x(s), u(s))\,\mathrm{d}W(s)$$

Denoting by Q_1 the cone (linear combination of vectors with non-negative coefficients) generated by all displacement vectors Δ^α and the vectors $g^\diamond(\pm 1)$, it is concluded that $K_1^\diamond = \bar{x}^\alpha(T) + Q_1$. Hence

$$v^1_\diamond(z^\alpha) = c^{\diamond,(\varepsilon)} \in K_{1\diamond} \tag{39}$$

(d) Substituting (36),(39) and (37) in to (33), *the transversality condition* (16) is obtained. Since, at least one of the vectors $v^0_\diamond(z^\alpha), v^1_\diamond(z^\alpha), v^{21}_\diamond(z^\alpha), \ldots, v^{2l}_\diamond(z^\alpha)$ should be distinct from zero at the point $z^\alpha = \bar{x}^\alpha(T)$, *the nontriviality condition* is obtained too. *The transversality condition* (17) can be satisfied by the corresponding selection of the matrices $C^{\alpha,(\varepsilon)}$. The statement 3 is also proven.

4.3 Proof of Property 4 (Maximality Condition)

This part of the proof seems to be more delicate and needs some additional constructions. In view of (32), (33), (36), (37) and (39), for $z = x^\alpha(T)$ the inequality (31) can be represented as follows

$$\begin{aligned} 0 \le F_\delta(u^\delta(\cdot)) &:= \mathbf{1}_0(\bar{x}^\diamond(T), x^\alpha(T)) + \mathbf{1}_1(\bar{x}^\diamond(T), x^\alpha(T)) + \sum_{j=1}^{l} \mathbf{1}_{2s}(\bar{x}^\diamond(T), x^\alpha(T)) \\ &= \sum_{\alpha\in\mathcal{A}} \Big[\mu^{(\varepsilon)}_\alpha \langle h^0_x(x^\alpha(T)), x^\alpha(T) - \bar{x}^\alpha(T)\rangle_E + \langle c^{\alpha,(\varepsilon)}, x^\alpha(T) - x^\alpha(T)\rangle_E \\ &\quad + \sum_{j=1}^{l} v^{(\varepsilon)}_{\alpha j} \langle h^j_x(x^\alpha(T)), x^\alpha(T) - \bar{x}^\alpha(T)\rangle_E\Big] \end{aligned} \tag{40}$$

valid for any s-feasible control $u^\delta(t)$.

As it has been shown in [56] and [53], any $u^\delta(t) \in \mathcal{U}^s_{\text{feas}}[0, T]$ provides the following trajectory variation

$$x^\alpha(t) - \bar{x}^\alpha(t) = y^{\delta\alpha}(t) + z^{\delta\alpha}(t) + o^{\delta\alpha}_\omega(t) \tag{41}$$

where $y^{\delta\alpha}(t)$, $z^{\delta\alpha}(t)$ and $o^{\delta\alpha}(t)$ are $\{\mathcal{F}_t\}_{t\geq 0}$-adapted stochastic vector processes satisfying (for the simplification of the calculations given below, the argument dependence is omitted) the following equations:

$$\begin{cases} dy^{\delta\alpha} = b_x^{\alpha} y^{\delta\alpha} dt + \sum_{j=1}^{m} [\sigma_x^{\alpha j} y^{\delta\alpha} + \Delta\sigma^{\alpha j} \chi_{T_\delta}] dW^j \\ y^{\delta\alpha}(0) = 0 \end{cases} \tag{42}$$

where

$$\begin{aligned} b_x^{\alpha} &:= b_x^{\alpha}(t, \bar{x}^{\alpha}(t), \bar{u}(t)), \quad \sigma_x^{\alpha j} := \sigma_x^{\alpha j}(t, \bar{x}^{\alpha}(t), \bar{u}(t)) \\ \Delta\sigma^{\alpha j} &:= [\sigma^{\alpha j}(t, \bar{x}^{\alpha}(t), u^{\varepsilon}(t)) - \sigma^{\alpha j}(t, \bar{x}^{\alpha}(t), \bar{u}(t))] \end{aligned} \tag{43}$$

(χ_{T_δ} is the characteristic function of the set T_δ),

$$\begin{cases} dz^{\delta\alpha} = [b_x^{\alpha} z^{\delta\alpha} + \frac{1}{2}\mathcal{B}^{\alpha}(t) + \Delta b^{\alpha} \chi_{T_\delta}] dt \\ \qquad + \sum_{j=1}^{m} [\sigma_x^{\alpha j} z^{\delta\alpha} - \frac{1}{2}\Xi^{\alpha j}(t) + \Delta\sigma_x^{\alpha j}(t) \chi_{T_\delta}] dW^j \\ z^{\delta\alpha}(0) = 0 \end{cases} \tag{44}$$

where

$$\begin{aligned} \mathcal{B}^{\alpha}(t) &:= \begin{pmatrix} \mathrm{tr}[b_{xx}^{\alpha 1}(t, \bar{x}^{\alpha}(t), \bar{u}(t)) Y^{\delta\alpha}(t)] \\ \vdots \\ \mathrm{tr}[b_{xx}^{\alpha n}(t, \bar{x}^{\alpha}(t), \bar{u}(t)) Y^{\delta\alpha}(t)] \end{pmatrix} \\ \Delta b^{\alpha} &:= b^{\alpha}(t, \bar{x}^{\alpha}(t), u^{\delta}(t)) - b^{\alpha}(t, \bar{x}^{\alpha}(t), \bar{u}(t)) \\ \sigma_x^{\alpha j} &:= \sigma_x^{\alpha j}(t, \bar{x}^{\alpha}(t), \bar{u}(t)) \end{aligned} \tag{45}$$

$$\begin{aligned} \div^{\alpha j}(t) &:= \begin{pmatrix} \mathrm{tr}[\sigma_{xx}^{\alpha 1 j}(t, \bar{x}^{\alpha}(t), \bar{u}(t)) Y^{\delta\alpha}(t)] \\ \vdots \\ \mathrm{tr}[\sigma_{xx}^{\alpha n j}(t, \bar{x}^{\alpha}(t), \bar{u}(t)) Y^{\delta\alpha}(t)] \end{pmatrix} (j = 1, \ldots, m) \\ \Delta\sigma_x^{\alpha j} &:= \sigma_x^{\alpha j}(t, \bar{x}^{\alpha}(t), u^{\delta}(t)) - \sigma_x^{\alpha j}(t, \bar{x}^{\alpha}(t), \bar{u}(t)) \\ Y^{\varepsilon\alpha}(t) &:= y^{\varepsilon\alpha}(t) y^{\varepsilon\alpha \mathrm{T}}(t) \end{aligned} \tag{46}$$

and

$$\begin{aligned} \sup_{t\in[0,T]} E\{\|x^{\alpha}(t) - \bar{x}^{\alpha}(t)\|^{2k}\} &= O(\delta^k) \\ \sup_{t\in[0,T]} E\{\|y^{\delta\alpha}(t)\|^{2k}\} &= O(\delta^k) \\ \sup_{t\in[0,T]} E\{\|z^{\delta\alpha}(t)\|^{2k}\} &= O(\delta^{2k}) \\ \sup_{t\in[0,T]} E\|o_{\omega}^{\delta\alpha}(t)\|^{2k} &= o(\delta^{2k}) \end{aligned} \tag{47}$$

hold for any $\alpha \in \mathcal{A}$ and $k \geq 1$. The structures (42)–(46) and the properties (47) are guaranteed by the assumptions A1–A4.

Taking into account these properties and the identity

$$h_x(x) = h_x(\bar{x}) + \int_{\theta=0}^{I} h_{xx}(\bar{x} + \theta(x - \bar{x}))(x - \bar{x})\, d\theta \tag{48}$$

valid for any $L_\phi(C^2)$-mapping $h(x)$, and substituting (41) into (40), it follows that

$$\begin{aligned} 0 \le F_\delta(u^\delta(\cdot)) = \int_{\alpha\in\mathcal{A}} & [\mu_\alpha^{(\varepsilon)}\langle h_x^0(\bar{x}^\alpha(T)), y^{\delta\alpha}(T) + z^{\delta\alpha}(T)\rangle_E + \langle c^{\alpha,(\varepsilon)}, y^{\delta\alpha}(T) + z^{\delta\alpha}(T)\rangle_E \\ & + v_{\alpha j}^{(\varepsilon)}\langle h_x^j(\bar{x}^\alpha(T)), y^{\delta\alpha}(T) + z^{\delta\alpha}(T)\rangle_E + \mu_\alpha^{(\varepsilon)}\langle h_{xx}^0(\bar{x}^\alpha(T))y^{\delta\alpha}(T), y^{\delta\alpha}(T)\rangle_E \\ & + v_{\alpha j}^{(\varepsilon)}\langle h_{xx}^j(\bar{x}^\alpha(T))y^{\delta\alpha}(T), y^{\delta\alpha}(T)\rangle_E]\, dm + o(\delta) \end{aligned} \tag{49}$$

In view of the transversality conditions, the last expression (49) can be represented as follows:

$$0 \le F_\delta(u^\delta(\cdot)) = -\int_{\alpha\in\mathcal{A}} E\{\mathrm{tr}[\Psi^{\alpha,(\varepsilon)}(T)Y^{\delta\alpha}(t)]\}\, dm + o(\delta) \tag{50}$$

The following fact (see Lemma 4.6 in [53] for quadratic matrix case) is used.

Lemma 8 *Let* $Y(\cdot), \Psi_j(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{n\times r}), P(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{r\times n})$ *satisfy*

$$\begin{cases} dY(t) = \Phi(t)Y(t) + \sum_{j=1}^{m} \Psi_j(t)\, dW^j \\ dP(t) = \Theta(t)P(t) + \sum_{j=1}^{m} Q_j(t)\, dW^j \end{cases}$$

with

$$\begin{aligned} &\Phi(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{n\times n}), \quad \Psi_j(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{n\times r}) \\ &Q_j(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{r\times n}), \quad \Theta(\cdot) \in L^2_{\mathcal{F}}(0, T; \mathbb{R}^{r\times r}) \end{aligned}$$

Then

$$\begin{aligned} & E\{\mathrm{tr}[P(T)Y(T)] - \mathrm{tr}[P(0)Y(0)]\} \\ & \quad = E\left\{\int_{t=0}^{T} \left(\mathrm{tr}[\Theta(t)Y(t)] + \mathrm{tr}[P(t)\Phi(t)] + \sum_{j=1}^{m} Q_j(t)\Psi_j(t)\right) dt\right\} \end{aligned} \tag{51}$$

The proof is based on the direct application of Ito's formula.

(a) *The evaluation of the term* $E\{\psi^{\alpha,(\varepsilon)}(T)^\top y^{\delta\alpha}(T)\}$. Applying directly (51) and taking into account that $y^{\delta\alpha}(0) = 0$, it follows that

$$\begin{aligned} E\{\psi^{\alpha,(\varepsilon)}(T)^\top y^{\delta\alpha}(T)\} &= E\{\mathrm{tr}[y^{\delta\alpha}(T)\psi^{\alpha,(\varepsilon)}(T)^\top]\} \\ &= E\left\{\int_{t=0}^{T} \mathrm{tr}\left[\sum_{j=1}^{m} q^{\alpha j,(\varepsilon)}(t)^\top \Delta\sigma^{\alpha j}\right]\chi_{T_\varepsilon}\, dt\right\} \\ &= E\left\{\int_{t=0}^{T} \mathrm{tr}[q^{\alpha,(\varepsilon)}(t)^\top \Delta\sigma^\alpha]\chi_{T_\delta}\, dt\right\} \end{aligned} \tag{52}$$

(b) *The evaluation of the term* $E\{\psi^{\alpha,(\varepsilon)}(T)^{\top}z^{\delta\alpha}(T)\}$. In similar way, applying directly (51) and taking into account that $z^{\delta\alpha}(0)=0$, it follows that

$$
\begin{aligned}
E\{\psi^{\alpha,(\varepsilon)}(T)^{\top}z^{\delta\alpha}(T) &= E\{\mathrm{tr}[z^{\delta\alpha}(T)\psi^{\alpha,(\varepsilon)}(T)^{\top}]\}\\
&= E\left\{\int_{t=0}^{T}\mathrm{tr}\left[\left(\frac{1}{2}\mathcal{B}^{\alpha}\psi^{\alpha,(\varepsilon)}(t)^{\top}+\frac{1}{2}\sum_{j=1}^{m}q^{\alpha j,(\varepsilon)\top\div\alpha j}\right)\right.\right.\\
&\quad\left.\left.+\left(\Delta b^{\alpha}\psi^{\alpha,(\varepsilon)\top}+\sum_{j=1}^{m}q^{\alpha j,(\varepsilon)\top}\Delta\sigma_{x}^{\alpha j}(t)y^{\delta\alpha}\right)\chi_{T_{\delta}}\right]\mathrm{d}t\right\}
\end{aligned}
$$

The equalities

$$
\mathrm{tr}\left[\mathcal{B}^{\alpha}(t)\psi^{\alpha,(\varepsilon)}(T)^{\top}+\sum_{j=1}^{m}q^{\alpha j,(\varepsilon)}(t)^{\top\div\alpha j}(t)\right]=\mathrm{tr}[H_{xx}^{\alpha}(t)Y^{\delta\alpha}(t)]
$$

$$
E\left\{\int_{t=0}^{T}\mathrm{tr}\left[\sum_{j=1}^{m}q^{\alpha j,(\varepsilon)}(t)^{\top}\Delta\sigma_{x}^{\alpha j}(t)y^{\delta\alpha}(t)\right]\chi T_{\delta}\,\mathrm{d}t\right\}=o(\delta)
$$

imply

$$
E\{\psi^{\alpha,(\varepsilon)}(T)^{\top}z^{\delta\alpha}(T)\}=E\left\{\int_{t=0}^{T}\mathrm{tr}\left[\frac{1}{2}H_{xx}^{\alpha}(t)Y^{\delta\alpha}(t)+\Delta b^{\alpha}(t)\psi^{\alpha,(\varepsilon)}(t)^{\top}\chi_{T_{\delta}}\right]\mathrm{d}t\right\}+o(\delta) \tag{53}
$$

(c) *The evaluation of the term* $(1/2)E\{\mathrm{tr}[\Psi^{\alpha,(\varepsilon)}(T)Y^{\delta\alpha}(T)]\}$. Using (42) and applying the Itô formula to $Y^{\delta\alpha}(t)=y^{\delta\alpha}(t)y^{\delta\alpha}(t)^{\top}$, it follows that (for the details see Ref. [53])

$$
\begin{cases}
\mathrm{d}Y^{\delta\alpha}(t)=\left[b_{x}^{\alpha}Y^{\delta\alpha}+Y^{\delta\alpha}b_{x}^{\alpha\top}+\sum_{j=1}^{m}(\sigma_{x}^{\alpha j}Y^{\delta\alpha}\sigma_{x}^{\alpha j\top}+B_{2j}^{\alpha}+B_{2j}^{\alpha\top})\right]\mathrm{d}t\\
\qquad+\sum_{j=1}^{m}(\sigma_{x}^{\alpha j}Y^{\delta\alpha}+Y^{\delta\alpha}\sigma_{x}^{\alpha j\top}+(\Delta\sigma^{\alpha j}y^{\delta\alpha\top}+y^{\delta\alpha}\Delta\sigma^{\alpha j\top})\chi_{T_{\delta}})\mathrm{d}W^{j}\\
Y^{\delta\alpha}(0)=0
\end{cases} \tag{54}
$$

where

$$
B_{2j}^{\alpha}:=(\Delta\sigma^{\alpha j}\Delta\sigma^{\alpha j\top}+\sigma_{x}^{\alpha j}y^{\delta\alpha}\Delta\sigma^{\alpha j\top})\chi_{T_{\delta}}
$$

Again, applying directly (51) and taking into account that $Y^{\delta\alpha}(0)=0$ and

$$
E\left\{\int_{t=0}^{T}\sum_{j=1}^{m}Q_{j}^{\alpha,(\varepsilon)}(t)(\Delta\sigma^{\alpha j}y^{\delta\alpha\top}+y^{\delta\alpha}\Delta\sigma^{\alpha j\top})\chi_{T_{\delta}}\mathrm{d}t\right\}=o(\delta)
$$

it follows that

$$
E\{\mathrm{tr}[\Psi^{\alpha,(\varepsilon)}(T)Y^{\delta\alpha}(T)]\}=E\int_{t=0}^{T}(-\mathrm{tr}[H_{xx}^{\alpha}Y^{\delta\alpha}(t)]+\mathrm{tr}[\Delta\sigma^{\alpha\top}\Psi^{\alpha,(\varepsilon)}\Delta\sigma^{\alpha}]\chi_{T_{\delta}})\mathrm{d}t+o(\delta) \tag{55}
$$

In view of the definition (20)

$$
\begin{aligned}
\delta\mathcal{H}&:=\mathcal{H}(t,\bar{x}^{\diamond}(t),u^{\delta}(t),\psi^{\diamond,(\varepsilon)}(t),\Psi^{\diamond,(\varepsilon)}(t),q^{\diamond,(\varepsilon)}(t))\\
&\quad-\mathcal{H}(t,\bar{x}^{\diamond}(t),\bar{u}(t),\psi^{\diamond,(\varepsilon)}(t),\Psi^{\diamond,(\varepsilon)}(t),q^{\diamond,(\varepsilon)}(t))\\
&=\int_{\alpha\in\mathcal{A}}\left(\Delta b^{\alpha\top}\psi^{(\varepsilon)}+\mathrm{tr}[q^{\alpha,(\varepsilon)\top}\Delta\sigma^{\alpha}]+\frac{1}{2}\mathrm{tr}[\Delta\sigma^{\alpha\top}\Psi^{\alpha,(\varepsilon)}\Delta\sigma^{\alpha}]\right)\mathrm{d}m
\end{aligned} \tag{56}
$$

Using (52),(53), (55) and (56), it follows that

$$E\left\{\int_{t=0}^{T}\delta\mathcal{H}(t)\chi_{T_{\delta_n}}\mathrm{d}t\right\}$$
$$=E\left\{\int_{t=0}^{T}\int_{\alpha\in\mathcal{A}}\left(\Delta b^{\alpha\top}\psi^{(\varepsilon)}+\mathrm{tr}[q^{\alpha,(\varepsilon)\top}\Delta\sigma^{\alpha}]+\frac{1}{2}\mathrm{tr}[\Delta\sigma^{\alpha\top}\Psi^{\alpha,(\varepsilon)}\Delta\sigma^{\alpha}]\right)\mathrm{d}m\chi_{T_{\delta_n}}\mathrm{d}t\right\}$$
$$=\langle\psi^{\diamond,(\varepsilon)}(T),y^{\delta\alpha}(T)+z^{\delta\alpha}(T)\rangle_E+\frac{1}{2}\int_{\alpha\in\mathcal{A}}E\{\mathrm{tr}[\Psi^{\alpha,(\varepsilon)}(T)Y^{\delta\alpha}(T)]\}\mathrm{d}m+o(\delta_n) \tag{57}$$

Since

$$y^{\delta\alpha}(T)+z^{\delta\alpha}(T)=\delta\Delta^{\alpha}+o^{\delta\alpha}(T)$$

where $\Delta^{\alpha}\in K_1^{\alpha}$ is a displacement vector, and $\psi^{\alpha,(\varepsilon)}(T)=c^{\alpha,(\varepsilon)}\in K_{1\alpha}$, then

$$\langle\psi^{\diamond,(\varepsilon)}(T),y^{\delta\alpha}(T)+z^{\delta\alpha}(T)\rangle_E=\delta\langle c^{\alpha,(\varepsilon)},\Delta^{\alpha}\rangle_E+o(\delta)\le 0 \tag{58}$$

for sufficiently small $\delta>0$ and any fixed $\varepsilon>0$. In view of (50) and (58), the right-hand side of (57) can be estimated as

$$E\left\{\int_{t=0}^{T}\delta\mathcal{H}(t)\chi_{T_{\delta_n}}\mathrm{d}t\right\}=\delta\langle c^{\diamond,(\varepsilon)},\Delta^{\diamond}\rangle_E+\frac{1}{2}\int_{\alpha\in\mathcal{A}}E\{\mathrm{tr}[\Psi^{\alpha,(\varepsilon)}(T)Y^{\varepsilon\alpha}(T)]\}\,\mathrm{d}m+o(\delta)\ \le o(\delta_n)$$

Dividing by δ_n, it follows that

$$\delta_n^{-1}E\left\{\int_{t=0}^{T}\delta\mathcal{H}(t)\chi_{T_\delta}\mathrm{d}t\right\}\le o(1) \tag{59}$$

Using the lemma 1 from ([37]) for

$$T_\delta=[t_0-\delta_n\beta_1,t_0+\delta_n\beta_2]\quad(\beta_1,\beta_2\ge 0;\beta_1+\beta_2>0)$$

and $\{\delta_n\}$ so that $\delta_n\to 0$, and in view of (59), it follows that

$$\delta_n^{-1}E\left\{\int_{t=0}^{T}\delta\mathcal{H}(t)\chi_{T_{\delta_n}}\mathrm{d}t\right\}\to(\beta_1+\beta_2)E\{\delta\mathcal{H}(t_0)\}\le 0 \tag{60}$$

for almost all $t_0\in[0,T]$. Here if $t_0=0$ then $\beta_1=0$ and if $t_0=T$ then $\beta_2=0$, but if $t_0\in(0,T)$ then $\beta_1,\beta_2>0$. The inequality (60) implies

$$E\{\delta\mathcal{H}(t)\}\le 0 \tag{61}$$

from which (22) follows directly. Indeed, assume that there exist the control $\check{u}(t)\in\mathcal{U}^s_{\text{feas}}[0,T]$ and a time $t_0\in(0,T)$ (not belonging to a set of null measure) such that

$$\mathbf{P}\{\omega\in\Omega_0(\rho)\}\ge p>0 \tag{62}$$

where $\Omega_0(\rho):=\{\omega\in\Omega:\delta\mathcal{H}(t_0)>\rho>0\}$. Then (61) can be rewritten as

$$0\ge E\{\delta\mathcal{H}(t)\}=E\{\chi(\omega\in\Omega_0(\rho))\delta\mathcal{H}(t)\}+E\{\chi(\omega\notin\Omega_0(\rho))\delta\mathcal{H}(t)\}$$
$$\ge\rho\mathbf{P}\{\omega\in\Omega_0(\rho)\}+E\{\chi(\omega\notin\Omega_0(\rho))\delta\mathcal{H}(t)\}\ge\rho p+E\{\chi(\omega\notin\Omega_0(\rho))\delta\mathcal{H}(t)\}$$

Since this inequality should be also valid for the control $\hat{u}(t)$ satisfying

$$\hat{u}(t)=\begin{cases}\check{u}(t) & \text{for almost all } \omega\in\Omega_0(\rho)\\ \bar{u}(t) & \text{for almost all } \omega\neq\Omega_0(\rho)\end{cases}$$

there is the contradiction

$$0 \geq E\{\delta\mathcal{H}(t)\} \geq \rho p + E\{\chi(\omega \notin \Omega_0(\rho))\delta\mathcal{H}(t)\} = \rho p > 0$$

This completes the proof. ■

5 DISCUSSIONS

5.1 The Important Comment on the Hamiltonian Structure

The Hamiltonian function $\mathcal{H}$ used for the construction of the robust optimal control $\bar{u}(t)$ is equal (see (19)) to the Lebesgue integral over the uncertainty set of the standard stochastic Hamiltonians $\mathcal{H}^{\alpha}$ corresponding to each fixed value of the uncertain parameter.

5.2 RSMP for the Control-Independent Diffusion Term

From the Hamiltonian structure (20) it follows that if $\sigma^{\alpha j}(t, x^{\alpha}(t), u(t))$ does not depend on $u(t)$, then

$$\begin{aligned}
&\arg\max_{u\in U} \mathcal{H}(t, \bar{x}^{\diamond}(t), u, \psi^{\diamond,(\varepsilon)}(t), \Psi^{\diamond,(\varepsilon)}(t), q^{\diamond,(\varepsilon)}(t)) \\
&\quad = \arg\max_{u\in U} \int_{\mathcal{A}} \mathcal{H}^{\alpha}(t, \bar{x}^{\alpha}(t), u, \psi^{\alpha,(\varepsilon)}(t), \Psi^{\alpha,(\varepsilon)}(t), q^{\alpha,(\varepsilon)}(t))\mathrm{d}m(\alpha) \\
&\quad = \arg\max_{u\in U} \int_{\mathcal{A}} H^{\alpha}(t, \bar{x}^{\alpha}(t), u, \psi^{\alpha,(\varepsilon)}(t), q^{\alpha,(\varepsilon)}(t))\mathrm{d}m(\alpha)
\end{aligned} \tag{63}$$

So, it follows that 2*nd order adjoint process* does not participate in the robust optimal constructions.

5.3 The Complete Information Case

If the stochastic plant is completely known, that is, there is no parametric uncertainty ($\mathcal{A} = \alpha_0$, $\mathrm{d}m(\alpha) = \delta(\alpha - \alpha_0)\mathrm{d}\alpha$), then from (63)

$$\begin{aligned}
&\arg\max_{u\in U} \mathcal{H}(t, \bar{x}^{\diamond}(t), u, \psi^{\diamond,(\varepsilon)}(t), \Psi^{\diamond,(\varepsilon)}(t), q^{\diamond,(\varepsilon)}(t)) \\
&\quad = \arg\max_{u\in U} \int_{\mathcal{A}} \mathcal{H}^{\alpha}(t, \bar{x}^{\alpha}(t), u, \psi^{\alpha,(\varepsilon)}(t), \Psi^{\alpha,(\varepsilon)}(t), q^{\alpha,(\varepsilon)}(t))\mathrm{d}m(\alpha) \\
&\quad = \arg\max_{u\in U} \mathcal{H}^{\alpha_0}(t, \bar{x}^{\alpha_0}(t), u, \psi^{\alpha_0,(\varepsilon)}(t), \Psi^{\alpha_0,(\varepsilon)}(t), q^{\alpha_0,(\varepsilon)}(t))
\end{aligned} \tag{64}$$

and if $\varepsilon \to 0$, it follows that, in this case, RSMP converts to *Stochastic Maximum Principle* (see [28, 53, 56]).

5.4 Deterministic Systems

In the deterministic case, when there is no any stochastics

$$(\sigma^{\alpha}(t, \bar{x}^{\alpha}(t), u(t)) \equiv 0),$$

the Robust Maximum Principle for minimax problems (in Mayer form) stated in [14] and [16] is obtained directly, that is, for $\varepsilon \to 0$ it follows

$$\begin{aligned} &\arg\max_{u\in U} \mathcal{H}(t, \bar{x}^{\diamond}(t), u, \psi^{\diamond,(\varepsilon)}(t), \Psi^{\diamond,(\varepsilon)}(t), q^{\diamond,(\varepsilon)}(t)) \\ &\qquad = \arg\max_{u\in U} \int_{\mathcal{A}} b^{\alpha}(t, \bar{x}(t), u)^{\mathrm{T}} \psi^{\alpha}(t) \mathrm{d}m(\alpha) \end{aligned} \tag{65}$$

When, in addition, there is no parametric uncertainties ($\mathcal{A} = \alpha_0$, $\mathrm{d}m(\alpha) = \delta(\alpha - \alpha_0)\,\mathrm{d}\alpha$), the *Classical Maximum Principle* for the optimal control problems (in Mayer form), is obtained [41], that is,

$$\begin{aligned} &\arg\max_{u\in U} \mathcal{H}(t, \bar{x}^{\diamond}(t), u, \psi^{\diamond,(0)}(t), \Psi^{\diamond,(0)}(t), q^{\diamond,(0)}(t)) \\ &\qquad = \arg\max_{u\in U} \mathcal{H}(t, \bar{x}(t), u, \psi(t), \Psi(t), q(t)) = \arg\max_{u\in U} b(t, \bar{x}(t), u)^{\mathrm{T}} \psi(t) \end{aligned} \tag{66}$$

5.5 Comment on Possible Non-fixed Horizon Extension

Consider the case when the function $h^0(x)$ is positive. Let us introduce a new variable x^{n+1} (associated with time t) with the equation

$$\dot{x}^{n+1} \equiv 1 \tag{67}$$

and consider the variable vector $\bar{x} = (x^1, \ldots, x^n, x^{n+1}) \in \mathbb{R}^{n+1}$. For the plant (2), added with (67), the initial conditions are as follows

$$x(t_0) = x_0 \in \mathbb{R}^n, \quad x^{n+1}(t_0) = 0 \text{ (for all } \alpha \in \mathcal{A})$$

Furthermore, we determine the terminal set $\mathcal{M}$ for the plant (2), (67) by the inequality

$$\mathcal{M} := \{x \in \mathbb{R}^{n+1} : h^{l+1}(x) = \tau - x^{n+1} \leq 0\}$$

assuming that the numbers t_0, τ are fixed ($t_0 < \tau$). Let now $u(t), \bar{x}(t), 0 \leq t \leq T$, be an admissible control that satisfies the terminal condition. Then $T \geq \tau$, since otherwise the terminal condition $x(t_1) \in \mathcal{M}$ wouldn't satisfied. The function $h^0(x)$ is defined only on $\mathbb{R}^n$, but we prolong it in to $\mathbb{R}^{n+1}$, setting

$$h^0(\bar{x}) = \begin{cases} h^0(x) & \text{for } x^{n+1} \leq \tau \\ h^0(x) + (x^{n+1} - \tau)^2 & \text{for } x^{n+1} > \tau \end{cases}$$

If now $T > \tau$, then (for every $\alpha \in \mathcal{A}$)

$$h^0(x(t_1)) = h^0(x(\tau)) + (t_1 - \tau)^2 > h^0(x(\tau))$$

Thus F^0 may attain its minimum only for $T = \tau$, that is, we have the problem with fixing time $T = \tau$. By this, the Theorem above gives the Robust Maximum Principle only for the problem with a fixed horizon. The non-fixed horizon case demands a special construction and implies another formulation of RMP.

5.6 The Case of Absolutely Continuous Measures for Uncertainty Set

Consider now the case of an absolutely continuous measure $m(\mathcal{A}_0)$, that is, consider the situation when there exists a summable (the Lebesgue integral $\int_{R^s} p(x)(\mathrm{d}x^1 \vee \cdots \vee \mathrm{d}x^n)$ is finite and s-fold) nonnegative function $p(x)$, given on $\mathbb{R}^s$ and named the density of a measure $m(\mathcal{A}_0)$, such that for every measurable subset $\mathcal{A}_0 \subset \mathbb{R}^s$ we have

$$m(\mathcal{A}_0) = \int_{\mathcal{A}_0} p(x)dx, \, \mathrm{d}x := \mathrm{d}x^1 \vee \cdots \vee \mathrm{d}x^n$$

By this initial agreement, $\mathbb{R}^s$ is a space with the countable additive measure. Now it is possible to consider controlled object (2) with the set of uncertainty $\mathcal{A} = \mathbb{R}^s$. In this case

$$\int_{\mathcal{A}_0} f(x)\mathrm{d}m = \int_{\mathcal{A}_0} f(x)p(x)\mathrm{d}x \tag{68}$$

The statements of the Robust Maximum Principle for this special case is obtained from the main Theorem with evident variation. It is possible also to consider a particular case when $p(x)$ is defined only on a ball $\mathcal{A} \subset \mathbb{R}^s$ (or on another subset of $\mathbb{R}^s$) and integral (68) is defined only for $\mathcal{A}_0 \subset \mathcal{A}$.

5.7 Uniform Density Case

If no *a priori* information on some or others parameter values and the distance on a compact $\mathcal{A} \subset \mathbb{R}^s$ is defined by the natural way as $\|\alpha_1 - \alpha_2\|$, then the *Maximum Condition* (22) can be formulated (and proved) as follows:

$$\begin{aligned} u(t) &\in \arg\max_{u \in U} H^{\diamond}(\psi(t), x(t), u) \\ &= \arg\max_{u \in U} \int_{\mathcal{A}} \mathcal{H}^{\alpha}(t, \bar{x}^{\alpha}(t), u, \psi^{\alpha,(\varepsilon)}(t), \Psi^{\alpha,(\varepsilon)}(t), q^{\alpha,(\varepsilon)}(t))\, \mathrm{d}\alpha \\ &\quad \text{almost everywhere on } [t_0, t_1] \end{aligned} \tag{69}$$

that represents, evidently, a partial case of the general condition (22) with an uniform absolutely continuous measure, that is, when

$$\mathrm{d}m(\alpha) = p(\alpha)\mathrm{d}\alpha = \frac{1}{m(\mathcal{A})}\mathrm{d}\alpha$$

with $p(\alpha) = m^{-1}(\mathcal{A})$.

5.8 Finite Uncertainty Set

If the uncertainty set $\mathcal{A}$ is *finite*, the Robust Maximum Principle, proved above, gives the result contained in [15, 43, 44]. In this case, the integrals may be replaced by finite sums. For example, formula (7) takes the form

$$F^0 = \max_{\alpha \in \mathcal{A}} h^0(x^{\alpha}(t_1))$$

and similar changes have to be done in the further formulas. Now, the number ε is superfluous and may be omitted, and in the complementary slackness condition we have the equalities, that is, the formulations in the main theorem should look (when $\varepsilon = 0$) as follows: *for every* $\alpha \in \mathcal{A}$ *the equalities*

$$\mu_{\alpha}^{(0)} \cdot (E\{h^0(\bar{x}^{\alpha}(T))\} - F^0) = 0, \quad \nu_{\alpha j}^{(0)} \cdot E\{h^j(\bar{x}^{\alpha}(T))\} = 0 \tag{70}$$

holds.

5.9 May the Complementary Slackness Inequalities be Replace by the Equalities?

It is naturally to ask: is it possible, in general case, to replace the inequalities by the equalities as it was done above or not? Below we present an example that gives the negative answer. Consider the case of the absolutely continuous measure for $s = 1 (R^s = R^1)$ with the density $p(x) = e^{-x^2}$. Furthermore, take, for the simplicity, $n = 1$. Consider the family of the simple controlled plants given by

$$\dot{x}^{\alpha,1} = f^{\alpha}(x, u) = -\frac{\alpha^2}{1+\alpha^2} + u$$

with $t_0 = 0, t_1 = 1/2, x^{\alpha,1}(0) = 1, \alpha \in [-1, 1], U = [-1, 1]$ and no noise at all. The terminal set $\mathcal{M}$ is defined by the inequality $h^1(x) \leq 0$ with $h^1(x) = x$. Finally, we take the cost function as $h^0(x) = 1 - x$. It is evident (applying the main theorem) that the optimal control is as follows: $u(t) \equiv -1, 0 \leq t \leq 1/2$ and $F^0 = 1$. But the complementary slackness condition in the form (70) implies that $\mu_{\alpha}^{(0)} = \nu_{\alpha}^{(0)} = 0$ for all α and any $\varepsilon = 0$. Consequently the transversality condition gives $\psi^{(0)}(t) \equiv 0$. But this contradicts to the nontriviality condition. Thus the inequalities in the main theorem cannot be replaces by equalities (70).

6 CONCLUSION

In this paper the Robust Stochastic Maximum Principle (in Mayer form) is presented for a class of nonlinear continuous-time stochastic systems containing an unknown parameter from a given measured set and subject to terminal constraints. Its proof is based on the use of the Tent Method with the special technique specific for stochastic calculus. The Hamiltonian function used for these constructions is equal to the Lebesgue integral over the given uncertainty set of the standard stochastic Hamiltonians corresponding to a fixed value of the uncertain parameter.

The future investigation may be focused on the Linear Quadratic Stochastic Problem which seems to us to be solvable by this technique also. However, it will be considered in a subsequent paper. Furthermore, it makes sense to continue this study in the following directions:

- formulate RSMP for minimax problem in the general Bolza form,
- consider the terminal constraints with the additional integral terms,
- consider the terminal constraints including the probability of some events,
- formulate RSMP for minimax problem not for a fixed horizon but for a random stopping time.

References

[1] Basar, T. and Bernhard, P. (1991) *H^∞ Optimal Control and Related Minimax Design Problems: A Dynamic Game Approach.* Boston, MA: Birkhauser.
[2] Basar, T. (1994) Minimax Control of Switching Systems Under Sampling. *Proceedings of the 33-rd Conference on Decision and Control.* Lake Buena Vista, Florida, December, pp. 716–721.
[3] Bellman, R. (1957) *Dynamic Programming*,Princeton University Press.
[4] Bensoussan, A. (1983) Lecture on stochastic control, part 1, *Lecture Notes in Math.*, **972**, 1–39.
[5] Bensoussan, A. (1992) *Stochastic Control of Partially Observable Systems*, Cambridge University Press.
[6] Bernhard, P. (1994). Minimax versus stochastic partial information control. *Proceedings of the 33-rd Conference on Decision and Control*, Lake Buena Vista, Florida, December, pp. 2572–2577.
[7] Blom, H. and Everdij, M. (1993) Embedding Adaptive JLQG into LQ Martingale with a Complete Observable Stochastic Control Matrix. *Proceedings of the 32-nd Conference on Decision and Control*, San Antonio, Texas. December, pp. 849–854.
[8] Bismut, J. M. (1977) On optimal control of linear stochastic equations with a linear quadratic criteria, *SIAM J. Control*, **15**(1).
[9] Bismut, J. M. (1978) An introductory approach to duality in stochastic control, *SIAM Review*, **20**, 62–78.
[10] Boltyansky, V. G. (1975) The method of tents in the theory of extremal problems, *Uspehi Mat. Nauk*, **30**(3), 3–65.
[11] Boltyansky, V. G. (1978) *Optimal Control of Discrete Systems.* New-York–Toronto–Ont.: A Halsted Press Book, John Willey & Sons.
[12] Boltyansky, V. G. (1987) The tent method in topological vector spaces, *Soviet. Math. Dokl.*, **34** (1), 176–179.
[13] Boltyansky, V. G. (2002) The tent method in Banach spaces (to be published).
[14] Boltyansky, V. G. and Poznyak, A. S. (1999) Robust maximum principle in minimax control. *Int. J. Control*, **72**(4), 305–314.
[15] Boltyansky, V. G. and Poznyak, A. S. (1999) Robust maximum principle for minimax Bolza problem with terminal set, *Proceedings of IFAC-99*, Beijing, F (2d-06), 263–268.
[16] Boltyansky, V. G. and Poznyak, A. S. (2002) Robust maximum principle for a measured space as uncertainty set, *Dynamic Systems and Applications*, **11**(2), 277–292.
[17] Boukas, E. K., Shi, P. and Andijani, A. (1999). Robust inventory-production control problem with stochastic demand, *Optimal Control Applications and Methods*, **20**(1), 1–20.
[18] Clarke, F. H. and Vinter, R. B. (1983) Local optimality conditions and Lipshitzian solutions to the Hamilton-Jacobi equations, *SIAM J. Control*, **21**, 856–870.
[19] Clarke, F. H. (1983) *Optimization and Nonsmooth Analysis.* NY: John Wiley.
[20] Crandall, M. G. and Lions, P. L. (1983) Viscosity solution of Hamilton-Jacobi equations, *Trans. Amer. Math. Soc.*, **277**, 1–42.
[21] Didinsky, C. and Basar, T. (1991) Minimax controller and filter designs for discrete–time linear systems under soft–constrained quadratic performance indices, *Proceedings of the 30-th Conference on Decision and Control.* Brighton, England, 585–590.
[22] Didinsky, G. and Basar, T. (1994) Minimax adaptive control of uncertain plants, *Proceedings of the 33-rd Conference on Decision and Control.*, Lake Buena Vista, Florida, 2839–2844.
[23] Dorato, P. and Drenick, R. F. (1966) Optimality, insensitivity and game theory. In: Radanovich, L. (Ed.), *Sensitivity Methods in Control Theory* New York, NY: Pergamon Press, pp. 78–102.
[24] Doyle, J., Glover, K., Khargonekar, P. and Francis, B. (1989) State–space solutions to standard H_2 and H^∞ control problems. *IEEE Transactions on Automatic Control*, **34**(8), 831–847.
[25] Duncan, T. E. (1967) *Doctoral Dissertation*, Dept. of El. Eng., Stanford University.
[26] Duncan, T. E., Guo, L. and Pasik-Duncan, B. (1999) Adaptive continuous-time linear quadratic Gaussian control, *IEEE Transaction on AC*, **44**(9), 1653–1662.
[27] Duncan, T. E. and Varaiya, P. P. (1971) On the solution of a stochastic control system, *SIAM J. Control*, **9**, 354–371.
[28] Fleming, W. H. and Rishel, R. W. (1975) *Optimal Deterministic and Stochastic Control.* Berlin: Springer-Verlag.
[29] Glover, K. and Doyle, J. (1988) State–space formulae for all stabilizing controllers that satisfy an H^∞-norm bound and relations to risk sensitivity. *Systems and Control Letters*, **11**, 167–172.
[30] Haussman, U. G. (1981) Some examples of optimal control, or: The stochastic maximum principle at work, *SIAM Review*, **23**, 292–307.
[31] Haussman, U. G. (1982) On the existence of optimal controls for partially observed diffusions, *SIAM J. Control*, **20**, 385–407.
[32] Kallianpur, G. (1980) *Stochastic Filtering Theory*, NY: Springer-Verlag.
[33] Khargonekar, P. P. (1991) State–space H^∞-control theory and the LQG control problem. In: Antoulas, A. C. (Ed.), *Mathematical System Theory: The Influence of R. E. Kalman*, Springer Verlag.
[34] Krsovskii, N. N. (1969) Game problem on movements correction, *Applied Mathematic and Mechanics*, **33**(3), (in Russian).
[35] Krylov, N. V. (1980) *Controlled Diffusion Processes.* NY: Springer-Verlag.
[36] Kurjanskii, A. B. (1977) *Control and Observations in Uncertain Conditions.* Moscow: Nauka (in Russian).
[37] Kushner, H. (1972) Necessary conditions for continuous parameter stochastic optimization problems, *SIAM J. Control*, **10**, 550–565.

[38] Lau, M. K., Boyd, S., Kosut, R. L. and Franklin, G. F. (1991) Robust control design for ellipsoidal plant set. *Proceedings of the 30-th Conference on Decision and Control*, Brighton: England, pp. 291–296.
[39] Limebeer, D., Anderson, B. D. O., Khargonekar, P. and Green, M. (1989) A game theoretic approach to H^{∞}-control for time varying systems, *Proceedings of Int. Symposium on the Mathematical Theory of Networks and Systems*, Amsterdam.
[40] Lions, P. L. (1983) Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations, part 1: Viscosity solutions and uniqueness, *Comm. Partial Diff. Equ.*, **11**, 1229–1276.
[41] Pontryagin, L. S., Boltyansky, V. G., Gamkrelidze, R. V. and Mishenko, E. F. (1962) *The Mathematical Theory of Optimal Processes*, New York: Interscience (translated from Russian, 1969).
[42] Poznyak, A. and Taksar, M. (1996) Robust control of linear stochastic systems with fully observable state. *Applicationes Mathematicae*, **24**(1), 35–46.
[43] Poznyak, A. S., Duncan, T. E., Pasik-Duncan, B. and Boltyansky, V. G. (2002) Robust stochastic maximum principle. *Proceedings of the American Control Conference*, Anchorage, AK, pp. 310–315.
[44] Poznyak, A. S., Duncan, T. E., Pasik-Duncan, B. and Boltyansky, V. G. (2002) Robust stochastic maximum principle, *International Journal of Control*, **175**(13), 1032–1048.
[45] Poznyak, A. S., Duncan, T. E., Pasik-Duncan, B. and Boltyansky, V. G. (2002) Robust optimal control for minimax stochastic linear quadratic Problem. *International Journal of Control*, **175**(14), 1054–1065.
[46] Pytlak, R. (1990) Strong variation algorithm for minimax control: Parallel approach, *Proceedings of the 29-th Conference on Decision and Control.* Honolulu, Hawaii, 2469–2474.
[47] Siljak, D. D. (1989) Parameter space methods for robust control design: A guided tour, *IEEE Transactions on Automatic Control*, **34**(7), 674–688.
[48] Taksar, M. I., Poznyak, A. S. and Iparraguirre, A. (1998) Robust output feedback control for linear stochastic systems in continuous time with time-varying parameters, *IEEE Transactions on AC*, **43**(8), 1133–1136.
[49] Taksar, M. I. and Zhou, X. Y. (1998) Optimal risk and divident control for a company with a debt liability, *Insurance: Math&Econom*, **22**, 105–122.
[50] Ugrinovskii, V. A. and Petersen, I. R. (1997) Finite horizon minimax optimal control of nonlinear continuous time systems with stochastic uncertainties. *Proceedings of 36th IEEE CUC*, San Diego, 2265–2270.
[51] Vinter, R. B. (1988) New results on the relationship between dynamic programming and their connection in deterministic control, *Math. Control. Sign. Syst.*, **1**, 97–105.
[52] Yaz, G. (1991) Minimax control of discrete nonlinear stochastic systems with white noise uncertainty, *Proceedings of the 30-th Conference on Decision and Control.* Brighton, England, pp. 1815–1816.
[53] Yong, J. and Zhou, X. Y. (1999) *Stochastic Controls: Hamiltonian Systems and HJB Equations*, Springer-Verlag.
[54] Yoshida, K. (1979) *Functional Analysis.* New Delhi: Narosa Publishing House.
[55] Zames, J. (1981) Feedback and optimality sensitivity: Model reference transformation, multiplicative seminorms and approximate inverses. *IEEE Transactions on Automatic Control*, **26**, 301–320.
[56] Zhou Xun Yu. (1991) A unified treatment of maximum principle and dynamic programming in stochastic controls, *Statistics and Stochastic Reports*, **36**, 137–161.
[57] Zakai, M. (1969) On the optimal filtering of diffusion processes, *Z. Wahrsch. Geb.*, **11**, 230–243.