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H_2 Guaranteed Cost Control of Discrete Linear Systems

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This paper presents necessary and sufficient conditions for the existence of a quadratically stabilizing output feedback controller which also assures H_2 guaranteed cost performance on a discrete linear uncertain system where the uncertainty is of the norm bounded type. The conditions are presented as a collection of linear matrix inequalities. The solution, however requires a search over a scalar parameter space.

Keywords: Guaranteed cost control; Uncertain discrete systems; Norm bounded uncertainty; H_2 control

AMS Classification Number: 93

1. PROBLEM STATEMENT

The robust control domain has experienced an exponential growth in recent years. In this research domain, new and better tools are developed to deal naturally with multi-input, multi-output systems, external disturbances, system's uncertainty and to include explicitly in the control design other common performance specification (Anderson and Moore [1], Barmish [2]).

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Among the robust approaches, one of the best known and practically implemented is the H_2-or LQR-. This approach searches for a control law that minimizes some performance index measuring the control efforts and profile of the states evolution. In turn, the performance may be expressed as a fictitious – commonly known as controllable – output (Doyle *et al.* [5]).

This paper addresses the problem of determining the existence of a H_2 controller for some discrete uncertain linear systems. The problem will be formulated as a collection of Linear Matrix Inequalities (LMIs) which will also imply a search over a parameter space. The derivations are based on a recent result (Scherer *et al.* [13]), where an elegant change of controller variables – in fact matrix change of variables – is used to obtain a linear set of matrix inequalities.

We consider uncertain discrete linear systems of the form:

$$\begin{aligned} x_{t+1} &= (A + DFE_1)x_t + (B + DFE_2)u_t + B_1 w_t \\ y_t &= Cx_t \\ z_t &= C_1 x_t + D_1 u_t \end{aligned} \tag{1}$$

where $x_t \in \mathbb{R}^n$, $u_t \in \mathbb{R}^m$ and $y_t \in \mathbb{R}^p$ are, respectively, the state, control and measurable output vectors. $w_t \in \mathbb{R}^{n_w}$ is the external disturbance and $z_t \in \mathbb{R}^{n_z}$ is the controlled output. A, B, C, B_1, C_1, D_1, are known constant matrices, determining the nominal system. D and E are known constant matrices that define the structure of the uncertainty (how it affects the system). The unknown matrix $F \in \mathbb{R}^{n_D \times n_E}$ represents the – norm bounded – uncertainty (Petersen [11]) and belongs to:

$$\mathbf{F} = \{F \in \mathbb{R}^{n_D \times n_E} : F^T F \leq I\}$$

The problem addressed is that of finding an observer-like controller of the form:

$$\begin{aligned} \hat{x}_{t+1} &= A_c \hat{x}_t + B_c y_t \\ u_t &= C_c \hat{x}_t \end{aligned} \tag{2}$$

such that when the system loop is closed with (2), the resulting system is stable and the H_2 norm of transfer function from w to z is less than or equal to some given $\sqrt{\gamma} > 0$ for all possible $F \in \mathbf{F}$, *i.e.*,

$$\|T_{wz}\|_2^2 \leq \gamma \quad \forall F \in \mathbf{F}$$

what is normally referred to as guaranteed cost control (Esfahai and Petersen [6]).

All results to be presented might be extended to pole placement in discs centered in some real scalar value α and radius r. Non strictly proper controllers might be equally considered. Without loss of generality, demonstrations are limited to systems of the form (1) and controllers (2) in order to keep the derivations much simpler.

When controller (2) is applied, the closed loop system obtained is:

$$\begin{pmatrix} x_{t+1} \\ \hat{x}_{t+1} \end{pmatrix} = \left\{ \underbrace{\begin{pmatrix} A & BC_c \\ B_c C & A_c \end{pmatrix}}_{\tilde{A}} + \underbrace{\begin{pmatrix} D \\ 0 \end{pmatrix}}_{\tilde{D}} F \underbrace{(E_1 \quad E_2 C_c)}_{\tilde{E}} \right\} \underbrace{\begin{pmatrix} x_t \\ \hat{x}_t \end{pmatrix}}_{\tilde{x}} + \underbrace{\begin{pmatrix} B_1 \\ 0 \end{pmatrix}}_{\tilde{B}_1} w_t \tag{3}$$

$$z_t = \underbrace{(C_1 \quad D_1 C_c)}_{\tilde{C}_1} \begin{pmatrix} x_t \\ \hat{x}_t \end{pmatrix}$$

or in simpler terms:

$$\begin{aligned} \tilde{x}_{t+1} &= (\tilde{A} + \tilde{D}F\tilde{E})\tilde{x}_t + \tilde{B}_1 w_t \\ z_t &= \tilde{C}_1 \tilde{x}_t. \end{aligned} \tag{4}$$

Our formulation of the problem is based on the quadratic stability concept, which is defined as follows:

Definition 1.1 (Garcia and Bernussou [7]) System (4) is quadratically stable if there exists a positive-definite symmetric matrix $P > 0$ such that:

$$(\tilde{A} + \tilde{D}F\tilde{E})^T P (\tilde{A} + \tilde{D}F\tilde{E}) - P < 0 \tag{5}$$

$\forall F \in \mathbf{F}$.

Remark 1.1 In general, disc stability (or *d*-stability) refers to the fact that the closed loop system has all its poles located in a disc of the s or z plane. In the case of (5) this disc is the unit disc of the z plane, *i.e.*, the one that assures asymptotic stability of the discrete system.

The H_2 guaranteed cost is given by:

DEFINITION 1.2 (Geromel *et al.* [8]) Let $\{A_c, B_c, C_c\}$ be a given quadratically stabilizing controller of system (1). Then it is a $\gamma(>0)$, H_2 guaranteed cost controller if:

$$\|T_{wz}\|_2^2 \leq \gamma \quad \forall F \in \mathbf{F}$$

where T_{wz} is the transfer function from w to z in (4) and is given by:

$$T_{wz} = \tilde{C}_1(\delta I - \tilde{A} - \tilde{D}F\tilde{E})^{-1}\tilde{B}_1, \quad \delta = \text{the advance or shift operator}$$

Let us recall now that (Garcia and Bernussou [7]):

$$\|T_{wz}\|_2^2 = \text{Trace}(\tilde{B}_1^T L_o(F)\tilde{B}_1)$$

where $L_o(F)$, – the Observability Grammian – satisfies:

$$(\tilde{A} + \tilde{D}F\tilde{E})^T L_o(F)(\tilde{A} + \tilde{D}F\tilde{E}) - L_o(F) + \tilde{C}_1^T\tilde{C}_1 = 0. \tag{6}$$

The following theorem gives us certain equivalent conditions for upper limits on the H_2 norm of system (4).

THEOREM 1.1 (Garcia and Bernussou [7]) *System* (4) *is quadratically stable if and only if there exists a matrix* $P > 0$ *and a scalar* $\varepsilon > 0$ *such that*

$$\tilde{A}^T(P^{-1} - \varepsilon\tilde{D}\tilde{D}^T)^{-1}\tilde{A} - P + \varepsilon^{-1}\tilde{E}^T\tilde{E} + \underbrace{\tilde{C}_1^T\tilde{C}_1}_{\tilde{Q}} < 0 \tag{7}$$

Remark 1.2 In Garcia and Bernussou [7] the result is formulated in terms of a matrix Q, which may be arbitrarily chosen, and a discrete Riccati equation. With an appropriate selection of Q (*e.g.*, $Q = \tilde{Q} +$ any positive-definite small matrix) (7) yields.

The upper limit on the H_2 norm is given in terms of the solution P of the Riccati inequality (7).

In fact, satisfaction of (7) is equivalent to (Garcia and Bernussou [7]):

$$(\tilde{A} + \tilde{D}F\tilde{E})^T P(\tilde{A} + \tilde{D}F\tilde{E}) - P + \tilde{C}_1^T\tilde{C}_1 < 0. \tag{8}$$

By comparing (6) and (8) it can be concluded that $P > L_o$ and therefore:

$$\text{Trace}(\tilde{B}_1^T L_o(F)\tilde{B}_1) \leq \text{Trace}(\tilde{B}_1^T P\tilde{B}_1).$$

In terms of Matrix Inequalities, the existence of a matrix $P > 0$ and a scalar $\varepsilon > 0$ such that:

$$\begin{pmatrix} P^{-1} & \tilde{B}_1 \\ \tilde{B}_1^T & \gamma I \end{pmatrix} > 0 \tag{9}$$

and

$$\begin{pmatrix} -P^{-1} + \varepsilon \tilde{D}\tilde{D}^T & \tilde{A} \\ \tilde{A}^T & -P + \varepsilon^{-1}\tilde{E}^T\tilde{E} + \tilde{C}_1^T\tilde{C}_1 \end{pmatrix} < 0 \tag{10}$$

implies that

$$\|T_{wz}\|_2^2 \le \gamma \quad \forall F \in \mathbf{F}$$

and that system (4) is quadratically stable. Now we are ready to present our main result.

2. MAIN RESULT

This section presents necessary and sufficient conditions for the existence of quadratic stabilizing controller with H_2 guaranteed cost.

To ease the presentation some elementary manipulation of previous results are featured.

LEMMA 2.1 *Inequality* (10) *is satisfied if and only if there exists a positive-definite symmetric matrix S such that*:

$$\begin{pmatrix} -S + \tilde{D}\tilde{D}^T & \tilde{A}S \\ S\tilde{A}^T & -S + S\tilde{E}^T\tilde{E}S + \varepsilon S\tilde{C}_1^T\tilde{C}_1 S \end{pmatrix} < 0 \tag{11}$$

Demonstration Pre and post multiply the left hand side of inequality (10) by the regular symmetric matrix:

$$\begin{pmatrix} I & 0 \\ 0 & P^{-1} \end{pmatrix}.$$

Then, by defining $S = \varepsilon^{-1}P^{-1}$, (11) is obtained. ■

Observe that with this change of variables, inequality (9) becomes:

$$\begin{pmatrix} S & \tilde{B}_1 \\ \tilde{B}_1^T & \varepsilon\gamma I \end{pmatrix} > 0. \tag{12}$$

LEMMA 2.2 *Let T be any regular square matrix of appropriate dimension, then inequality* (11) *is satisfied if and only if there exists a positive-definite symmetric matrix S such that*:

$$\begin{pmatrix} -I & 0 & \tilde{E}ST^T & 0 & 0 \\ 0 & -\varepsilon^{-1}I & \tilde{C}_1ST^T & 0 & 0 \\ TS\tilde{E}^T & TS\tilde{C}_1^T & -TST^T & TS\tilde{A}^TT^T & 0 \\ 0 & 0 & T\tilde{A}ST^T & -TST^T & T\tilde{D} \\ 0 & 0 & 0 & \tilde{D}^TT^T & -I \end{pmatrix} < 0 \tag{13}$$

Demonstration (13) is obtained by pre and post multiplying the left hand side of (11) by the regular matrix:

$$\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$$

and its transpose respectively and then applying the Schur complement equivalence (Boyd *et al.* [3]) to expand the dimensions of the matrix. ∎

LEMMA 2.3 *Let T be any regular square matrix of appropriate dimension, then inequality* (12) *is satisfied if and only if there exists a positive-definite symmetric matrix S such that*:

$$\begin{pmatrix} TST^T & TB_1 \\ B_1^TT^T & \varepsilon\gamma I \end{pmatrix} > 0. \tag{14}$$

Demonstration (14) is obtained by pre and post multiplying the left hand side of (12) by the regular matrix:

$$\begin{pmatrix} T & 0 \\ 0 & I \end{pmatrix}$$

and its transpose respectively. ∎

We now present our main result:

THEOREM 2.1 *System* (4) *is quadratically stable with $\gamma(>0)$ H_2 guaranteed cost control if and only if there exist positive-definite* *symmetric matrices X and Y and matrices H, L, Z and a scalar $\varepsilon > 0$ such that the following set of Linear Matrix Inequalities is satisfied:*

$$\begin{pmatrix} Y & I & YB_1 \\ I & X & B_1 \\ B_1^T Y & B_1^T & \varepsilon\gamma I \end{pmatrix} > 0 \tag{15}$$

and

$$\begin{pmatrix} -I & 0 & E_1 & E_1X + E_2L & 0 & 0 & 0 \\ \star & -\varepsilon^{-1}I & C_1 & C_1X + D_1L & 0 & 0 & 0 \\ \star & \star & -Y & -I & A^TY + C^TH^T & A^T & 0 \\ \star & \star & \star & -X & Z & XA^T + L^TB^T & 0 \\ \star & \star & \star & \star & -Y & -I & YD \\ \star & \star & \star & \star & \star & -X & D \\ \star & \star & \star & \star & \star & \star & -I \end{pmatrix} < 0 \tag{16}$$

furthermore, an H_2 guaranteed cost controller is given by:

$$\begin{aligned} B_c &= V^{-1}H \\ C_c &= L(U^T)^{-1} \\ A_c &= V^{-1}(Z^T - YAX - HCX - YBL)(U^T)^{-1} \end{aligned} \tag{17}$$

where V is any regular matrix (arbitrarily chosen by the designer) and U satisfies:

$$XY + UV^T = I.$$

Observe that the left hand side of (16) being a symmetric matrix, a "$\star$" has been introduced to avoid a very cumbersome matrix description.

Demonstration We partition S in (13) as:

$$S = \begin{pmatrix} X & U \\ U^T & \hat{X} \end{pmatrix} \quad S^{-1} = P = \begin{pmatrix} Y & V \\ V^T & \hat{Y} \end{pmatrix}$$

and define the regular matrix T of the form:

$$T = \begin{pmatrix} Y & V \\ I & 0 \end{pmatrix}$$

where it may be assumed with no loss of generality that matrix V is a regular matrix (Scherer *et al.* [13]).

We obtain the results of (15) and (16) with this choice of matrix T on (13) and by simply defining:

$$H = VB_c;\ L = C_cU^T;\ Z = (YAX + HCX + YBL + VA_cU^T)^T.$$

It only remains to show that matrix S is positive definite. Observe that, being T a regular matrix, $S > 0$ if and only if $TST^T > 0$ and then $S > 0$ if and only if:

$$\begin{pmatrix} Y & I \\ I & X \end{pmatrix} > 0$$

such a condition is implicitly satisfied when either (15) or (16) is met. This set of linear matrix inequalities then assure the H_2 guaranteed cost performance of system (4). ■

3. SOME COMMENTS ON THE RESULT

In Theorem 2.1 the convexity of the problem is lost due to the scalar variable ε. Even though there exist methods to deal with this specific structure directly (see for instance Iwasaki and Skelton [10]), it does seem much simpler to exploit our knowledge of the problem and the meaning of the inequalities.

In fact, LMI (16) stands for the robust stability condition. Since it is a necessary condition, it could be solved alone first – this is a convex programming problem – and look for the maximum ε^*, for which there exists robust stability; then – recalling (Petersen [11]) – it may be assured that for any ε in the interval $(0, \varepsilon^*)$ there will also exist a solution to LMI (16). No ε other than those in the mentioned interval could be a solution to the robust stability problem. Furthermore, for each ε there exist a number – probably infinite – of γ guaranteed cost performance for some positive values of γ.

Once the feasible "ε-*interval*" is known, one could start a search of "γ-*suboptimal* H_2 controllers" solutions of (15) and (16) – again a convex programming problem if ε is fixed in each iteration – and then choose the best performance among them, *i.e.*, $\gamma^* < \gamma$, for all γ solutions of the second problem.

With this approach, standard LMI tools (such as those in MATLAB) may be used. In the next section such an approach is developed with a numerical example.

Even though it appears that the final controller is very sensitive to the choice of V, it is not so. In fact, it is very easy to show that the transfer function of the controller (17) is, indeed, independent of the choice of V and only depends on the solution of (15) and (16), *i.e.*, on X, Y, L, H, Z.

4. NUMERICAL EXAMPLE

This section presents a small numerical example, from Huei and Fong [9] slightly modified.

The system considered is:

$$\begin{aligned} x_{t+1} &= \left\{ \begin{pmatrix} -1.00 & -1.20 \\ 0.10 & -0.15 \end{pmatrix} + \begin{pmatrix} 0.1 \\ 0.2 \end{pmatrix} F (1 \quad 0) \right\} x_t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u_t \\ &\quad + \begin{pmatrix} 0 \\ 0.3 \end{pmatrix} w_t \\ y_t &= (1.2 \quad -1.5) x_t \\ z_t &= (0 \quad 1) x_t + u_t \end{aligned} \tag{18}$$

The maximum ε^* for which there exists a solution – readily obtained by using MATLAB `mincx` instruction – is: $\varepsilon^* = 1.7679$.

Figure 1 shows the evolution of best γ for $\varepsilon \in (1, 1.76)$. Smaller values of ε give higher valuer of γ. Minimum $\gamma^* = 1.3033$ is obtained at $\varepsilon = 1.55$.

And the rest of – matrix – variables, solution of (15) and (16) for the same ε are:

$$X = \begin{pmatrix} 0.9104 & -0.3122 \\ -0.3122 & 0.2967 \end{pmatrix}; \quad Y = \begin{pmatrix} 3.3681 & -0.9385 \\ -0.9385 & 22.4426 \end{pmatrix};$$

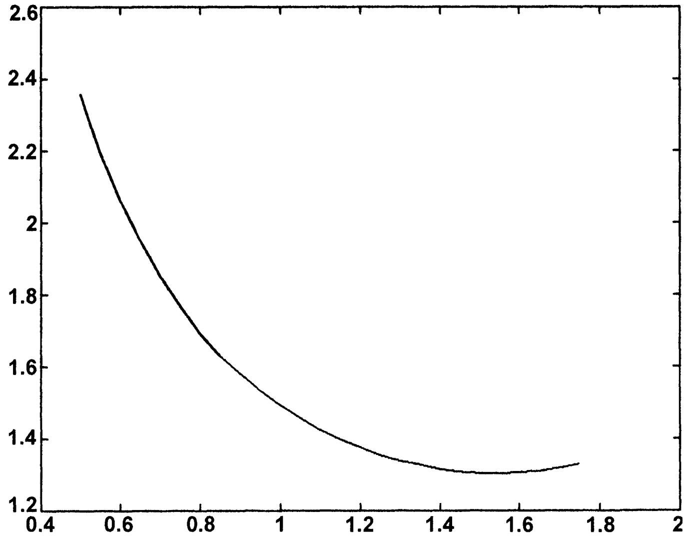

FIGURE 1 Best gamma *vs*. epsilon.

and

$$L = (0.3396 \quad -0.0443); \; Z = \begin{pmatrix} -0.0477 & 0.0747 \\ -0.3692 & 0.0769 \end{pmatrix}; \; H = \begin{pmatrix} 2.1716 \\ -2.2378 \end{pmatrix}.$$

With this set of values and by simply choosing V as the identity matrix the controller is:

$$\hat{x}_{t+1} = \begin{pmatrix} 0.8053 & -0.0950 \\ -0.1715 & -0.0147 \end{pmatrix} \hat{x}_t + \begin{pmatrix} 2.1716 \\ -2.2378 \end{pmatrix} y_t$$

$$u_t = (-0.4664 \quad -0.0968)\hat{x}_t.$$

Figure 2, features the closed loop pole location when the uncertain parameter F ranges from -1 to 1.

Figure 3 includes the impulse response of the nominal system as well as the 2 extremal systems ($F = -1$ and $F = 1$), of the transfer function from w_t to z_t (T_{wz}) in (1).

It is very easy to construct a Lyapunov matrix P for the closed loop system, as a function of X, Y and V or U.

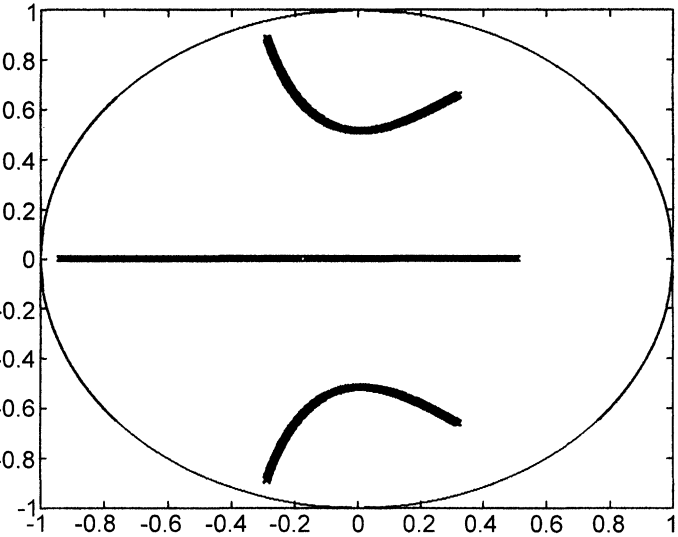

FIGURE 2 Closed loop pole location. F uniformly distributed between -1 and 1.

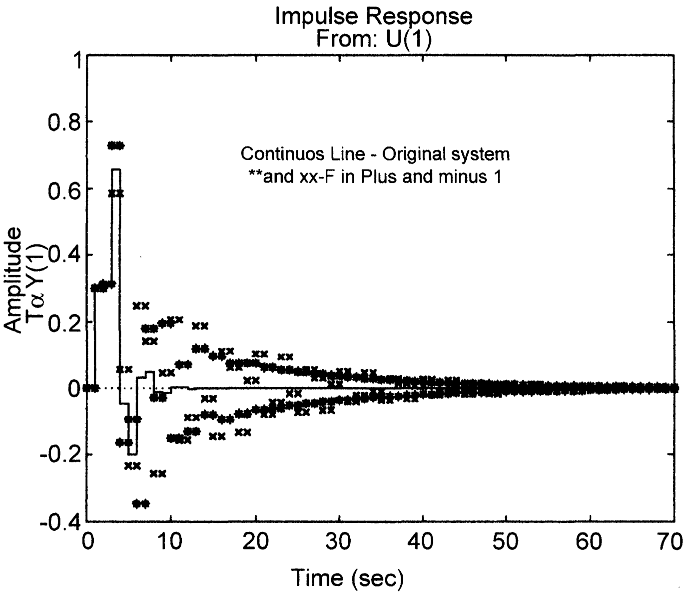

FIGURE 3 Impulse response of T_{wz}. $F=0, -1, 1$.

5. CONCLUSIONS

This paper presents necessary and sufficient conditions for the existence of a H_2 guaranteed cost output feedback control for linear discrete uncertain systems.

Although the conditions are presented as a set of linear matrix inequalities, the convexity of the problem is lost due to a scalar variable "ε". Solution of the problem implies a search over this parameter space. The latter does not seem to be a great limitation if an *a priori* search of the solution space of that parameter, as it has been done in the example, is performed.

The results can, easily, be extended the root clustering problem in disc regions – $D(\alpha, r)$ – (discs centered in $(-\alpha, 0)$ and radius r), from the fact that all the eigenvalues of a given matrix A are located in $D(\alpha, r)$ if and only if $(A+\alpha I)/r$ is stable in the discrete time sense (Chilali *et al.* [4]).

Finally, extension to the robust stability problem of descriptor discrete time systems is currently under way, following the ideas in Rehm and Allgower [12].

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