MPE — Volume 1, pp. 95-137
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MATHEMATICAL THEORY OF IMPROVABILITY FOR PRODUCTION SYSTEMS*

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(Received 15 October 1994)

A mathematical model for continuous improvement processes in production systems is formulated. Both constrained and unconstrained cases are addressed. A solution for the case of a serial production line with finite buffers and a Bernoulli model of machines reliability is given. In particular, it is shown that a production line is unimprovable under constraints if each buffer is on the average half full and each machine has equal probability of blockages and starvations. Based on this result, guidelines for continuous improvement processes are formulated.

AMS Nos.: 90B30, 60J20, 93A30
KEYWORDS: Serial production lines, Bernoulli machines, indicators of improvability

1. INTRODUCTION

1.1. Manufacturing Considerations

The process of continuous improvement is a route necessary to achieve and maintain competitive positions in any type of manufacturing environment—mass, lean, or agile. Unfortunately, no formal methods are available to guide this process on the factory floor. Typically, continuous improvement projects are conducted using managerial intuition, manufacturing gurus [1]–[3], and/or discrete event simulations [4]. Given this situation, knowledge of the basic properties which govern the process of continuous improvement is of importance. This paper is devoted to the analysis of these properties. More specifically, we introduce and analyze the property of *improvability* in production systems. Roughly speaking, a production system is improvable (under constraints) if the limited resources involved in its operation can be redistributed so that a performance index is improved.

Improvability is related to optimality. Indeed, an unimprovable system is optimal. However, we use the term improvability to indicate that the goal here is not necessarily to render the system optimal, but rather to determine whether it can or cannot be improved and indicate directions which lead to this improvement. In addition, given the lack of

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*This work has been supported by NSF under grant No. DDM-9105086.

precise information available on the factory floor, optimality may not be practically achievable, whereas improvability still may be characterized by simple *indicators* robust with respect to imprecise information.

If a system is unimprovable in the sense mentioned above, the only route for continuous improvement is the relaxation of the constraints, that is, bottleneck elimination. It turns out that the notion of the bottleneck is closely related to the property of improvability, and this relationship is also explored in this paper.

Based on the results obtained for improvability and bottleneck analysis, this paper formulates guidelines for the process of continuous improvement which are applicable, we believe, to a wide class of production systems in large volume manufacturing. Although these guidelines are quite informal consequences of the theoretical results derived in this paper, they have proven to be useful in a number of practical applications which we have recently carried out in the automotive industry.

1.2. Problem Formulation

Consider a production system of M unreliable machines and B finite buffers interconnected by a material handling system. Assume that each machine is characterized by its average production rate in isolation, p_i, $i = 1, \dots, M$, and each buffer is characterized by its capacity, N_i, $i = 1, \dots, B$. Assume that the N_i's and p_i's are constrained as follows:

$$\sum_{i=1}^{B} N_i = N^*, \tag{1.1}$$

$$\prod_{i=1}^{M} p_i = p^*. \tag{1.2}$$

Constraint (1.1) implies that the total work-in-process (WIP) available in the system cannot exceed N^*. Constraint (1.2) is interpreted as a bound on the workforce (WF). Indeed, in many systems, assignment of the workforce (both machine operators and skilled trades for repair and maintenance) defines the production rate and the average up-time of each machine. Therefore, the total WF available can be conceptually mapped into constraint (1.2).

Let $PI(p_1, \dots, p_M, N_1, \dots, N_B)$ be the performance index of interest. Examples of PI are the average production rate, the due-time performance, product quality, and so forth.

Definition 1.1 A production system is called improvable with respect to WIP if there exists a sequence $N_1^*, \dots, N_B^*$ such that $\Sigma_{i=1}^{B} N_i^* = N^*$ and

$$PI(p_1, \dots, p_M, N_1^*, \dots, N_B^*) > PI(p_1, \dots, p_M, N_1, \dots, N_B).$$

Definition 1.2 A production system is called improvable with respect to WF if there exists a sequence $p_1^*, \dots, p_M^*$ such that $\Pi_{i=1}^{M} p_i^* = p^*$ and

$$PI(p_1^*, \dots, p_M^*, N_1, \dots, N_B) > PI(p_1, \dots, p_M, N_1, \dots, N_B).$$

Definition 1.3 A production system is called improvable with respect to WIP & WF simultaneously if there exist sequences $N_1^*, \ldots, N_B^*$ and $p_1^*, \ldots, p_M^*$ such that $\Sigma_{i=1}^{B} N_1^* = N^*, \Pi_{i=1}^{M} p_i^* = p^*$, and

$$PI(p_1^*, \ldots, p_M^*, N_1^*, \ldots, N_B^*) > PI(p_1, \ldots, p_M, N_1, \ldots, N_B).$$

The main problem considered in this paper can be formulated as follows:

Problem 1.1 Given a production system described above, find both quantitative and qualitative indicators of improvability with respect to WIP, WF, and WIP & WF simultaneously.

This is the first problem addressed in this paper.

When a system is unimprovable under (1.1) and (1.2), constraint relaxation (i.e. increase p^* or N^*) is necessary to improve *PI*. The question arises: Which p_i and/or N_i should be increased so that the most benefits are obtained? To formulate this question precisely, introduce

Definition 1.4 Machine i is the bottleneck machine if

$$\frac{\partial PI(p_1, \ldots, p_M, N_1, \ldots, N_B)}{\partial p_i} > \frac{\partial PI(p_1, \ldots, p_M, N_1, \ldots, N_B)}{\partial p_j}, \forall j \neq i.$$

Definition 1.5 Buffer i is the bottleneck buffer if

$$PI(p_1, \ldots, p_M, N_1, \ldots, N_i + 1, \ldots, N_B) > PI(p_1, \ldots, p_M, N_1, \ldots, N_j + 1, \ldots, N_B), \forall j \neq i.$$

Contrary to the popular belief, the machine with the smallest production rate and the buffer with the smallest capacity are not necessarily the bottleneck machine and the bottleneck buffer, respectively. In some cases, the most productive machine, that is, the machine with the largest p_i, is the bottleneck (see section 4 for an example). This happens because the inequalities in Definitions 1.4 and 1.5 depend not on a particular machine or buffer, but rather on the system as a whole. Therefore, the problem arises:

Problem 1.2 Given a production system defined by machines $p_1 \ldots p_M$ and buffers $N_1, \ldots, N_B$ interconnected by a material handling system, identify the bottleneck machine and the bottleneck buffer.

This is the second problem addressed in this paper.

Finally, in some cases it is important to determine the total work-in-process N^*, such that all $N > N^*$ give practically no increase in *PI*. To formulate this property, introduce

$$PI(N) = \max_{\substack{N_1, \ldots, N_B \\ \Sigma_{i=1}^{B} N_i = N}} PI(p_1, \ldots, p_M, N_1, \ldots, N_B).$$

Definition 1.6 The total WIP, N^*, is said to be ϵ-adapted to $p_1, \ldots, p_M$ if

$$\frac{|PI(N^*) - PI(N)|}{PI(N^*)} < \epsilon, \forall N \geq N^*.$$

Problem 1.3. Given a production system and $\epsilon > 0$, find the smallest N^* which is ϵ adapted to the p_i's.

This is the third problem addressed in this work.

The organization of this paper is as follows: In section 2, we introduce a specific production system and performance index which are studied throughout this work. Section 3 is devoted to improvability under constraints. In section 4, constraint relaxation and ϵ-adaptation are discussed. The guidelines for the process of continuous improvement are formulated in section 5. Finally, the conclusions are given in section 6. The facts established in this paper are either proven mathematically or verified numerically. All proofs of a mathematical nature are presented in Appendices A–C. Due to the size limitation, no results of practical applications are included in this paper; they will be described elsewhere.

2. PRODUCTION SYSTEM

2.1 Model

Although Problems 1.1–1.3 are of interest in a large variety of manufacturing situations, for the purposes of this study we analyze the simplest, but archetypical, production system—the serial production line. A number of models for such lines are available in [5]–[9]. The following model is considered throughout this work.

(i) The system consists of M machines arranged serially, and $M - 1$ buffers separating each consecutive pair of machines.
(ii) The machines have identical cycle time T_c. The time axis is slotted with the slot duration T_c. Machines begin operating at the beginning of each time slot.
(iii) Each buffer is characterized by its capacity, N_i, $1 \le i \le M - 1$.
(iv) Machine i is starved during a time slot if buffer $i - 1$ is empty at the beginning of the time slot. Machine 1 is never starved.
(v) Machine i is blocked during a time slot if buffer i has N_i parts at the beginning of the time slot and machine $i + 1$ fails to take a part during the time slot. Machine M is never blocked.
(vi) Machine i, being neither blocked nor starved during a time slot, produces a part with probability p_i and fails to do so with probability $q_i = 1 - p_i$. Parameter p_i is referred to as the *production rate* of machine *i in isolation*.

Remark 2.1 Assumption (vi) implies that each machine's reliability obeys the Bernoulli model. This model is appropriate when disturbances occur only for short periods of time, comparable with the cycle time T_c. This is often the case for large volume assembly and painting operations where the perturbations are due to the quality requirements (*i.e.*, the operational conveyors are stopped for a short period of time in order to accomplish the operation with the highest possible quality). The Bernoulli model may not be applicable to machining operations where the perturbations are due to machine break-downs which occur for periods of time much longer than the cycle time. In this situation, Markovian models of machine reliability are more appropriate. The improvability properties for serial production lines with Markovian machines, although quite similar to the Bernoulli case, will be addressed elsewhere.

Remark 2.2 Model (i) – (vi) is a generalization of the model considered in [10] and [11], where $p_i = 1 - \epsilon k_i$, $0 < \epsilon << 1$. Here, therefore, we treat the general case. Another generalization of [10], [11] has been described in [12].

The performance index analyzed in this work is the average production rate (*PR*), that is, the average number of parts produced by the *M*th machine in the steady state of the system's operation; we denote this quantity as

$$PR = PR(p_1, \dots, p_M, N_1, \dots, N_{M-1}).$$

Unfortunately, this function cannot be calculated in closed form if $M \geq 3$. Therefore, below we derive an estimate, PR_{est}, of the production rate, and evaluate its accuracy.

2.2 Production Rate Estimate

Although a number of recursive algorithms for production rate evaluation in serial lines have been described in the literature ([7]–[9],[13]–[16]), analytical justification of their convergence and accuracy seems to be lacking. Since the derivation of the improvability conditions pursued in this work requires these properties, we present below a recursive procedure developed for model (i)–(vi), prove its convergence, and provide an estimate (however weak) of its accuracy.

Consider the following recursive procedure:

$$p_i^b(s+1) = p_i[1 - Q(p_{i+1}^b(s+1), p_i^f(s), N_i)], \qquad 1 \leq i \leq M-1.$$

$$p_i^f(s+1) = p_i[1 - Q(p_{i-1}^f(s+1), p_i^b(s+1), N_{i-1})], \qquad 2 \leq i \leq \mathrm{M}. \qquad (2.1)$$

$$p_1^f(s) = p_1, \qquad p_M^{\mathrm{b}}(\mathrm{s}) = \mathrm{p}_M,$$

$$s = 1, 2, 3, \dots,$$

with initial conditions

$$p_i^f(0) = p_i, \qquad i = 1, \dots, M,$$

where

$$Q(x, y, N) = \begin{cases} \dfrac{(1-x)(1-\alpha)}{1 - \dfrac{x}{y}\alpha^N}, & x \neq y \\[2ex] \dfrac{1-x}{N+1-x}, & x = y \end{cases} \qquad (2.2)$$

and

$$\alpha = \frac{x(1-y)}{y(1-x)}.$$

LEMMA 2.1 *The recursive procedure (2.1) is convergent, so that the limits*

$$p_i^f = \lim_{s \to \infty} p_i^f(s),$$

$$p_i^b = \lim_{s \to \infty} p_i^b(s), \tag{2.3}$$

$$i = 1, \dots, M$$

exist.

Proof See Appendix A.

Procedure (2.1) represents an aggregating technique consisting of two principal components, a forward and a backward aggregation. In the forward aggregation, the first two machines and the intervening buffer are repeatedly replaced by a single machine, thereby reducing the length of the line, until the entire line has been reduced to a single machine. Parameter p_i^f is the machine parameter of the single machine replacing the first i machines and $i - 1$ buffers. Similarly, in the backward aggregation the last two machines and the intervening buffer are repeatedly replaced by a single machine until the entire line has again been reduced to a single machine. Parameter p_i^b is the machine parameter of the single machine replacing machines $i, \dots, M$ and buffers $i + 1, \dots, M - 1$. Parameters p_i^f and p_i^b can be interpreted as

$$p_i^f \approx \text{Prob}\{\text{machine } i \text{ produces a part} \mid \text{machine } i \text{ is not blocked}\}.$$

$$p_i^b \approx \text{Prob}\{\text{machine } i \text{ produces a part} \mid \text{machine } i \text{ is not starved}\}.$$

Therefore, since the last machine is never blocked, the production rate estimate for the line (i)–(vi) is defined as

$$PR_{est}(p_1, \dots, p_M, N_1, \dots, N_{M-1}) = p_M^f. \tag{2.4}$$

To evaluate the accuracy of this estimate, consider the joint steady state probability, $X_{i,\dots,j}(h_i, \dots, h_j)$, that the consecutive buffers $i, i + 1, \dots, j$, $1 \le i < j \le M - 1$, contain h_i, $h_{i+1}, \dots, h_j$ parts, respectively. In general, one cannot expect that this joint probability is close to the product of its marginals, that is, $X_{i,\dots,j}(h_i, \dots, h_j) \neq X_i(h_i)\, X_{i+1,\dots,j}(h_{i+1}, \dots, h_j)$, where $X_i(h_i)$ is the probability that the ith buffer contains h_i parts. It turns out, however, that for certain values of $h_i, h_{i+1}, \dots, h_j$, related to blockages and starvations, they are indeed close. Specifically, define

$$\delta_{i,j}(b) = |X_{i,\dots,j}(0, b, N_{i+2}, \dots, N_j) - X_i(0)\, X_{i+1,\dots,j}(b, N_{i+2}, \dots, N_j)|$$

$$\delta^{i,j}(a) = |X_{i,\dots,j}(a, N_{i+1}, \dots, N_j) - X_i(a)\, X_{i+1,\dots,j}(N_{i+1}, \dots, N_j)| \tag{2.5}$$

$$\delta = \max_{i,j} \max_{a,b} \{\delta_{i,j}(b), \delta^{i,j}(a)\}.$$

Then, as it follows from extensive numerical experimentation, δ is always small. An illustration is given in Table I for several lines with $N_i = 3, i = 1, 2, 3$. At present, we do not have an analytical proof that $\delta << 1$, although we believe that such a proof is possible. Therefore, we formulate

NUMERICAL FACT 2.1 *For serial production lines defined by assumptions (i)–(vi),*

$$\delta << 1.$$

It should be pointed out that δ turns out to be small only in lines where the first machine is never starved and the last machine is never blocked, which is the case in model (i)–(vi). If the first machine can be starved (say, by CONWIP raw material dispatch [17]) or the last machine can be blocked (for instance, by the empty carriers buffer [18]), δ is no longer much smaller than one. We suspect that many heuristic algorithms for production rate evaluation work well for open lines and much worse for closed lines precisely due to this property.

THEOREM 2.1 *Under assumptions (i)–(vi), production rate estimate (2.4) results in* $\mathcal{O}(\delta)$ *accuracy, that is,*

$$Error = | PR_{est}(p_1, \dots, p_M, N_1, \dots, N_{M-1}) - PR(p_1, \dots, p_M, N_1, \dots, N_{M-1}) | \sim \mathcal{O}(\delta),$$

where δ *is defined in (2.5).*

Proof See Appendix A.

Although this estimate is quite weak (since δ is not an asymptotic parameter) numerical experiments, illustrated in Table I, show that the proportionality constant $\mathcal{O}(\delta)$ is quite small, and the estimate (2.4) results in high accuracy.

In what follows, the analysis of the process of continuous improvement, that is, the solution of Problems 1.1–1.3, is carried out in terms of the production rate estimate—$PR_{est}(p_1, \dots, p_M, N_1, \dots, N_{M-1})$, defined by (2.1)–(2.4).

To conclude this section, we cite the following two structural properties of PR_{est}:

THEOREM 2.2 PR_{est} *possesses the reversibility property, that is,*

$$PR_{est}(p_1, \dots, p_M, N_1, \dots, N_{M-1}) = PR_{est}(p_M, \dots, p_1, N_{M-1}, \dots, N_1).$$

Proof See Appendix A.

Reversibility properties of this type have been known for a long time [19].

THEOREM 2.3 PR_{est} *possesses the monotonicity property, that is, function* $PR_{est}(p_1, \dots, p_M, N_1, \dots, N_{M-1})$ *is monotonically increasing with respect to all arguments.*

Table I Behavior of δ and estimation error

p_1	p_2	p_3	p_4	*PR*	δ	*Error*
0.80	0.80	0.80	0.80	0.7109	0.0073	0.0008
0.70	0.80	0.70	0.80	0.6352	0.0233	0.0047
0.70	0.90	0.70	0.90	0.6562	0.0568	0.0144
0.60	0.99	0.99	0.60	0.5705	0.1181	0.0283
0.99	0.60	0.60	0.99	0.5294	0.0083	$2*10^{-7}$

Proof *See* Appendix A.

This theorem is a particular example of a general monotonicity result discovered in [20].

3. IMPROVABILITY UNDER CONSTRAINTS

3.1 Improvability with Respect to WF

In compliance with Definition 1.2, serial production line (i)–(vi) is improvable with respect to WF if there exists $p_1^*, \ldots, p_{\mathrm{M}}^*$ such that $\Pi_{i=1}^{M} p_i^* = p^*$ and

$$PR_{est}(p_1^*, \ldots, p_M^*, N_1, \ldots, N_{M-1}) > PR_{est}(p_1, \ldots, p_M, N_1, \ldots, N_{M-1}).$$

THEOREM 3.1 *Serial production line (i)–(vi) is unimprovable with respect to WF if and only if*

$$p_i^f = p_{i+1}^b, \qquad i = 1, \ldots, M-1. \tag{3.1}$$

Proof *See* Appendix B.

COROLLARY 3.1 *If condition (3.1) is satisfied,*

(a) each machine i is blocked with almost the same frequency as machine i + 1 is starred, that is,

$$\begin{aligned} &|b_i - s_{i+1}| \backsim \mathcal{O}(\delta), i = 1, \ldots, M-1. \\ &b_i \;=\; \text{Prob}\{\text{machine } i \text{ is blocked,}\} \\ &s_i \;=\; \text{Prob}\{\text{machine } i \text{ is starved,}\} \end{aligned} \tag{3.2}$$

where δ is defined in (2.5);

(b) each buffer is on the average close to being half full in the following sense:

$$E\,[h_i] = \frac{N_i}{2}\;\frac{N_i + 1}{N_i + 1 - p_i^f} + \mathcal{O}(\delta) \approx \frac{N_i}{2}, \qquad i = 1, \ldots, M-1, \tag{3.3}$$

where h_i is the steady-state occupancy of the ith buffer and E[·] denotes the expectation.

Proof *See* Appendix B.

Remark 3.1 Although condition (3.3) seems somewhat unexpected it is, in retrospect, quite logical. Indeed, the buffer between machines i and $i + 1$ is used to prevent the blockage of machine i and the starvation of machine $i + 1$. Therefore, if this buffer is half full, it offers equal possibilities for alleviating the perturbations for both machines. If the buffer is too full, the production rate of machine i is too high, and p_i can be decreased and reallocated to another machine, possibly machine $i + 1$, so that the PR_{est} of the whole system is increased. Analogously, if buffer i is too empty, machine $i + 1$ works "too fast" and a fraction of p_{i+1} can be transferred to another machine so that PR_{est} is increased. Thus, the status of the buffers—which ones are empty and which ones are full—offers guidance for potential improvements. Therefore, (3.3) is an *indicator of improvability with respect to WF.*

To characterize p_i^*, $i = 1, \ldots, M$, which render the system unimprovable, introduce

$$PR_{est}^* = \max_{\substack{p_1, \ldots, p_M \\ \Pi_{i=1}^M p_i = p^*}} PR_{est}(p_1, \ldots, p_M N_1, \ldots, N_{M-1}). \tag{3.4}$$

Consider the following recursive procedure:

$$x(n+1) = (p^*)^{\frac{1}{M}} \prod_{i=1}^{M-1} \left(\frac{N_i + x(n)}{N_i + 1} \right)^{\frac{2}{M}}. \tag{3.5}$$

THEOREM 3.2 *Assume* $\sum_{i=1}^{M-1} N_i^{-1} \leq M/2$. *Then recursive procedure (3.5) is a contraction on [0, 1]. Its steady state,* x^*, *satisfies the property*

$$x^* = \lim_{n \to \infty} x(n) = PR_{est}^*. \tag{3.6}$$

Moreover, the sequence $p_1^*, \ldots, p_M^*$ *which renders the serial production line (i)–(vi) unimprovable with respect to WF is defined by*

$$p_1^* = \left(\frac{N_1 + 1}{N_1 + PR_{est}^*} \right) PR_{est}^*,$$

$$p_i^* = \left(\frac{N_{i-1} + 1}{N_{i-1} + PR_{est}^*} \right) \left(\frac{N_i + 1}{N_i + PR_{est}^*} \right) PR_{est}^*, \qquad i = 2, \ldots, M-1, \tag{3.7}$$

$$p_M^* = \left(\frac{N_{M-1} + 1}{N_{M-1} + PR_{est}^*} \right) PR_{est}^*.$$

Proof See Appendix B.

THEOREM 3.3 *Serial production line (i)–(vi), with* $N_i = N$, $i = 1, \ldots, M-1$, *and (3.1) satisfied, satisfies the bowl phenomenon:*

$$p_1 = p_M < p_2, \ldots, p_{M-1}.$$

Proof Follows directly from (3.7) taking into account that $PR_{est}^* < 1$.

The bowl phenomenon as an indicator of optimality is well known [21]–[22].

3.2 Improvability with Respect to WF and WIP Simultaneously

As it follows from Definition 1.3, serial production line (i)–(vi) is improvable with respect to WF and WIP simultaneously, if there exist $p_1^*, \ldots, p_M^*$ and $N_1^*, \ldots, N_{M-1}^*$ such that $\Pi_{i=1}^M p_i^* = p^*$ and $\Sigma_{i=1}^{M-1} N_i^* = N^*$ and

$$PR_{est}(p_1^*, \ldots, p_M^*, N_1^*, \ldots, N_{M-1}^*) > PR_{est}(p_1, \ldots, p_M, N_1, \ldots, N_{M-1}).$$

THEOREM 3.4 *Let N^* be an integer multiple of $M-1$. Then serial production line (i)–(vi) is unimprovable with respect to WF and WIP simultaneously, if and only if (3.1) takes place and, in addition,*

$$p_i^f = p_i^b, \qquad i = 2, \ldots, M-1 \tag{3.8}$$

Proof *See* Appendix B.

COROLLARY 3.2 *If condition (3.8) is satisfied, each intermediate machine is blocked and starved with practically the same frequency, that is,*

$$|b_i - s_i| \backsim \mathcal{O}(\delta), \quad i = 2, \ldots, M-1. \tag{3.9}$$

where b_i, s_i, and δ are defined in (3.2) and (2.5), respectively.

Proof *See* Appendix B.

Remark 3.2 Condition (3.9) also can be given a simple interpretation. Buffers $i-1$ and i serve to prevent starvations and blockages of machine i, respectively. Thus, if machine i is blocked more often than it is starved, buffer $i-1$ can be reduced and the excess capacity could be added to another buffer, possibly buffer i. Analogously, if machine i is starved more often than blocked, N_i should be reduced and N_{i-1} increased. Conditions (3.3) and (3.9) are *indicators of improvability with respect to WF and WIP, simultaneously.*

The values of N_i^* and p_i^* which render the system unimprovable with respect to WF and WIP simultaneously can be characterized as follows:

THEOREM 3.5 *Let N^* be an integer multiple of $M-1$, and let*

$$PR_{est}^{**} = \max_{\substack{p_1, \ldots, p_M;\ \prod_{i=1}^{M} p_i = p^* \\ N_1, \ldots, N_{M-1};\ \sum_{i=1}^{M} N_i = N^*}} PR_{est}(p_1, \ldots, p_M, N_1, \ldots, N_{M-1})$$

Then conditions (3.1) and (3.8) are satisfied if and only if

$$\begin{aligned} p_1^* &= p_M^* = \left(\frac{N_1^* + 1}{N_1^* + PR_{est}^{**}}\right) PR_{est}^{**}, \\ p_i^* &= \left(\frac{N_i^* + 1}{N_i^* + PR_{est}^{**}}\right) PR_{est}^{**}, \qquad i = 2, \ldots, M-1, \\ N_i^* &= \frac{N^*}{M-1}, \qquad i = 1, \ldots, M-1. \end{aligned} \tag{3.10}$$

Proof *See* Appendix B.

Thus, a system is unimprovable with respect to WF and WIP simultaneously, if and only if all buffers are of equal capacity (no bowl phenomenon occurs). The isolation production rates of machines i, $i = 2, \ldots, M-1$, are also the same; the isolation production rates of machines 1 and M are somewhat smaller than those for the other machines, as defined by the first two expressions in (3.10).

3.3 Improvability with Respect to WIP

From Definition 1.1, serial production line (i)–(vi) is improvable with respect to WIP if there exists $N_1^*, \ldots, N_{M-1}^*$ such that $\Sigma_{i=1}^{M-1} N_i^* = N^*$ and

$$PR_{est}(p_1, \ldots, p_M, N_1^*, \ldots, N_{M-1}) > PR_{est}(p_1, \ldots, p_M, N_1, \ldots, N_{M-1}).$$

Theorem 3.6 *Serial production line (i)–(vi) is unimprovable with respect to WIP if and only if the quantity*

$$\min_{i=1\ldots,M} p_i \left(\min \left\{ \frac{p_i^f}{p_i^b}, \frac{p_i^b}{p_i^f} \right\} \right) \tag{3.11}$$

is maximized over all sequences $N_1, \ldots, N_{M-1}$ such that $\Sigma_{i=1}^{M-1} N_i = N^*$.

Proof See Appendix B.

Unfortunately, this result is of little practical significance, mainly due to the fact that the first and the last machines (which are never starved or blocked, respectively) are included in (3.11). In addition, the interpretation of (3.11) is much less obvious than that of (3.2). To alleviate these difficulties, we modify (3.11) to a form similar to (3.9) with the provision, however, that the p_i' s, $i = 2, \ldots, M - 1$, may not necessarilty be equal to each other:

Criterion 3.1 Serial production line (i)–(vi) is practically unimprovable with respect to WIP if the quantity

$$\max_{i=2,\ldots,M-1} \frac{1}{p_i} |b_i - s_i| \tag{3.12}$$

is minimized over all sequences N_i, $i = 1, \ldots, M - 1$, such that $\Sigma_{i=1}^{M-1} N_i = N^*$, where b_i and s_i are defined in (3.2).

The term "practically unimprovable" is used here in the following sense:

Numerical Fact 3.1 *Let $N_1^*, \ldots, N_{M-1}^*$ be the sequence which maximizes (3.11). Let $N_1^{**}, \ldots, N_{M-1}^{**}$ be the sequence which minimizes (3.12). Then*

$$|PR_{est}(p_1, \ldots, p_M, N_1^*, \ldots, N_{M-1}^*) - PR_{est}(p_1, \ldots, p_M, N_1^{**}, \ldots, N_{M-1}^{**})| =: \delta_A << 1.$$

This fact has been established on the basis of extensive computer simulation. Several illustrations are given in Table II for systems with $M = 4$, $N^* = 10$, p_i as indicated, and where $PR_{est}^* = PR_{est}(p_1, \ldots, p_4, N_1^*, N_2^*, N_3^*)$.

Remark 3.3 Criterion 3.1 can be given a simple interpretation: A system is practically unimprovable with respect to WIP if the weighted (by the inverse of the isolation production rates, $1/p_i$) differences between blockages and starvations for each intermediate machine are as close to each other as possible. This, in particular, implies that machines

Table II Illustration of Numerical Facts 3.1 and 3.2

p_1	p_2	p_3	p_4	PR^*_{est}	δ_A	δ_B
0.80	0.80	0.80	0.80	0.7211	0	0
0.70	0.80	0.70	0.80	0.6496	0	0
0.70	0.90	0.70	0.90	0.6795	0.0016	0.0016
0.60	0.99	0.99	0.60	0.6000	0.0024	0.0201
0.99	0.60	0.60	0.99	0.5679	0	0

with smaller p_i's should have $|b_i - s_i|$ smaller than those with larger p_i's (protection of the "bottlenecks"). Condition (3.12), rewritten in the form

$$\frac{1}{p_i} | b_i - s_i | \approx \text{const}, \qquad i = 2, \dots, M-1, \tag{3.13}$$

is referred to as an *indicator of improvability with respect to WIP.*

At present, we do not have an analytical expression for the sequence $N_1^{**}, \dots, N_{M-1}^{**}$ which minimizes (3.12). A procedure, however, has been developed which approaches this distribution:

Procedure 3.1

(a) Consider the line defined by (i)–(vi) and, using (2.1), calculate $\tilde{b}_i = 1 - p_i^b/p_i$ *and* $\tilde{s}_i = 1 - p_i^f/p_i$, $i = 2, \dots, M-1$. Note that due to Lemma A. 6, $|\tilde{b}_i - b_i| \sim \mathcal{O}(\delta)$ and $|\tilde{s}_i - s_i| \sim \mathcal{O}(\delta)$, $i = 1, \dots, M$. Calculate

$$t_i = \frac{1}{p_i} | \tilde{b}_i - \tilde{s}_i |, \qquad i = 2, \dots, M-1.$$

Let machine i^* be the machine with the largest t_i.

(b) If machine i^* is blocked more often than starved (*i.e.*, $p_{i^*}^b < p_{i^*}^f$), transfer one buffer slot from buffer $i^* - 1$ to buffer i^*. If machine i^* is starved more often than blocked (*i.e.*, $p_{i^*}^b > p_{i^*}^f$), transfer one buffer slot from buffer i^* to buffer $i^* - 1$.

(c) Calculate PR'_{est} defined by the new buffer distribution. If $PR'_{est} > PR_{est}$ go to step (a). If $PR'_{est} \leq PR_{est}$ then stop, retaining the previous buffer distribution.

NUMERICAL FACT 3.2 *Let* $N_1^*, \dots, N_{M-1}^*$ be the sequence which maximizes (3.11). Let $N_1^{***}, \dots, N_{M-1}^{***}$ be the sequence obtained from Procedure 3.1. Then

$$| PR_{est}(p_1, \dots, p_M, N_1^*, \dots, N_{M-1}^*) - PR_{est}(p_1, \dots, p_M, N_1^{***}, \dots, N_{M-1}^{***}) | =: \delta_B << 1.$$

This fact has been established through extensive computer simulation. Several illustrations are given in Table II, starting in each case from the initial buffer distribution $N_1 = 1$, $N_2 = 1$, $N_3 = 8$.

4. CONSTRAINT RELAXATION

4.1. General Considerations

In the previous section, improvability under the constraints $\Sigma_{i=1}^{M-1} N_i = N^*$ and $\Pi_{i=1}^{M} p_i = p^*$ has been analyzed. From Theorem 2.3 it is clear that an increase in a particular N_i

or p_i, so that $\Sigma_{i=1}^{M-1} N_i > N^*$ or $\Pi_{i=1}^{M} p_i > p^*$, never leads to a decrease in PR_{est}. The questions, however, arise: Which particular p_i should be increased so that the largest possible increase in PR_{est} is obtained? Which N_i should be increased so that the best possible effect is obtained? Finally, how much should N^* be increased so that an appreciable increase in PR_{est} is observed? These are the questions addressed in this section.

4.2. Bottleneck Machine

In section 1, the bottleneck machine has been defined (Definition 1.4) as the machine which leads to the largest incremental increase of the system performance index when the production rate of the machine is increased. In terms of the serial production line (i)–(vi), machine i is the bottleneck if

$$\frac{\partial PR_{est}}{\partial p_i} \geq \frac{\partial PR_{est}}{\partial p_j}, \ \forall j \neq i.$$

As it has been alluded to in section 1, the machine with the smallest p_i is not necessarily the bottleneck. An example is given in Table III, where not the worst but the best machine turns out to be the bottleneck. This happens because the bottleneck property depends not only on the machines, but also on the buffers. Therefore, in general, the determination of the bottleneck machine is a non-trivial problem. In systems unimprovable with respect to WF, however, this problem becomes quite simple due to the following property:

THEOREM 4.1 *In serial production lines unimprovable with respect to WF, the following property holds:*

$$p_i \frac{\partial PR_{est}}{\partial pi} = const, \qquad i = 1, \dots, M. \tag{4.1}$$

Proof See Appendix C.

Relationship (4.1) implies that the machine with the smallest p_i corresponds to the largest $\partial \mathrm{PR}_{est} / \partial p_i$. Therefore, in lines unimprovable with respect to WF the bottleneck machine can be found easily—it is the machine with the smallest production rate in isolation.

4.3. Bottleneck Buffer

As defined in section 1, buffer i^* is the bottleneck buffer of the line (i)–(vi) if

$$PR_{est}(p_1, \dots, p_M, N_1, \dots, N_{i^*} + 1, \dots, N_{M-1}) > PR_{est}(p_1, \dots, p_M, N_1, \dots, N_j + 1, \dots, N_{M-1}), \forall j \neq i^*.$$

As it is the case with the bottleneck machines, the smallest buffer is not necessarily the bottleneck. However, in lines unimprovable with respect to WIP the search for the

TABLE III Bottleneck example

i	1	2	3	4
p_i	0.80	0.83	0.77	0.80
N_i	2	2	5	—
$\partial PR_{est} / \partial p_i$	0.369	0.452	0.443	0.022

bottleneck buffer becomes quite simple. Indeed, assume that a line is unimprovable in the sense that Procedure 3.1 has been carried out. Let machine i^* be the machine with the largest t_i. Then either buffer $i^* - 1$ (if $p^b_{i^*} > p^f_{i^*}$) or buffer i^* (if $p^b_{i^*} < p^f_{i^*}$) is the bottleneck buffer in the following sense:

NUMERICAL FACT 4.1 Let $N^*_1, \ldots, N^*_{M-1}$ be the assignment of N^* buffer spaces according to Procedure 3.1 and let machine i^* be the machine with the largest t_i. Let $N^{**}_1, \ldots, N^{**}_{M-1}$ be the assignment of $N^* + 1$ buffer spaces according to Procedure 3.1. Then

$$\begin{aligned} &| PR_{est}(p_1, \ldots, p_M, N^*_1, \ldots, N^*_{i^*-1} + 1, \ldots, N^*_{M-1}) - \\ &PR_{est}(p_1, \ldots, p_M, N^{**}_1, \ldots, N^{**}_{M-1}) | =: \delta_C << 1, \end{aligned}$$

if $p^b_{i^*} > p^f_{i^*}$, or

$$\begin{aligned} &| PR_{est}(p_1, \ldots, p_M, N^*_1, \ldots, N^*_{i^*} + 1, \ldots, N^*_{M-1}) - PR_{est}(p_1, \ldots, p_M, N^{**}_1, \ldots, N^{**}_{M-1}) | \\ &=: \delta_C << 1, \end{aligned}$$

if $p^b_{i^*} < p^f_{i^*}$.

This fact has been arrived at through numerical studies. It is illustrated in Table IV for several lines with $N^* = 10$, and starting Procedure 3.1 in each case with $N_1 = 1, N_2 = 1$, and all remaining buffer slots allocated to N_3.

4.4. The Choice of N^*

As it is clear from the above, an increase in N^* leads to an increase in PR_{est}. However, there exists an N^* such that any further increase in its value leads to an insignificant increase in PR_{est}. This phenomenon is captured in Definition 1.6 of N^* being ϵ-adapted to $p_1, \ldots, p_M$ (section 1). For serial production line (i)–(vi), N^* is ϵ-adapted to $p_1, \ldots, p_M$ if

$$\frac{| PR_{est}(N^*) - PR_{est}(N) |}{PR_{est}(N^*)} < \epsilon, \qquad \forall N \geq N^*,$$

where $PR_{est}(N^*) = PR_{est}(p_1, \ldots, p_M, N_1, \ldots, N_{M-1}), N^* = \Sigma^{M-1}_{i=1} N_i$, and $N_1, \ldots, N_{M-1}$ are calculated according to Procedure 3.1. To characterize this N^*, define τ as the time that each part spends, on the average, from the moment it enters the first machine until the moment it is processed by the last machine. From Little's Law and Corollary 3.1, for a line (i)–(vi) unimprovable with respect to WF,

Table IV Illustration of Numerical Fact 4.1

p_1	p_2	p_3	p_4	PR^{**}_{est}	δ_c
0.80	0.80	0.80	0.80	0.7272	0.0004
0.70	0.80	0.70	0.80	0.6575	0.0020
0.70	0.90	0.70	0.90	0.6787	0.0053
0.60	0.99	0.99	0.60	0.5817	0.0130
0.99	0.60	0.60	0.99	0.5709	0

$$\tau \approx \frac{N^*}{2\, PR_{est}(p_1^*, \ldots, p_M^*, N_1, \ldots, N_{M-1})}.$$

For a line unimprovable with respect to WIP,

$$\tau \leq \frac{N^*}{PR_{est}(p_1, \ldots, p_M, N_1^*, \ldots, N_{M-1}^*)} = T(N^*).$$

Typical behaviors of $T(N^*)$ and $PR_{est}(N^*)$ are illustrated in Figures 1 and 2, respectively. Since $PR_{est}(N^*)$ exhibits saturation, the increase of N^* beyond a certain value leads only to a practically linear increase of the residence time estimate $T(N^*)$ without a meaningful increase in PR_{est}. When the work-in-process is allowed to become infinite, the average production rate of the line becomes equal to the average production rate of the slowest machine, that is, the machine with the smallest p_i. Thus, N^* which is ϵ-adapted to $p_1, \ldots, p_M$, can be determined from the equation

$$PR_{est}(N^*) = \frac{1}{1+\epsilon} \min_{i=1,\ldots,M} p_i. \tag{4.2}$$

This N^*, being distributed according to Procedure 3.1, represents a compromise between the competing goals of ensuring a high average production rate and maintaining low work-in-process inventory.

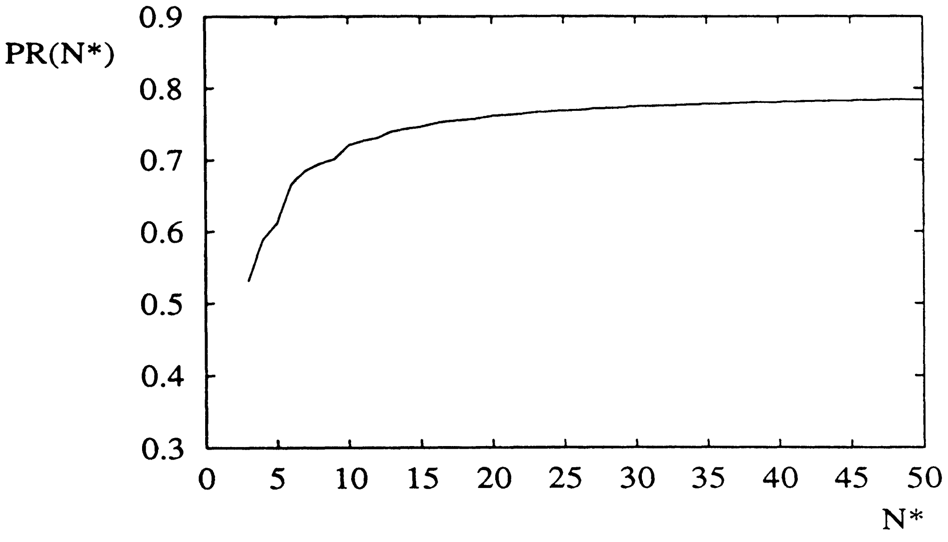

Figure 1 Production rate of the line with buffers allocated according to Criterion 3.1, $M = 4$, and $p_i = 0.8$, $i = 1, \ldots, 4$.

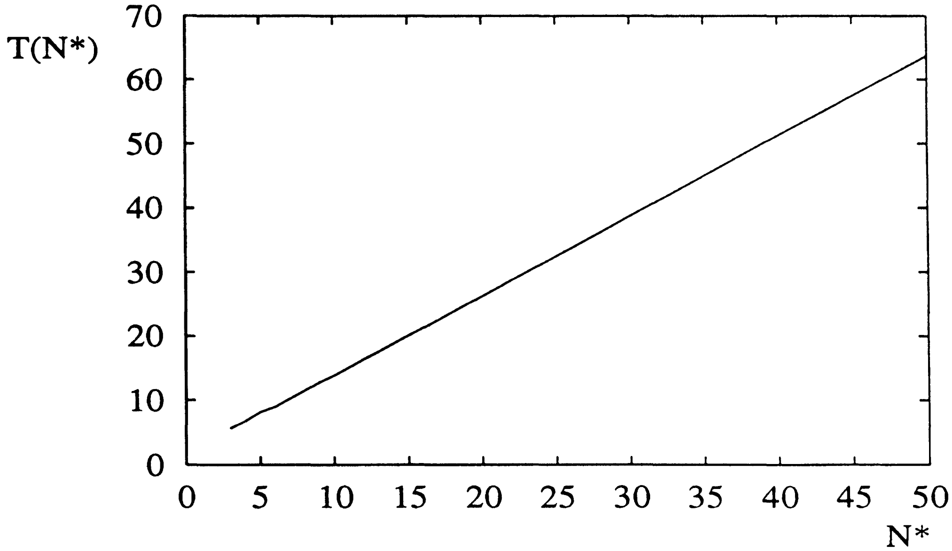

Figure 2 Time constant upper bound for the line with buffers allocated according to Criterion 3.1, $M = 4$, and $p_i = 0.8$, $i = 1, \ldots, 4$.

5. RECOMMENDATIONS FOR THE PROCESS OF CONTINUOUS IMPROVEMENT

Based on the properties of improvability described above and on the experience gained in several applications at automotive plants, we formulate the following guidelines for the process of continuous improvement in large volume production systems:

1. Verify assumptions (i)–(vi). If they are not met, the methods prescribed here are, strictly speaking, not applicable. If they are, proceed with the next step.
2. Verify whether
   (α) Each buffer is, on the average, close to being half full,
   (β) $1/p_i$ (starvation frequency of machine i—blockage frequency of machine i) ≈ const for all $i = 2, \ldots M-1$.
   If either (α) or (β) or both are violated, the line is improvable (under constraints) in the appropriate sense (WF, WIP, or WF and WIP).
3. Identify parameters p_i and N_i. The p_i's can be obtained from the up-time records. The N_i's can be evaluated as the capacity of buffers, conveyors, accumulators, and so on.
4. If only the p_i's are assignable (*i.e.*, the N_i's are fixed) and (α) is not satisfied, using the p_i's and N_i's identified and Theorem 3.2, calculate $p^*_1, \ldots, p^*_M$.
5. If (β) is not satisfied then either use Theorem 3.5 to calculate $p^*_1, \ldots, p^*_M$ and $N^*_1, \ldots, N^*_{M-1}$ if the p_i's and N_i's are assignable, or Procedure 3.1 to redistribute the WIP if only the N_i's are assignable.
6. If both (α) and (β) are satisfied, the system can be improved only by relaxing the constraints. To accomplish this

(α) Using Definition 1.4 and Theorem 4.1, find the bottleneck machine; improve its performance in isolation (*i.e.*, increase the corresponding p_i) by any means available.

(β) Choose an appropriate ϵ, and using (4.2), determine N^* so that the WIP is ϵ-adapted to $p_1, \ldots, p_M$. Increase or decrease N^* appropriately, and determine $N_1^*, \ldots, N_{M-1}^*$ using either Theorem 3.5 or Procedure 3.1.

7. Implement any of the recommendations 4–6 and, if necessary, go back to step 2.

Although these recommendations represent quite an arbitrary way of utilization of the improvability indicators, they have proven to be quite useful in a number of practical applications.

6. CONCLUSIONS

In this paper, we described the laws which govern the behavior of serial production lines defined by assumptions (i)–(vi). According to these laws,

(α) A production system is well designed if each buffer is, on the average, half full.

(β) A production system is well designed if all intermediate machines have weighted (by the inverse of the isolation production rate) differences between the frequencies of blockage and starvation as small as possible.

(γ) In a well-designed system the maximal WIP is adapted to the machines' isolation production rates.

If these indicators are violated, the system performance can be improved by, first, redistributing WIP and WF and, second, by eliminating the bottleneck machines and buffers. The methods and guidelines for these continuous improvement measures are described.

APPENDIX A. PROOFS FOR SECTION 2

The logic of the proof of Lemma 2.1 and Theorem 2.1 is as follows:

First, if the distribution of the last buffer occupancy, $X_{M-1}(\cdot)$, is known, the production rate can be calculated immediately as

$$PR = (1 - X_{M-1}(0))\, p_M.$$

Second, $X_i(\cdot)$, $i = 1, \ldots, M-1$, can be evaluated, under Numerical Fact 2.1, with accuracy $\mathcal{O}(\delta)$ if the conditional probabilities $\tilde{p}_i^f = \text{Prob}\{m_i \text{ produces a part} \mid m_i \text{ is not blocked}\}$ and $\tilde{p}_i^b = \text{Prob}\{m_i \text{ produces a part} \mid m_i \text{ is not starved}\}$, $i = 1, \ldots, M$, are known (Lemma A.7).

Third, these conditional probabilities are $\mathcal{O}(\delta)$ close to p_i^f and p_i^b, $i = 1, \ldots, M$, the limits of the sequences $p_i^f(s)$ and $p_i^b(s)$, $s = 0, 1, \ldots$, generated by the recursive procedure (2.1) (Lemma A.10). Finally, these limits do exist (Lemma 2.1).

To prove Lemma 2.1, we need the following facts:

LEMMA A.1 *Function* $Q(x, y, N)$, $0 < x < 1$, $0 < y < 1$, $N \in Z_+$, *defined in (2.2), has the following properties:*

(a) *monotonically decreasing in x,*
(b) *monotonically increasing in y,*
(c) *monotonically decreasing in N,*
(d) *takes values in (0, 1).*

Proof For $x \neq y$, re-write $Q(x, y, N)$ as follows:

$$Q(x, y, N) = \frac{1-x}{\left[\dfrac{1-\frac{x}{y}\alpha^N}{1-\alpha}\right]} = \frac{1-x}{\left[\dfrac{1-\alpha^N}{1-\alpha}\right]+\left[\dfrac{(1-\frac{x}{y})\,\alpha^N}{1-\alpha}\right]}.$$

Substituting the expression for α we obtain:

$$Q(x, y, N) = \frac{1-x}{1+\alpha+\alpha^2+\cdots+\alpha^{N-1}+\dfrac{(y-x)\,y(1-x)}{y(y(1-x)-x(1-y))}\alpha^N}$$

$$= \frac{1-x}{1+\alpha+\alpha^2+\cdots+\alpha^{N-1}+(1-x)\alpha^N} \qquad \text{(A.1)}$$

$$= \frac{1-x}{1+\alpha+\alpha^2+\cdots+\alpha^{N-2}+(1+\dfrac{x(1-y)}{y})\,\alpha^{N-1}}.$$

Statement (a) (or (b) or (c)) follows from this expression since the numerator is monotonically decreasing in x, (constant in y and N), and the denominator is monotonically increasing in x (decreasing in y and increasing in N). Statement (d) follows from this expression since the numerator is in (0,1) and the denominator is a positive number greater than 1.

For $x = y$, expression (A.1) holds again, since in this case $\alpha = 1$. Therefore, properties (*a*)–(*d*) hold in this case as well.

LEMMA A.2 *Consider* $p_i^f(s)$ *and* $p_i^b(s)$, $i = 1, \ldots, M$, *defined by recursive procedure (2.1). If for all* $j = 2, \ldots, M$, $p_j^f(s) < p_j^f(s-1)$, *then for all* $j = 1, \ldots, M-1$, $p_j^b(s+1) > p_j^b(s)$.

Proof By induction: For $j = M-1$, using Lemma A.1, from (2.1) and the assumptions of Lemma A.2, we obtain:

$$\begin{aligned} p_{M-1}^b(s+1) &= p_{M-1}\,[1 - Q(p_M, p_{M-1}^f(s), N_{M-1})] \\ &> p_{M-1}\,[1 - Q(p_M, p_{M-1}^f(s-1), N_{M-1})] \\ &= p_{M-1}^b(s). \end{aligned}$$

For $j = M - 2, M - 3, \ldots, 2, 1$, we write

$$\begin{aligned} p_j^b(s+1) &= p_j[1 - Q(p_{j+1}^b(s+1), p_j^b(s), N_j)] \\ &> p_j[1 - Q(p_{j+1}^b(s), p_j^f(s), N_j)] \\ &> p_j[1 - Q(p_{j+1}^b(s), p_j^f(s-1), N_j)] \\ &= p_j^b(s). \end{aligned}$$

Lemma A.3 *If for all* $j = 1, \ldots, M - 1$, $p_j^b(s+1) > p_j^b(s)$, *then for all* $j = 2, \ldots, M$, $p_j^f(s+1) < p_j^f(s)$.

Proof Similar to that of Lemma A.2.

Lemma A.4 *Sequences* $p_j^f(s)$ *and* $p_j^b(s)$, $s = 1,2,3, \ldots$, *are monotonically decreasing and increasing, respectively.*

Proof By induction: For $s = 1$, due to Lemma A.1, we have

$$p_j^f(1) = p_j[1 - Q(p_{j-1}^f(1), p_j^b(1), N_j)] < p_j = p_j^f(0), \quad 2 \leq j \leq M.$$

Assume that for $s > 0$,

$$p_j^f(s) < p_j^f(s-1), \quad 2 \leq j \leq M.$$

Then by Lemma A.2,

$$p_j^b(s+1) > p_j^b(s), \quad 1 \leq j \leq M - 1.$$

So, by Lemma A.3

$$p_j^f(s+1) < p_j^f(s), \quad 2 \leq j \leq M.$$

Proof of Lemma 2.1 Since the sequences $p_j^f(s)$ and $p_j^b(s)$, $1 < j < M$, are monotonic (Lemma A.4) and bounded from above and below (Lemma A.1) they are convergent.

The proof of Theorem 2.1 requires the following lemmas:

Lemma A.5 *Serial production line (i)–(vi) with* $M = 2$ *has production rate*

$$PR = p_2[1 - Q(p_1, p_2, N_1)] = p_1[1 - Q(p_2, p_1, N_1)].$$

which is a monotonically increasing function of p_1, p_2, *and* N_1.

Proof Let $X(j, s)$ denote the probability that the buffer contains j parts at time moment s. This is a closed irreducible Markov chain, which therefore converges to a unique equilibrium distribution. Let

$$X(j) = \lim_{s \to \infty} X(j, s), \qquad 0 \leq j \leq N_1.$$

This equilibrium distribution must satisfy the following equilibrium equation of the Markov transition equation:

$$\begin{aligned} X(0) &= (1-p_1)X(0) + (1-p_1)p_2X(1) \\ X(1) &= p_1X(0) + [p_1p_2 + (1-p_1)(1-p_2)]X(1) + (1-p_1)p_2X(2) \\ X(j) &= p_1(1-p_2)X(j-1) + [p_1p_2 + (1-p_1)(1-p_2)]X(j) \\ &\quad + (1-p_1)p_2X(j+1), \qquad 2 \le j \le N_1 - 1 \\ X(N_1) &= p_1(1-p_2)X(N_1-1) + [1-p_2+p_1p_2]X(N_1). \end{aligned} \tag{A.2}$$

Solving equation (A.2), we obtain

$$X(j) = X(0)\left(\frac{1}{1-p_2}\right)\alpha^i, \qquad 1 \le j \le N_1, \tag{A.3}$$

where $\alpha = p_1(1-p_2)/p_2(1-p_1)$. Since $X(0) + X(1) + \cdots + X(N_1) = 1$.

$$X(0)\left[1 + \frac{\alpha}{1-p_2} + \frac{\alpha^2}{1-p_2} + \cdots + \frac{\alpha^{N_1}}{1-p_2}\right] = 1$$

and

$$X(0) = \frac{1-p_2}{\alpha + \cdots + \alpha^{N_1} + 1 - p_2}.$$

After some algebra this simplifies to

$$X(0) = \begin{cases} \dfrac{(1-p_1)(1-\alpha)}{1-\dfrac{p_1}{p_2}\alpha^{N_1}}, & p_1 \ne p_2 \\[2ex] \dfrac{1-p_1}{N_1+1-p_1}, & p_1 = p_2. \end{cases} \tag{A.4}$$

For the line produce a part during a cycle, the second machine must be operational and not starved. Therefore, the production rate, PR, can be calculated as follows:

$$PR = p_2(1 - X(0)) = p_2[1 - Q(p_1, p_2, N_1)], \tag{A.5}$$

where $Q(x, y, N)$ is defined by (2.2).

From (A.3) and (A.4), after simplifying, we have:

$$X(N_1) = \begin{cases} \dfrac{1 - \dfrac{1}{\alpha}}{1 - \dfrac{p_2}{p_1}\left(\dfrac{1}{\alpha}\right)^{N_1}}, & p_1 \neq p_1 \\[2ex] \dfrac{1}{N_1 + 1 - p_1}, & p_1 = p_2. \end{cases} \tag{A.6}$$

Since the first machine produces a part if it is operational and not blocked,

$$PR = p_1(1 - (1 - p_2)X(N_1)) = p_1[1 - Q(p_2, p_1, N_1)]. \tag{A.7}$$

The monotonicity of the production rate in p_1 (or p_2 or N_1) follows directly from (A.5) (or (A.7)) and Lemma A.1.

Introduce the following conditional probabilities:

$$\begin{aligned} \tilde{p}_i^f &= \text{Prob}\{m_i \text{ produces} \mid m_i \text{ is not blocked}\}, \\ \tilde{p}_i^b &= \text{Prob}\{m_i \text{ produces} \mid m_i \text{ is not starved}\}. \end{aligned} \tag{A.8}$$

These probabilities play a crucial role in the proof of Theorem 2.1 Specifically, we show below (Lemma A.7) that if $\tilde{p}_i^f$ and $\tilde{p}_{i+1}^b$ are known, then the stationary probability distribution of buffer occupancy, $X_i(\cdot)$, can be calculated with the error $\mathcal{O}(\delta)$. Further, Lemma A.10 shows that $\tilde{p}_i^f$*and* $\tilde{p}_{i+1}^f$ can be calculated from the steady state of recursive procedure (2.1) with the error $\mathcal{O}(\delta)$. Therefore, since the production rate can be calculated from $PR = (1 - X_{M-1}(0))p_M$, the claim of Theorem 2.1 will follow.

LEMMA A.6 *The conditional probabilities* $\tilde{p}_i^f$, $\tilde{p}_i^b$, *take the following forms:*

(*a*) $\tilde{p}_i^f = p_i[1 - X_{i-1}(0)] + \mathcal{O}(\delta), \quad i = 2, \dots, M,$

(*b*) $\tilde{p}_i^b = p_i\left[1 - \sum_{j=i+1}^{M} \left(\prod_{r=i+1}^{j-1} p_r\right)(1 - p_j) X_{i,\dots,j-1}(N_i, \dots, N_{j-1})\right]$
$+ \mathcal{O}(\delta), i = 1, \dots, M - 1,$

where $X_{i,\dots,j}(h_i, \dots, h_j)$ *is the steady-state probability that consecutive buffers* $i, \dots, j$ *in production line (i)–(vi) contain* $h_i, \dots, h_j$ *parts, respectively, and* δ *is defined in (2.5).*

Proof The probability that machine i is blocked can be expressed as follows:

$$\text{Prob}\{m_i \text{ is blocked}\} = \sum_{j=i+1}^{M} \left(\prod_{r=i+1}^{j-1} p_r\right)(1 - p_j) X_{i,\dots,j-1}(N_i, \dots, N_{j-1}). \tag{A.9}$$

Since machine i is not starved when buffer $i - 1$ contains one or more parts, using the conditional probability formula and the definition of δ, we write:

$$\mathrm{Prob}\{m_i \text{ is blocked} \mid m_i \text{ is not starved}\} =$$

$$\begin{aligned}
&= \sum_{j=i+1}^{M} \left(\prod_{r=i+1}^{j-1} p_r \right) (1 - p_j) \frac{\sum_{c=1}^{N_{i-1}} X_{i-1,\ldots,j-1}(c, N_i, \ldots, N_{j-1})}{1 - X_{i-1}(0)}, \\
&= \sum_{j=i+1}^{M} \left(\prod_{r=i+1}^{j-1} p_r \right) (1 - p_j) \frac{X_{i,\ldots,j-1}(N_i, \ldots, N_{j-1}) - X_{i-1,\ldots,j-1}(0, N_i, \ldots, N_{j-1})}{1 - X_{i-1}(0)}, \\
&= \sum_{j=i+1}^{M} \left(\prod_{r=i+1}^{j-1} p_r \right) (1 - p_j) \frac{X_{i,\ldots,j-1}(N_i, \ldots, N_{j-1}) - X_{i-1}(0)\, X_{i,\ldots,j-1}(N_i, \ldots, N_{j-1})}{1 - X_{i-1}(0)} \\
&\quad + \mathcal{O}(\delta), \\
&= \sum_{j=i+1}^{M} \left(\prod_{r=i+1}^{j-1} p_r \right) (1 - p_j)\, X_{i,\ldots,j-1}(N_i, \ldots, \mathrm{N}_{j-1}) + \mathcal{O}(\delta).
\end{aligned} \tag{A.10}$$

From here and equation (A.9) we obtain

$$\mathrm{Prob}\{m_i \text{ is blocked} \mid m_i \text{ is not starved}\} = \mathrm{Prob}\{m_i \text{ is blocked}\} + \mathcal{O}(\delta).$$

Using repeatedly the conditional probability formula, the definition of $\tilde{p}_i^f$, and equation (A.10), we obtain

$$\begin{aligned}
\tilde{p}_i^f &= \mathrm{Prob}\{m_i \text{ produces} \mid m_i \text{ is not blocked}\} \\
&= \mathrm{Prob}\{m_i \text{ is up, not blocked, and not starved} \mid m_i \text{ is not blocked}\} \\
&= \frac{\mathrm{Prob}\{m_i \text{ is up, not blocked, and not starved}\}}{\mathrm{Prob}\{m_i \text{ is not blocked}\}} \\
&= \mathrm{Prob}\{m_i \text{ is not starved}\} \frac{\mathrm{Prob}\{m_i \text{ is up not blocked} \mid m_i \text{ is not starved}\}}{\mathrm{Prob}\{m_i \text{ is not blocked}\}} \\
&\qquad \bullet\ \mathrm{Prob}\{m_i \text{ is up} \mid m_i \text{ is not blocked or starved}\} \\
&= p_i (1 - X_{i-1}(0)) \frac{\mathrm{Prob}\{m_i \text{ is not blocked} \mid m_i \text{ is not starved}\}}{\mathrm{Prob}\{m_i \text{ is not blocked}\}}
\end{aligned}$$

$$= p_i (1 - X_{i-1}(0)) \frac{1 - \text{Prob}\{m_i \text{ is blocked} \mid m_i \text{ is not starved}\}}{1 - \text{Prob}\{m_i \text{ is blocked}\}}$$

$$= p_i (1 - X_{i-1}(0)) + \mathcal{O}(\delta).$$

This proves statement (a) of the lemma. Statement (b) is proved analogously.

Consider now $(M-1)$ two machine, one buffer lines $\tilde{L}_i$, $i = 1, \ldots, M-1$, where the first machine is defined by $\tilde{p}_i^f$, the second by $\tilde{p}_{i+1}^b$, and the buffer is of capacity N_i. Let $\tilde{X}_i(\cdot)$ be the equilibrium probability distribution of buffer occupancy of line $\tilde{L}_i$. Along with these $M-1$ lines, consider the line (i)–(vi) with M machines. Let $X_i(\cdot)$, as before, be the equilibrium probability distribution of buffer occupancy of buffer i. Then, we have

Lemma A.7 *The following property holds:*

$$|\tilde{X}_i(j) - X_i(j)| \sim \mathcal{O}(\delta), \qquad i = 1, \ldots, M-1, \qquad j = 0, \ldots, N_i,$$

where δ is defined by (2.5).

Proof Consider line (i)–(vi) with M machines. Let $K_i = [k_i \ldots k_{i-1}, k_{i+1}, \ldots k_{M-1}]^T$, $1 \le i \le M-1$, $0 \le k_j \le N_j$, $j \neq i$, be an $(M-2)$-dimensional vector. Let $Y_i(h_i, K_i)$, $1 \le i \le M-1$, denote the probability that there are h_i parts in buffer i and k_j, parts in buffer j, $\forall j \neq i$. Since line (i)–(vi) can be described by an ergodic Markov chain with states $Y_i(h_i, K_i)$, in the steady state we write:

$$Y_i(0, K_i) = \sum_{K_i'} Y_i(0, K_i') \text{Prob}\{m_i \text{ does not produce} \mid 0, K_i'\} \text{ Prob }\{K_i' \to K_i \mid 0 \to 0\}$$

$$+ \sum_{K_i'} Y_i(1, K_i') \text{ Prob }\{m_i \text{ does not produce, } m_{i+1} \text{ produces} \mid 1, K_i'\} \text{ Prob }\{K_i' \to K_i \mid 1 \to 0\}.$$

where $\text{Prob}\{m_i \text{ does not produce} \mid h_i, K_i\}$ denotes the conditional probability that machine i does not produce a part during a cycle, given that buffer i contains h_i parts and buffer j contains k_j parts, $\forall j \neq i$, and $\text{Prob}\{K_i' \to K_i \mid h_i' \to h_i'\}$ denotes the conditional probability of the transition from the state where buffer j, $j \neq i$, contains k_j' parts to the state where buffer j contains k_j parts, given that the number of parts in buffer i changes from h_i' to h_i. Summation over all $K_i \in \mathbb{R}^{M-2}$ yields

$$X_i(0) = \sum_{K_i'} Y_i(0, K_i') \text{Prob}\{m_i \text{ does not produce} \mid 0, K_i'\} \sum_{K_i} \text{Prob }\{K_i' \to K_i \mid 0 \to 0\} +$$

$$\sum_{K_i'} Y_i(1, K_i') \text{Prob}\{m_i \text{ does not produce, } m_{i+1} \text{ produces} \mid 1, K_i'\} \sum_{K_i} \text{Prob }\{K_i' \to K_i \mid 1 \to 0\}.$$

Since $\sum_{K_i} \text{Prob }\{K_i' \to K_i \mid 0 \to 0\} = 1$,

$$X_i(0) = \sum_{K_i'} Y_i(0, K_i') \operatorname{Prob}\{m_i \text{ does not produce} \mid 0, K_i'\} \tag{A.11}$$

$$+ \sum_{K_i'} Y_i(1, K_i') \operatorname{Prob}\{m_i \text{ does not produce}, m_i +_1 \text{ produces} \mid 1, K_i'\}.$$

Consider now the first term on the right hand side of equation (A.11):

$$\sum_{K_i'} Y_i(0, K_i') \operatorname{Prob}\{m_i \text{ does not produce} \mid 0, K_i'\}$$

$$= \sum_{\substack{K_i' \text{ such that} \\ k_{i-1} \geq 1}} Y_i(0, K_l') \operatorname{Prob}\{m_i \text{ does not produce} \mid 0, K_i'\} \tag{A.12}$$

$$+ \sum_{\substack{K_1' \text{ such that} \\ k_{i-1=0}}} Y_i(0, K_i') \operatorname{Prob}\{m_i \text{ does not produce} \mid 0, K_i'\}.$$

When buffer $i - 1$ contains at least one part, machine i is not starved, and when buffer i contains zero parts machine i is not blocked. Therefore, the probability in the first term on the right hand side of equation (A.12) is equal to $1 - p_i$. When buffer $i - 1$ contains zero parts, machine i is starved, and the probability in the second term on the right hand side of equation (A.12) is equal to one. Consequently,

$$\sum_{K_i'} Y_i(0, K_i') \operatorname{Prob}\{m_i \text{ does not produce} \mid 0, K_i'\}$$

$$= (1 - p_i)[X_i(0) - X_{i-1,i}(0, 0)] + X_{i-1,i}(0, 0)$$

$$= X_i(0)(1 - p_i) + X_{i-1,i}(0, 0)\, p_i. \tag{A.13}$$

Using (2.5), this can be rewritten as

$$\sum_{K_i'} Y_i(0, K_i') \operatorname{Prob}\{m_i \text{ does not produce} \mid 0, K_i'\}$$

$$= X_i(0)(1 - p_i) + X_{i-1}(0)\, X_i(0)\, p_i + \mathcal{O}(\delta)$$

$$= X_i(0)[1 - p_i(1 - X_{i-1}(0))] + \mathcal{O}(\delta).$$

By Lemma A.6, we finally obtain:

$$\sum_{K_i'} Y_i(0, K_i') \operatorname{Prob}\{m_i \text{ does not produce} \mid 0, K_i'\} = X_i(0)(1 - \tilde{p}_i^f + \mathcal{O}(\delta)).$$

Analysis of the second term on the right–hand side of equation (A.11) proceeds analogously and results in

$$X_i(0) = X_i(0)\,(1 - \tilde{p}_i^f + \mathcal{O}(\delta)) + X_i(1)\,(1 - \tilde{p}_i^f)\,\tilde{p}_{i+1}^b.$$

Similar arguments can be used to obtain equations for $X_i(j)$, $j = 1, \ldots, N_i$. As a result, we obtain the following set of equations:

$$\begin{aligned}
X_i(0) &= (1 - \tilde{p}_i^f + \mathcal{O}(\delta))X_i(0) + (1 - \tilde{p}_i^f)\,\tilde{p}_{i+1}^b\,X_i(1)\\
X_i(1) &= \tilde{p}_i^f X_i(0) + [\tilde{p}_i^f \tilde{p}_{i+1}^b + (1 - \tilde{p}_i^f)(1 - \tilde{p}_{i+1}^b) + \mathcal{O}(\delta)]\,X_i(1)\\
&\quad + (1 - \tilde{p}_i^f)\,\tilde{p}_{i+1}^b\,X_i(2) \qquad\qquad (\mathrm{A.14})\\
X_i(j) &= \tilde{p}_i^f(1 - \tilde{p}_{i+1}^b)\,X_i(j-1) + [\tilde{p}_i^f \tilde{p}_{i+1}^b + (1 - \tilde{p}_i^f)\,(1 - \tilde{p}_{i+1}^b) + \mathcal{O}(\delta)]X_i(j)\\
&\quad + (1 - \tilde{p}_i^f)\,\tilde{p}_{i+1}^b\,X_i(j+1) \qquad 2 \le j \le N_i - 1\\
X_i(N_i) &= \tilde{p}_i^f(1 - \tilde{p}_{i+1}^b)\,X_i(N_i - 1) + [\,1 - \tilde{p}_{i+1}^b + \tilde{p}_i^f \tilde{p}_{i+1}^b + \mathcal{O}(\delta]\,X_i(N_i).
\end{aligned}$$

These equations can be written in matrix form as

$$X_i = (A + \Delta A)\,X_i, \qquad X_i = [X_i(0), \ldots, X_i(N_i)]^T \qquad (\mathrm{A.15})$$

where

$$A = \begin{pmatrix}
1 - \tilde{p}_i^f & (1 - \tilde{p}_i^f)\,\tilde{p}_{i+1}^b & & & \\
\tilde{p}_i^f & \tilde{p}_i^f\,\tilde{p}_{i+1}^b + (1 - \tilde{p}_i^f)(1 - \tilde{p}_{i+1}^b) & \ddots & & \\
 & \tilde{p}_i^f(1 - \tilde{p}_{i+1}^b) & \ddots & \ddots & \\
 & & \ddots & \tilde{p}_i^f\,\tilde{p}_{i+1}^b + (1 - \tilde{p}_i^f)\,(1 - \tilde{p}_{i+1}^b) & (1 - \tilde{p}_i^f)\,\tilde{p}_{i+1}^b \\
 & & & \tilde{p}_i^f(1 - \tilde{p}_{i+1}^b) & 1 - \tilde{p}_{i+1}^b + \tilde{p}_{i+1}^b\,\tilde{p}_i^f
\end{pmatrix}$$

$$(\mathrm{A.16})$$

and ΔA is a diagonal matrix with diagonal elements all of the order $\mathcal{O}(\delta)$, and therefore $\|\Delta A\| \sim \mathcal{O}(\delta)$.

As it follows from equation (A.2) of Lemma A.5, the equilibrium distribution of parts $\tilde{X}_i(\cdot)$ of line $\tilde{L}_i$ is described by $\tilde{X}_i = A\tilde{X}_i$, where A is given in equation (A.16). Since A is the state transition matrix of an ergodic Markov chain, $\lambda = 1$ is an eigenvalue of multiplicity 1 of A. Therefore, using the perturbation theory (*see, for instance*, [23]) we obtain:

$$| \tilde{X}_i(j) - X_i(j)| \sim \mathcal{O}(\delta), \qquad 1 \le i \le M-1, \quad 0 \le j \le N_{i.}$$

Lemma A.7 showed that if the conditional probabilities $\tilde{p}^f_i$ and $\tilde{p}^b_i$ are known, then they may be used to estimate the probability distributions to buffer occupancy $X_i(\cdot)$ of line (i)–(vi). Our next goal is to show how p^f_i and p^b_i, the steady-state values determined via recursive procedure (2.1), can be used to determine $\tilde{p}^f_i$ and $\tilde{p}^b_i$. Before we do so, we will need a few preliminary results.

As it follows from Lemma 2.1, the steady state equation of recursive procedure (2.1), that is,

$$\begin{aligned} p^f_i &= p_i[1 - Q(p^f_{i-1}, p^b_i, N_{i-1})], \qquad 2 \le i \le M, \\ p^b_i &= p_i[1 - Q(p^b_{i+1}, p^f_i, N_i)], \qquad 1 \le i \le M-1, \\ p^f_1 &= p_1, \qquad p^b_M = p_M, \end{aligned} \tag{A.17}$$

has at least one solution $P_{agg} = [p^f_1, \ldots, p^f_M, p^b_1, \ldots, p^b_M]^T$. We prove below that this solution is, in fact, unique. To accomplish this we introduce $(M-1)$ two machine one buffer serial production lines, L_i, $i = 1, \ldots, M-1$, where the first machine has the isolation production rate p^f_i, the second p^b_{i+1}, and the buffer capacity is N_i. The following properties hold:

Lemma A.8 *Let PR_i be the production rate of line L_i, $i = 1, \ldots, M-1$, and let $PR_M = p^f_M$. Then $PR_i = p^f_i p^b_i / p_i$, $i = 1, \ldots, M$. Moreover, $PR_i = const$, $\forall i = 1, \ldots, M$.*

Proof From Lemma A.5 and eq. (A.17), for $1 \le i \le M-1$,

$$PR_i = p^f_i[1 - Q(p^b_{i+1}, p^f_i, N_i)] = p_i[1 - Q(p^b_{i+1}, p^f_i, N_i)]\frac{p^f_i}{p_i} = \frac{p^b_i p^f_i}{p_i}, \qquad i = 1, \ldots, M-1,$$

and

$$PR_M = p^f_M = \frac{p^f_M p_M}{p_M} = \frac{p^b_M p^f_M}{p_M}.$$

This proves the first statement of the lemma. Moreover,

$$PR_i = \frac{p^f_i p^b_i}{p_i} = \frac{p^b_i}{p_i} p_i[1 - Q(p^f_{i-1}, p^b_i, N_{i-1})] = p^b_i[1 - Q(p^f_{i-1}, p^b_i, N_{i-1})] = PR_{i-1}, \qquad 1 = 2, \ldots, M.$$

Lemma A.9 *The equilibrium equation (A.17) of recursive procedure (2.1) has a unique solution.*

Proof By contradiction: Assume that along with the solution $P_{agg} = [p_1^f, \ldots, p_M^f, p_1^b, \ldots, p_M^b]^T$ to eq. (A.17), there exists another solution denoted by $\bar{P}_{agg} = [\bar{p}_1^f, \ldots, \bar{p}_M^f, \bar{p}_1^b, \ldots, \bar{p}_M^b]^T$. Suppose that $\bar{p}_1^b > p_1^b$. Then, by Lemma A.8.

$$\overline{PR}_i > PR_i, \; 1 \le i \le M. \tag{A.18}$$

Since $\overline{PR}_1(p_1, \bar{p}_2^b, N_1) > PR_1(p_1, p_2^b, N_1)$, by Lemma A.5 $\bar{p}_2^b > p_2^b$. Therefore, by Lemma A.1,

$$\bar{p}_2^f = p_2\,[1 - Q(p_1, \bar{p}_2^b, N_1)] < p_2\,[1 - Q(p_1, p_2^b, N_1)] = p_2^f$$

Now proceed inductively. Assume $\bar{p}_j^b > p_j^b$ and $\bar{p}_j^f < p_j^f$. The base case ($j = 2$) has already been established. By eq. (A.18), $\overline{\mathrm{PR}}_j(\bar{p}_j^f, \bar{p}_{j+1}^b, N_j) > PR_j(p_j^f, p_{j+1}^b, N_j)$. Since $\bar{p}_j^f < p_j^f$, by Lemma A.5 $\bar{p}_{j+1}^b > p_{j+1}^b$. Using Lemma A.1, and the assumptions that $\bar{p}_j^f < p_j^f$ and $\bar{p}_{j+1}^b > p_{j+1}^b$,

$$\bar{p}_{j+1}^f = p_{j+1}[1 - Q(\bar{p}_j^f, \bar{p}_{j+1}^b, N_j)] < p_{j+1}[1 - Q(p_j^f, p_{j+1}^b, N_j)] = p_{j+1}^f.$$

Thus the inductive hypothesis is established, and therefore $\bar{p}_j^b > p_j^b$ and $\bar{p}_j^f < p_j^f$, $2 \le j \le M$. In particular, $\bar{p}_M^f < p_M^f$, so by Lemma A.8, $\overline{PR}_M < PR_M$, which contradicts eq. (A.18). We therefore conclude that $\bar{p}_1^b \le p_1^b$.

Assuming that $\bar{p}_1^b < p_1^b$, and proceeding analogously, yields $\bar{p}_1^b \ge p_1^b$. Therefore, $\bar{p}_1^b = p_1^b$.

The equality of the remaining components of $\bar{P}_{agg} = P_{agg}$ will be shown by induction. Note that $p_1^f = p_1 = \bar{p}_1^f$, and that $p_1^b = \bar{p}_1^b$. Assume that $p_i^b = \bar{p}_i^b$ and $p_i^f = \bar{p}_i^f$. Let $f(x) = p_j(1 - Q(x, p_j^f, N_j)) - p_j^b$. By Lemma A.1, $Q(x, y, N)$ is a monotonic function of x, so $f(x)$ is also a monotonic function of x. Therefore, $f(x)$ can have at most one root. By the inductive hypothesis, $p_j^f = \bar{p}_j^f$ and $p_j^b = \bar{p}_j^b$, and therefore both $\bar{p}_{j+1}^b$ and p_{j+1}^b must be roots of $f(x)$, which proves $\bar{p}_{j+1}^b = p_{j+1}^b$. It may now be calculated that $\bar{p}_{j+1}^f = p_{j+1}(1 - Q(\bar{p}_j^f, \bar{p}_{j+1}^b, N_j)) = p_{j+1}(1 - Q(p_j^f, p_{j+1}^b, N_j)) = p_{j+1}^f$, which establishes the inductive hypothesis.

Lemma A.7 showed that if the conditional probabilities $\tilde{p}_i^f$ and $\tilde{p}_i^b$, $i = 1, \ldots, M$, are known, then it is possible to determine, approximately, the steady-state buffer occupancy probability distributions $X_i(\cdot)$, $i = 1, \ldots, M - 1$. The task of determining the values of these conditional probabilities, however, remains. Lemma A.10 shows that they are given, approximately, by recursive procedure (2.1).

LEMMA A.10 *The following relationships hold*:

$$|\tilde{p}_i^f - p_i^f| \sim \mathcal{O}(\delta),$$
$$|\tilde{p}_i^b - p_i^b| \sim \mathcal{O}(\delta),$$
$$i = 1, \ldots, M,$$

where p_i^f and p_i^b are given in (2.3) and δ is defined in (2.5).

Proof Let $\tilde{X}_i(\cdot)$ be the equilibrium probability distribution of buffer occupancy of line $\tilde{L}_i$, $i = 1, \ldots, M-1$, as described earlier, and let $X_i(\cdot)$ be the equilibrium probability distribution of buffer occupancy for buffer i of line (i)–(vi). Let the conditional probabilities $\tilde{p}_i^f$ and $\tilde{p}_i^b$, $i = 1, \ldots, M$, be as defined in eq. (A.8). Then, by Lemma A.6, $\tilde{p}_i^f$ can be expressed in terms of $X_{i-1}(0)$ as

$$\tilde{p}_i^f = p_i(1 - X_{i-1}(0)) + \mathcal{O}(\delta), \qquad i = 2, \ldots, M.$$

By Lemma A.7, this can be approximated with the distribution of parts on line $\tilde{L}_i$ by

$$\tilde{p}_i^f = p_i(1 - \tilde{X}_{i-1}(0)) + \mathcal{O}(\delta), \qquad i = 2, \ldots, M.$$

Using Lemma A.5, this can be rewritten as

$$\tilde{p}_i^f = p_i(1 - Q(\tilde{p}_{i-1}^f, \tilde{p}_i^b, N_{i-1})) + \mathcal{O}(\delta) \quad i = 2, \ldots, M. \tag{A.19}$$

Analogously, by Lemma A.6,

$$\tilde{p}_i^b = p_i\left[1 - \sum_{j=i+1}^{M} \left(\prod_{r=i+1}^{j-1} p_r\right)(1 - p_j)\, X_{i,\ldots,j-1}(N_i, \ldots, N_{j-1})\right] + \mathcal{O}(\delta)$$

$$= p_i\left[1 - (1 - p_{i+1})X_i(N_i) - \sum_{j=i+2}^{M} \left(\prod_{r=i+1}^{j-1} p_r\right)(1 - p_j)X_{i,\ldots,j-1}(N_i, \ldots, N_{j-1})\right] + \mathcal{O}(\delta).$$

Using (2.5), this can be approximated by

$$\tilde{p}_i^b = p_i\Bigg[1 - (1 - p_{i+1})X_i(N_i)$$

$$- X_i(N_i) \sum_{j=i+2}^{M} \left(\prod_{r=i+1}^{j-1} p_r\right)(1 - p_j)\, X_{i+1,\ldots,j-1}(N_{i+1}, \ldots, N_{j-1})\Bigg] + \mathcal{O}(\delta).$$

By Lemma A.7, this may be rewritten as

$$\tilde{p}_i^b = p_i\Bigg[1 - (1 - p_{i+1})\tilde{X}_i(N_i)$$

$$- \tilde{X}_i(N_i) \sum_{j=i+2}^{M} \left(\prod_{r=i+1}^{j-1} p_r\right)(1 - p_j)\, X_{i+1,\ldots,j-1}(N_{i+1}, \ldots, N_{j-1})\Bigg] + \mathcal{O}(\delta).$$

Rearranging, using Lemma A.6, we obtain

$$\tilde{p}_i^b = p_i[1 - (1 - p_{i+1})\tilde{X}_i(N_i) - \tilde{X}_i(N_i)(p_{i+1} - \tilde{p}_{i+1}^b)] + \mathcal{O}(\delta)$$

$$= p_i[1 - (1 - \tilde{p}_{i+1}^b)\tilde{X}_i(N_i)] + \mathcal{O}(\delta).$$

Using Lemma A.5, this may be written as

$$\tilde{p}_i^b = p_i[1 - Q(\tilde{p}_{i+1}^b, \tilde{p}_i^f, N_i)] + \mathcal{O}(\delta). \tag{A.20}$$

By Lemma A.9, the equilibrium eq. (A.17) has a unique solution p_i^f, p_i^b, $i = 1, \dots, M$. Equations (A.19) and (A.20) show that the conditional probabilities $\tilde{p}_i^f$, $\tilde{p}_i^b$, $i = 1, \dots, M$, solve eq. (A.17) with error $\mathcal{O}(\delta)$. Therefore, we conclude that

$$|\tilde{p}_i^f - p_i^f| \sim \mathcal{O}(\delta),$$

$$|\tilde{p}_i^b - p_i^b| \sim \mathcal{O}(\delta),$$

$$i = 1, \dots, M.$$

Proof of Theorem 2.1 Using Lemma A.7, the production rate may be calculated as

$$PR = (1 - X_{M-1}(0))p_M = (1 - \tilde{X}_{M-1}(0))p_M + \mathcal{O}(\delta).$$

Using Lemma A.5, this may be expressed as

$$PR = [1 - Q(\tilde{p}_{M-1}^f, \tilde{p}_M, N_{M-1})]p_M + \mathcal{O}(\delta).$$

By Lemma A.10, we obtain

$$PR = (1 - Q(p_{M-1}^f, p_M, N_{M-1}))\, p_M + \mathcal{O}(\delta).$$

By Lemma A.9, we may finally conclude that

$$PR = p_M^f + \mathcal{O}(\delta).$$

Proof of Theorem 2.2 Let p_j^f and p_j^b, $1 \le j \le M$, denote the steady state of recursive procedure (2.1) applied to the original line. Let $\bar{p}_j^f = p_{M-j}^b$ and $\bar{p}_j^b = p_{M-j}^f$. Observe that $\bar{p}_j^f$ and $\bar{p}_j^b$ solve the equilibrium equations of recursive procedure (2.1) for the reversed line. By Lemma A.9, the equilibrium equations possess a unique solution, so $\bar{p}_j^f$ and $\bar{p}_j^b$ must be the limiting values obtained by recursive procedure (2.1) for the reversed line. Therefore, $PR_{est}(p_M, \dots, p_1, N_{M-1}, \dots, N_1) = \bar{p}_M^f = p_1^b$. By Lemma A.8, $p_1^b = p_M^f$, which completes the proof.

Proof of Theorem 2.3 Let $PR(p_1, p_2, N_1)$ denote the production rate of serial production line (i)–(vi) with $M = 2$. Then the following three facts hold:

(α) Function $PR(p_1, p_2, N_1)$ is monotonically increasing in p_1, p_2, and N_1 (Lemma A.5).

(β) Function $Q(x, y, N)$, introduced in (2.2), is monotonically decreasing in x and N, and monotonically increasing in y (Lemma A.1).

(γ) $PR(p^f_i, p^b_{i+1}, N_i) = p^f_M$, $i = 1, \ldots, M-1$, where p^f_i and p^b_i are defined in (2.1)–(2.3) (Lemma A.8).

Consider two serial production lines (i)–(vi), the first of which is described by parameters p_i, $i = 1, \ldots, M$, and N_i, $i = 1, \ldots, M-1$, and the second by parameters $\tilde{p}_i \geq p_i$, $i = 1, \ldots, M$, and $\tilde{N}_i \geq N_i$, $i = 1, \ldots, M-1$. Let p^f_i, p^b_i, $\tilde{p}^f_i$, $\tilde{p}^b_i$, $i = 1, \ldots, M$, denote the steady state of recursive procedure (2.1) for the first and second lines, respectively. We prove Theorem 2.3 by contradiction.

Assume

$$\tilde{p}^f_M < p^f_M. \tag{A.21}$$

Then, using (γ),

$$PR(\tilde{p}^f_1, \tilde{p}^b_2, \tilde{N}_1) < PR(p^f_1, p^b_2, N_1).$$

Since, by (α), $PR(p_1, p_2, N_1)$ is a monotonically increasing function of each of its arguments, and by construction $\tilde{p}^f_1 \geq p^f_1$, $\tilde{N}_1 \geq N_1$, it follows that $\tilde{p}^b_2 < p^b_2$. Therefore, using (2.1) and the monotonicity property (β),

$$\tilde{p}^f_2 = \tilde{p}_2[1 - Q(\tilde{p}_1, \tilde{p}^b_2, \tilde{N}_1)] > p_2[1 - Q(p_1, p^b_2, N_1)] = p^f_2.$$

Now proceed inductively. Assume $\tilde{p}^b_i < p^b_i$ and $\tilde{p}^f_i > p^f_i$. The base case ($i = 2$) has already been established. From (γ) and (A.21), $PR(\tilde{p}^f_i, \tilde{p}^b_{i+1}, \tilde{N}_i) < PR(p^f_i, p^b_{i+1}, N_i)$. Since $\tilde{p}^f_i > p^f_i$ and $\tilde{N}_i > N^f_i$, it follows from (α) that $\tilde{p}^b_{i+1} < p^b_{i+1}$. Equation (2.1) and the monotonicity property (β) then yield

$$\tilde{p}^f_{i+1} = \tilde{p}_{i+1}(1 - Q(\tilde{p}^f_i, \tilde{p}^b_{i+1}, \tilde{N}_i)) > p_{i+1}(1 - Q(p^f_i, p^b_{i+1}, N_i)) = p^f_{i+1}.$$

The inductive hypothesis is therefore established, and $\tilde{p}^b_i < p^b_i$, $\tilde{p}^f_i > p^f_i$, $i = 2, \ldots, M$. In particular, $\tilde{p}^f_M > p^f_M$, which contradicts the assumption (A.21). Therefore, $\tilde{p}^f_M \geq p^f_M$ and, using (2.4), $PR_{est}(p_1, \ldots, p_M, N_1, \ldots, N_{M-1})$ is a monotomically increasing function of its arguments.

APPENDIX B. PROOFS FOR SECTION 3

Define function $f_N(x, y)$ as follows:

$$f_N(x, y) = [1 - Q(x, y, N)][1 - Q(y, x, N)].$$

where $Q(x, y, N)$ is given in (2.2).

LEMMA B.1 *Function $f_N(x,y)$ can be represented as follows:*

$$f_N(x, y) = \left[\frac{g(1-\alpha^N)}{1-\alpha^N g^2}\right]^2 ,$$

where

$$g = \sqrt{\frac{x}{y}},$$

$$\alpha = \frac{x(1-y)}{y(1-x)}.$$

Proof From eq. (2.2), we obtain

$$1 - Q(x, y, N) = 1 - \frac{(1-x)(1-\alpha)}{1-\frac{x}{y}\alpha^N}$$

$$= \frac{\alpha + x(1-\alpha) - \frac{x}{y}\alpha^N}{1-\frac{x}{y}\alpha^N}$$

$$= \left(\frac{x}{y}\right)\left[\frac{1-\alpha^N}{1-\frac{x}{y}\alpha^N}\right].$$

By Lemma A.5,

$$x(1 - Q(y, x, N)) = y(1 - Q(x, y, N)),$$

and therefore

$$f_N(x, y) = [1 - Q(x, y, N)]\,[1 - Q(y, x, N)]$$

$$= [1 - Q(x, y, N)]^2\,\frac{y}{x}$$

$$= \left(\frac{x}{y}\right)\left[\frac{1-\alpha^N}{1-\frac{y}{x}\alpha^N}\right]^2$$

$$= g^2 \left[\frac{1-\alpha^N}{1-\alpha^N g^2}\right]^2.$$

LEMMA B.2 *Under the constraint $xy = p = const$, function $f_N(x,y)$, achieves its maximum value if and only if $x = y$. Furthermore, $f_N(\sqrt{p},\sqrt{p}) = [N/(N+1-\sqrt{\mathrm{p}})]^2$.*

Proof Define $g = \sqrt{x/y}$, $p = xy$, and $\alpha = x(1-y)/y(1-x)$. Then $x = g\sqrt{p}$, and $y = \sqrt{p}/g$. Suppose $x < y$, which implies $0 < g < 1$. Then

$$\alpha = \frac{g\sqrt{p}\left(1-\dfrac{\sqrt{p}}{g}\right)}{\dfrac{\sqrt{p}}{g}(1-g\sqrt{p})}.$$

Observe that α is a monotonically increasing function of g. Using Lemma B.1, $f_N(x,y)$ can be expressed as

$$f_N\left(g\sqrt{p},\frac{\sqrt{p}}{g}\right) = \left(\frac{g(1-\alpha^N)}{1-\alpha^N g^2}\right)^2$$
$$= \exp 2\,[\ln(g) + \ln(1-\alpha^N) - \ln(1-\alpha^N g^2)].$$

Using the series expansion

$$\ln(1-x) = -\sum_{i=1}^{\infty}\frac{x^i}{i}, \qquad |x| < 1,$$

function $f_N(g\sqrt{p}, \sqrt{p}/g)$ may be expressed, for $0 < g < 1$, as

$$f_N(g\sqrt{p},\frac{\sqrt{p}}{g}) = \exp 2\left[\ln(g) - \sum_{i=1}^{\infty}\frac{(\alpha^N)^i}{i} + \sum_{i=1}^{\infty}\frac{(\alpha^N g^2)^i}{i}\right]$$

$$= \exp 2\left[\ln(g) + \sum_{i=1}^{\infty}\frac{(\alpha^N g^2)^i - (\alpha^N)^i}{i}\right]$$

$$= \exp 2\left[\ln(g) + \sum_{i=1}^{\infty}\frac{1}{i}\alpha^{Ni}(g^{2i}-1)\right]. \tag{B.1}$$

Since each term in the exponent of eq. (B.1) is a monotonically increasing function of g, it can be concluded that $f_N(g\sqrt{p},\sqrt{p}/g)$ is a monotonically increasing function of g for $0 < g < 1$.

Now suppose $y < x$. Define $\bar{g} = 1/g$ and observe $0 < \bar{g} < 1$. Because of the symmetry of its definition, $f_N(x,y) = f_N(y,x)$. So $f_N(g\sqrt{p}, \sqrt{p}/g,) = f_N(\sqrt{p}/\bar{g}, \bar{g}\sqrt{p}) = f_N(\bar{g}\sqrt{p}, \sqrt{p}/\bar{g})$. By the previous arguments, this function is monotonically increasing in $\bar{g}$ for $0 < \bar{g} < 1$, and therefore monotonically decreasing in g for $1 < g < \infty$. Thus, $f_N(g\sqrt{p}, \sqrt{p}/g)$, $0 < g < \infty$, attains its maximum at $g = 1$, which corresponds to $x = y = \sqrt{p}$.

Define $c_i = \sqrt{p_i^f p_{i+1}^b}$, and $PR = PR_{est}$.

LEMMA B.3 *The following is true*:

$$c_i \geq PR\left(\frac{N_i + 1}{N_i + PR}\right)$$

The equality takes place if and only if $p_i^f = p_{i+1}^b$.

Proof Using Lemma A.8,

$$\begin{aligned} c_i^2 &= p_i^f p_{i+1}^b \\ &= \left(\frac{PR\, p_i}{p_i^b}\right)\left(\frac{PR\, p_{i+1}}{p_{i+1}^f}\right) \\ &= \frac{PR^2\ p_i p_{i+1}}{p_i(1 - Q(p_{i+1}^b, p_i^f, N_i))\ p_{i+1}(1 - Q(p_i^f, p_{i+1}^b, N_i))} \\ &= \frac{PR^2}{f_{N_i}(p_i^f, p_{i+1}^b)}. \end{aligned}$$

By Lemma B.2, and the definition of c_i,

$$f_{N_i}(p_i^f, p_{i+1}^b) \leq \left[\frac{N_i}{N_i + 1 - c_i}\right]^2,$$

with equality if and only if $p_i^f = p_{i+1}^b$, so

$$c_i^2 \geq \frac{PR^2}{\left[\dfrac{N_i}{N_i + 1 - c_i}\right]^2},$$

$$c_i \geq \frac{PR(N_i + 1 - c_i)}{N_i},$$

$$c_i\left(1 + \frac{PR}{N_i}\right) \geq \frac{PR\,(N_i + 1)}{N_i},$$

$$c_i \geq \frac{PR\,(N_i + 1)}{N_i + PR}.$$

LEMMA B.4 *The total workforce, p^*, necessary to achieve the production rate value PR, is bounded by*

$$p^* \geq \prod_{j=1}^{M-1} \left(\frac{N_j + 1}{N_j + PR} \right)^2 PR^M.$$

The equality holds if and only if $p_i^f = p_{i+1}^b$, $i = 1, \ldots, M - 1$.

Proof By Lemma A.8,

$$PR^M = \left(\frac{p_1^f p_1^b}{p_1} \right) \cdots \left(\frac{p_i^f p_i^b}{p_i} \right) \cdots \left(\frac{p_M^f p_M^b}{p_M} \right),$$

$$PR^M p^* = p_1^b (p_1^f p_2^b) \cdots (p_{M-1}^f p_M^b) p_M^f.$$

Since, by Lemma A.8, $p_1^b = p_M^f = PR$, we obtain

$$p^* = \frac{c_1^2 c_2^2 \cdots c_{M-1}^2}{PR^{M-2}}.$$

Using Lemma B.3 we obtain, with equality if and only if $p_i^f = p_{i+1}^b$, $i = 1, \ldots, M - 1$,

$$p^* \geq \frac{\prod_{j=1}^{M-1} \left(\frac{N_i + 1}{N_i + PR} \right)^2}{PR^{M-2}}$$

$$= \prod_{j=1}^{M-1} \left(\frac{PR\,(N_i + 1)}{N_i + PR} \right)^2 PR^M.$$

Lemma B.4 provides a lower bound on the workforce necessary to achieve a desired production rate. We now show that this bound is achievable.

LEMMA B.5: *The condition* $p_i^f = p_{i+1}^b$, $i = 1, \ldots, M - 1$, *is achieved if and only if the workforce is distributed as*

$$p_1 = \left(\frac{N_1 + 1}{N_1 + PR} \right) PR,$$

$$p_j = \left(\frac{N_{j-1} + 1}{N_{j-1} + PR} \right) \left(\frac{N_j + 1}{N_j + PR} \right) PR, \qquad 2 \leq j \leq M - 1, \tag{B.2}$$

$$p_M = \left(\frac{N_{M-1} + 1}{N_{M-1} + PR} \right) PR,$$

where PR is the production rate of the line.

Proof Suppose $p_i^f = p_{i+1}^b$, $i = 1, \ldots, M - 1$. Then, by Lemmas A.5 and A.8,

$$\begin{aligned} PR &= p_i^f (1 - Q(p_{i+1}^b, p_i^f, N_i)) \\ &= p_i^f (1 - Q(p_i^f, p_i^f, N_i)) \\ &= p_i^f (1 - \frac{1 - p_i^f}{N_i + 1 - p_i^f}) \\ &= \frac{p_i^f N_i}{N_i + 1 - p_i^f}. \end{aligned}$$

Solving this equation for p_i^f, and recalling the assumption that $p_i^f = p_{i+1}^b$, we obtain

$$p_i^f = p_{i+1}^b = \left(\frac{N_i + 1}{N_i + PR} \right) \text{PR}. \tag{B.3}$$

Using Lemma A.8, for $i = 2, \ldots, M - 1$,

$$\begin{aligned} PR &= \frac{p_i^f p_i^b}{p_i} \\ &= \left(\frac{N_{i-1} + 1}{N_{i-1} + PR} \right) \left(\frac{N_i + 1}{N_i + PR} \right) \frac{PR^2}{p_i}, \end{aligned}$$

which can be rearranged into

$$p_i = \left(\frac{N_{i-1} + 1}{N_{i-1} + PR} \right) \left(\frac{N_i + 1}{N_i + PR} \right) PR, \qquad i = 2, \ldots, M - 1. \tag{B.4}$$

The expressions for $p_1 = p_1^f$ and $p_M = p_M^b$ are obtained from equation (B.3).

Now suppose that the workforce is distributed as in eq. (B.2). We next show that this implies that $p_i^f = p_{i+1}^b$, $i = 1, \ldots, M - 1$. By Lemma A.9, there is a unique solution to the equilibrium equation (A.17) or recursive procedure (2.1) We claim that the solution is

$$p_1^b = p_M^f = PR, \qquad p_i^f = p_{i+1} = \left(\frac{N_i + 1}{N_i + PR} \right) PR, \qquad i = 1, \ldots, M - 1. \tag{B.5}$$

Since eq. (A.17) has exactly one solution, we only need to show that eq. (B.5) is indeed a solution. Consider, for $i = 2, \ldots, M$, using eq. (B.4),

$$\begin{aligned} p_i(1 - Q(p^f_{i-1}, p^b_i, N_{i-1})) &= PR\left(\frac{N_{i-1}+1}{N_{i-1}+PR}\right)\left(\frac{N_i+1}{N_i+PR}\right)\left(1 - \frac{1-p^b_i}{N_{i-1}+1-p^b_i}\right) \\ &= PR\left(\frac{N_{i-1}+1}{N_{i-1}+PR}\right)\left(\frac{N_i+1}{N_i+PR}\right)\left(\frac{N_{i-1}}{N_{i-1}+1-p^b_i}\right) \\ &= PR\left(\frac{N_{i-1}+1}{N_{i-1}+PR}\right)\left(\frac{N_i+1}{N_i+PR}\right)\frac{1}{p^b_i}\left(\frac{N_{i-1}}{\frac{N_{i-1}+1}{p^b_i} - 1}\right) \\ &= \left(\frac{N+1}{N+PR}\right)PR \\ &= p^f_i\,. \end{aligned}$$

The proof for p^b_i $i = 1, \ldots, M-1$, is similar.

LEMMA B.6 *The minimum workforce* p^*_{min} *required to achieve production rate PR is given by*

$$p^*_{min} = \prod_{j=1}^{M-1}\left(\frac{N_j+1}{N_j+PR}\right)^2 PR^M.$$

Moreover, this production rate is achieved if and only if p^* is distributed among $p_i, \ldots, p_M$, $\prod_i p_i = p^*$, *so that* $p^f_i = p^b_{i+1}$, $i = 1, \ldots, M-1$.

Proof By Lemma B.4,

$$p^*_{min} \geq \prod_{j=1}^{M-1}\left(\frac{N_j+1}{N_j+PR}\right)^2 PR^M,$$

and equality is achieved if and only if $p^f_i = p^b_{i+1}$, $i = 1, \ldots, M-1$. By Lemma B.5, lower bound is attainable with the workforce distribution as specified in eq. (B.2).

LEMMA B.7 *The minimum workforce* p^*_{min} *necessary to achieve the production rate PR is a monotonically increasing function of PR.*

Proof From Lemma B.6, p^*_{min} is given by

$$p^*_{min} = PR\prod_{j=1}^{M-1}\left[\left(\frac{N_j+1}{N_j+PR}\right)^2 PR^M\right].$$

Consider one term of the product,

$$T_j = \left(\frac{N_j + 1}{N_j + PR}\right)^2 PR^M, \qquad j = 1, \ldots, M - 1.$$

Differentiation of T_j with respect to PR yields

$$\frac{\partial T_j}{\partial PR} = \frac{(N_j + 1)^2 (N_j^2 - PR^2)}{(N_j + PR)^4} > 0.$$

Since p^*_{min} is the product of positive, monotonically increasing functions of PR, the lemma follows.

Proof of Theorem 3.1 Suppose the line is unimprovable, but that there exists an i such that $p^f_i \neq p^b_{i+1}$. Then by Lemma B.6, $p^* > p^*_{min}$. Then by Lemma B.7, workforce p^* optimally distributed can achieve a larger production rate, which is a contradiction.

Proof of Corollary 3.1 By Lemmas A.7 and A.10, the distribution of parts in buffer i can be approximated with error $\mathcal{O}(\delta)$ by the distribution of parts in the buffer of the two machine line $L = \{p^f_i, N_i, p^b_{i+1}\}$. From Lemma A.5 applied to line L_i, Prob$\{m_i$ is starved$\} = Q(p^f_i, p^b_{i+1}, N_i)$ and Prob$\{m_i$ is blocked$\} = Q(p^b_{i+1}, p^f_i, N_i)$. Since $p^f_i = p^b_{i+1}$, the result of part (a) follows. From eq. (A.3) of Lemma A.5, when applied to line L_i.

$$X_i(j) = \frac{X_i(0)}{1 - p^f_i}, \qquad 1 \leq j \leq N_i, \tag{B.6}$$

and so, using eq. (A.4),

$$\begin{aligned} E[h_i] &= \sum_{j=0}^{N_i} j X_i(j) \\ &= \sum_{j=1}^{N_i} j \left(\frac{1}{1 - p^f_i}\right)\left(\frac{1 - p^f_i}{N_i + 1 - p^f_i}\right) \\ &= \frac{N_i(N_i + 1)}{2(N_i + 1 - p^f_i)}, \end{aligned}$$

from which the result of part (b) of the corollary follows.

Proof of Theorem 3.2 It was established in Lemma B.5 that (3.1) is satisfied if and only if

$$p_1 = \left(\frac{N_l + 1}{N_1 + PR^*_{est}}\right) PR^*_{est},$$

$$p_i = \left(\frac{N_{i-1}+1}{N_{i-1}+PR^*_{est}}\right)\left(\frac{N_i+1}{N_i+PR^*_{est}}\right)PR^*_{est}, \quad i = 2, \ldots, M-1, \tag{B.7}$$

$$p_M = \left(\frac{N_{M-1}+1}{N_{M-1}+PR^*_{est}}\right)PR^*_{est},$$

where $PR^*_{est} = p^f_m$ is defined in (3.4). Therefore, for a line satisfying (3.1),

$$p^* = \prod_{i=1}^{M} p_i = (PR^*_{est})^M \prod_{i=1}^{M-1}\left(\frac{N_i+1}{N_i+PR^*_{est}}\right)^2.$$

This may be rearranged as

$$PR^*_{est} = (p^*)^{\frac{1}{M}} \prod_{i=1}^{M-1}\left(\frac{N_i+PR^*_{est}}{N_i+1}\right)^{\frac{2}{M}}. \tag{B.8}$$

Define function $f(.)$ by

$$\mathrm{f(x)} = (p^*)^{\frac{1}{M}} \prod_{i=1}^{M-1}\left(\frac{N_i+x}{N_i+1}\right)^{\frac{2}{M}},$$

and observe that

$$PR^*_{est} = f(PR^*_{est}). \tag{B.9}$$

We next show that $x(n+1) = f(x(n))$ is a contraction on [0, 1], from which it follows that (B.9) has exactly one solution and that recursive procedure (3.5) always converges to this solution. Using the relationship

$$\frac{d[\Pi^N_{i=1} y_i(x)]}{dx} = \left[\prod_{i=1}^{N} y_i(x)\right]\left[\sum_{i=1}^{N}\frac{y'_i(x)}{y_i(x)}\right],$$

where $y'_i(x) = dy_i(x)/dx$, calculate

$$\frac{df(x)}{dx} = (p^*)^{\frac{1}{M}}\left(\frac{2}{M}\right)\left[\prod_{i=1}^{M-1}\left(\frac{N_i+x}{N_i+1}\right)\right]^{\frac{2}{M}-1}\left[\prod_{i=1}^{M-1}\left(\frac{N_i+x}{N_i+1}\right)\right]\left[\sum_{i=1}^{M-1}\frac{1}{N_i+x}\right]$$

$$= (p^*)^{\frac{1}{M}}\left(\frac{2}{M}\right)\left[\prod_{i=1}^{M-1}\left(\frac{N_i+x}{N_i+1}\right)\right]^{\frac{2}{M}}\left[\sum_{i=1}^{M-1}\frac{1}{N_i+x}\right]$$

Since $p^* < 1$ and $(N_i + x)/(N_i + 1) \leq 1$ for $x \in [0, 1]$, using the assumption $\sum_{i=1}^{M-1} 1/N_i \leq M/2$ we obtain

$$\left|\frac{df(x)}{dx}\right| < 1, \qquad x \in [0, 1].$$

The Mean Value Theorem guarantees that for all $x,y \in [0, 1]$, there exists a $c \in [x, y]$ such that

$$f(x) - f(y) = \frac{df(c)}{dx}(x - y),$$

and therefore $|f(x) - f(y)| < |x - y|$. This implies that $x(n + 1) = f(x(n))$ is a contraction on $[0, 1]$, which establishes Theorem 3.2.

Proof of Theorem 3.4 Define

$$f(N_1, \ldots, N_{M-1}, x) = (p^*)^{\frac{1}{M}} \prod_{i=1}^{M-1} \left(\frac{N_i + x}{N_i + 1}\right)^{\frac{2}{M}} \tag{B.10}$$

and

$$f^*(x) = \max_{\substack{N_1, \ldots, N_{M-1} \\ \sum_{i=1}^{M-1} N_i = N^*}} f(N_1, \ldots, N_{M-1}, x). \tag{B.11}$$

The values $N_1^*, \ldots, N_{M-1}^*$ which solve (B.11) can be determined by the Lagrange multiplier technique. The Lagrange function is:

$$F(N_1, \ldots, N_{M-1}, \lambda) = f(N_1, \ldots, N_{M-1}, x) + \lambda(N_1 + \cdots + N_{M-1} - N^*)$$

$$= (p^*)^{\frac{1}{M}} \prod_{i=1}^{M-1} \left(\frac{N_i + x}{N_i + 1}\right)^{\frac{2}{M}} + \lambda(N_1 + \cdots + N_{M-1} - N^*).$$

Therefore, the optimality condition

$$\frac{\partial F(N_1, \ldots, N_{M-1}, \lambda)}{\partial N_i} = (p^*)^{\frac{1}{M}} \left(\frac{2}{M}\right) \left[\prod_{i=1}^{M-1} \left(\frac{N_i + x}{N_i + 1}\right)^{\frac{2}{M}}\right] \left[\frac{1 - x}{(N_i + 1)(N_i + x)}\right] + \lambda = 0,$$

$$i = 1, \ldots, M - 1,$$

is satisfied if and only if $N_i = N_j, \forall i,j$. Thus (B.11) is solved by

$$N_i^* = \frac{N^*}{M - 1}, \qquad \mathrm{i} = 1, \ldots, M - 1. \tag{B.12}$$

Consider now recursive procedure (3.5):

$$x(n+1) = f(N_1, \ldots, N_{M-1}, x(n)),$$

and recall that, according to Theorem 3.2, $\lim_{n\to\infty} x(n) = PR_{est}(p_1^*, \ldots, p_M^*, N_1, \ldots, N_{M-1})$, where $p_i^*, i = 1, \ldots, M$, are defined by (3.7). Define the recursive procedure (3.5) for two sequences of N_i's, the optimal one and any other:

$$x^*(n+1) = f(N_1^*, \ldots, N_{M-1}^*, x^*(n)), \tag{B.13}$$

$$x'(n+1) = f(N_1', \ldots, N_{M-1}', x'(n)), \tag{B.14}$$

where N_i^* is defined by (B.12) and N_i', $i = 1, \ldots, M-1$, is any sequence satisfying $\sum_{i=1}^{M-1} N_i' = N^*$. Assume that the initial conditions for (B.13) and (B.14) are the same:

$$x^*(0) = x'(0) \in [0, 1].$$

We show below that

$$(\alpha).\ x^*(n) \geq x'(n), \forall\, n > 0, \textit{i.e.}$$
$$PR_{est}(p_1^*, \ldots, p_M^*, N_1^*, \ldots, N_{M-1}^*) \geq PR_{est}(p_1', \ldots, p_M', N_1', \ldots, N_{M-1}')$$

and

$$(\beta).\ p_i^f = p_i^b, \qquad i = 2, \ldots, M-1.$$

This would prove Theorem 3.4.

Fact (α) is proved by induction. For $n = 1$, the result $x^*(1) \geq x'(1)$ follows immediately from the fact that the sequence N_i^*, $i = 1, \ldots, M-1$, solves (B.11). Now assume that $x^*(n) \geq x'(n)$. Because

$$\frac{df(N_1, \ldots, N_{M-1}, x)}{dx} = (p^*)^{\frac{1}{M}} \left(\frac{2}{M}\right) \left[\prod_{i=1}^{M-1} \left(\frac{N_i + x}{N_i + 1}\right)\right]^{\frac{2}{M}} \left[\sum_{i=1}^{M-1} \frac{1}{N_i + x}\right] > 0, \tag{B.15}$$

that is, $f(N_1, \ldots, N_{M-1}, x)$ is a monotonically increasing function of x, and since the sequence N_i^*, $i = 1, \ldots, M-1$, solves (B.11),

$$\begin{aligned} x^*(n+1) &= f(N_1^*, \ldots, N_{M-1}^*, x^*(n)) \\ &\geq f(N_1^*, \ldots, N_{M-1}^*, x'(n)) \\ &\geq f(N_1', \ldots, N_{M-1}', x'(n)) \\ &= x(n+1). \end{aligned}$$

Statement (β) follows from (B.12) and the fact that the unique solution to the equilibrium equation (A.17) of recursive procedure (2.1), when the workforce is distributed according to (3.7), is given in eq. (B.5) by

$$p_1^b = p_M^f = PR, \qquad p_i^f = p_{i+1}^b = \left(\frac{N_i + 1}{N_i + PR}\right) PR, \quad i = 1,\dots, M - 1. \qquad \text{(B.16)}$$

Proof of Corollary 3.2 By Lemma A.10,

$$p_i^f = \tilde{p}_i^f + \mathcal{O}(\delta), \qquad p_i^b = \tilde{p}_i^b + \mathcal{O}(\delta), \qquad i = 1, \dots, M,$$

where p_i^f, p_i^b are the limiting values obtained from recursive procedure (2.1), and $\tilde{p}_i^f$, $\tilde{p}_i^b$ are the conditional probabilities defined in (A.8). Using Lemma A.6 and eq. (A.9), these may be rewritten as

$$p_i^f = p_i\,[1 - \text{Prob}\,\{\,m_i \text{ is starved}\}] + \mathcal{O}(\delta),$$

$$p_i^b = p_i\,[1 - \text{Prob}\,\{m_i \text{ is blocked}\}] + \mathcal{O}(\delta), \qquad i = 2, \dots, M - 1.$$

By condition (3.8), $p_i^f = p_i^b$, $i = 2, \dots, M - 1$, and we may therefore conclude that

$$|\,\text{Prob}\{m_i \text{ is starved}\} - \text{Prob}\,\{\,m_i \text{ is blocked}\}\,| \sim \mathcal{O}(\delta).$$

Proof of Theorem 3.5 Similar to the proof of Theorem 3.4. Eq. (3.10) follows directly from (B.12) and (3.7).

Proof of Theorem 3.6 Using the identity

$$\min\{a, b\} = \sqrt{ab}\,\min\left\{\sqrt{\frac{a}{b}}, \sqrt{\frac{b}{a}}\right\}, \qquad 0 < a, \quad 0 < b,$$

in

$$PR_{est} = \min_i\,\{p_i^f, p_i^b\},$$

we obtain

$$PR_{est} = \min_i \sqrt{p_i^f p_i^b}\,\min\left\{\sqrt{\frac{p_i^f}{p_i^b}}, \sqrt{\frac{p_i^f}{p_i^b}}\right\}.$$

By Lemma A.8, this can be written as

$$PR_{est} = \min_i \sqrt{PR_{est}\,p_i}\,\min\left\{\sqrt{\frac{p_i^f}{p_i^b}}, \sqrt{\frac{p_i^f}{p_i^b}}\right\}.$$

Squaring both sides, we obtain

$$PR_{est} = \min_i \left\{ p_i \min \left\{ \frac{p_i^f}{p_i^b}, \frac{p_i^b}{p_i^f} \right\} \right\}.$$

APPENDIX C. PROOFS FOR SECTION 4

Proof of Theorem 4.1 Consider a line (i)–(vi) with (3.1) satisfied. Let the workforce distribution be denoted by $p_i = p_i^*, i = 1, \ldots, M$. Suppose that the workforce distribution is modified by $p_i = gp_i^*$ and $p_j = (1/g)p_j^*$, $1 \leq i \leq M$ and $1 \leq j \leq M$. Observe that the total workforce p^* does not depend on g, but that the line is unimprovable when $g = 1$. Therefore, the production rate achieves its maximum value when $g = 1$. Letting $PR = PR(g)$, we observe

$$\frac{\partial PR(1)}{\partial g} = 0. \tag{C.1}$$

Using the chain rule,

$$\frac{\partial PR(1)}{\partial g} = \frac{\partial PR(1)}{\partial p_i}\frac{\partial(gp_i)}{\partial g} + \frac{\partial PR(1)}{\partial p_j}\frac{\partial\left(\frac{p_i}{g}\right)}{\partial g} = p_i \frac{\partial PR(1)}{\partial p_i} - p_j \frac{\partial PR(1)}{\partial p_j}. \tag{C.2}$$

Since i and j were chosen arbitrarily, we therefore conclude, from eq. (C.1) and (C.2), that

$$p_i \frac{\partial PR(1)}{\partial p_i} = p_j \frac{\partial PR(1)}{\partial p_j}, \qquad \forall i, j.$$

References

1. E. Deming, *Out of the Crisis,* MIT Press, Cambridge, MA. (1986).
2. E. Goldratt and J. Cox, *The Goal,* North Rivers Press, New York (1986).
3. G. Taguchi, *System of Experimental Design: Engineering Methods to Optimize Quality and Minimize Cost,* UNIPUB/Kraus International Publications, Dearborn, MI (1987).
4. A. M. Law and W. D. Kelton, *Simulation Modeling and Analysis,* McGraw-Hill, New York (1991).
5. E. Koenigsberg, Production Lines and Internal Storage—A Review, *Management Sci.,* **5**, 410–433 (1959).
6. G. M. Buxey, N. D. Slack, and R. Wild, Production Flow Line Systems Design—A Review, *AIIE Trans.,* **5**, 37–48 (1973).
7. B. A. Sevastyanov, Influence of Storage Bin Capacity on the Average Standstill Time of a Production Line, *Theory Prob. Appl.,* **7**, 429–438 (1962).
8. J. A. Buzacott and J. G. Shanthikumar, *Stochastic Models of Manufacturing Systems,* Prentice Hall, New York (1993).
9. S. B. Gershwin, *Manufacturing Systems Engineering,* Prentice Hall, New York (1994).
10. J.-T. Lim, S. M. Meerkov, and F. Top, Homogeneous, Asymptotically Reliable Serial Production Lines: Theory and a Case Study, *IEEE Trans. Automat. Contr.,* **35**, 524–534 (1990).
11. S. Meerkov and F. Top, Asymptotically Reliable Serial Production Lines: Analysis, Synthesis, and a Case Study, *IFAC World Congress Proceedings,* Talin, Estonia, **9**, 2–7 (1990).

12. G. Olsder and E. van Bracht, Analysis and Synthesis of Serial Production Lines, *Proc. of the 30th IEEE CDC.*, 2168–2173, Brighton, U.K. (Dec. 1991).
13. J. Wijngaard, The Effect of Interstage Buffer Storage on the Output of Two Unreliable Production Units in Series with Different Production Rates, *AIIE Trans.*, **11**, 42–46 (1979).
14. C. Terracol and R. David, An Aggregation Technique for Performance Evaluation of Transfer Lines with Unreliable Machines and Finite Buffers, *IEEE Conf. Robot. Automat.*, **3**, 1333–1338 (1987).
15. Y. Dallery, R. David, and X. Xie, An Efficient Algorithm for Analysis of Transfer Lines with Unreliable Machines and Finite Buffers, *IIE Trans.*, **20**, 280–283 (1988).
16. Y. Dallery, R. David, and X. Xie, Approximate Analysis of Transfer Lines with Unreliable Machines and Finite Buffers, *IEEE Trans. Automat. Contr.*, **34**, 943–953 (1989).
17. M. Spearman, D. Woodruff, and W. Hopp, CONWIP: A Pull Alternative to Kanban, *Int. J. Prod. Res.*, **28**, 879–894 (1990).
18. J.-T. Lim and S. M. Meerkov, On Asymptotically Reliable Closed Serial Production Lines, *Contr. Eng. Pract.*, **1**, 147–152 (1993).
19. E. Muth, The Reversability Property of Production Lines, *Management Sci.*, **25**, 152–158 (1979).
20. P. Glasserman and D. Yao, Algebraic Structure of some Stochastic Discrete Event Systems, with Applications, *Dyn. Sys.: Theory and App.*, **1**, 7–36 (1991).
21. R. Conway, W. Maxwell, J. McClain, and L. Thomas, The Role of Work-In-Process Inventory in Serial Production Lines, *Oper. Res.*, **36**, 229–241 (1988).
22. G. Yamazaki, H. Sakasegawa, and J. G. Shantikumar, On Optimal Arrangement of Stations in a Tandem Queuing System with Blocking, *Management Sci.*, **38**, 137–153 (1992).
23. R. Bellman, *Introduction to Matrix Analysis,* McGraw-Hill, New York (1960).