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# NONLINEAR SINGULARLY PERTURBED SYSTEMS OF DIFFERENTIAL EQUATIONS: A SURVEY

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In this paper a survey of the most effective methods in singular perturbations is presented. Many applied problems can be modeled by nonlinear singularly perturbed systems, as, for example, problems in kinetics, biochemistry, semiconductors theory, theory of electrical chains, economics, and so on. In this survey we consider averaging and constructive methods that are very useful from the point of view of their numerical and computer realizations.

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## 1. INTRODUCTION

A classical example of a singularly perturbed problem that is easy to visualize and that illustrates the general behavior of almost all singularly perturbed problems is the flow of fluid at low viscosity past a body. In viscous flow, the tangential velocity must be zero at the boundary of the body, while in nonviscous flow the fluid can slide along the boundary. Thus, if viscosity is neglected, the solution of this reduced problem will not approximate the actual viscous flow near the boundary. However, for low viscosity the actual viscous flow will be closely approximated by the nonviscous flow, except in a narrow strip near the boundary. This narrow strip is often called *Prandtl's boundary layer*. For this example, the modeling differential equation is a nonlinear partial differential equation—the Navier-Stokes equation. We can give the following informal definitions.

Definition 1. A *regular perturbation* problem, $P_\epsilon(y_\epsilon) = 0$, depends on its small parameter $\epsilon$ in such a way that its solution $y_\epsilon(x)$ converges as $\epsilon \rightarrow 0$ (uniformly with respect to the independent variable $x$ in the relevant domain) to the solution $y_0(x)$ of the reduced problem $P_0\ (y_0)$.

The parameter $\epsilon$ typically represents the influence of many nearly negligible physical influences. When there is sufficient smoothness (with respect to $y$, $x$ and $\epsilon$), the solution of a regular perturbation problem can be approximated by a formal asymptotic power series expansion in $\epsilon$ having the leading term (i.e., asymptotic limit) $y_0$.

Definition 2. A *singular perturbation* is said to occur whenever the regular perturbation limit $y_\epsilon(x) \rightarrow y_0(x)$ fails.

Such a breakdown typically can occur in narrow intervals of space or short intervals of time (although secular problems with nonuniform behavior at infinity, as we found for the harmonic oscillator, can exhibit breakdowns).

In other words, problem $P_\epsilon$ depends on a small positive parameter $\epsilon = 0$. Since the reduced equation is of lower order than the full equation, the solution of the reduced problem cannot be expected to satisfy all of the boundary conditions of the full problem. Thus, even if the limiting solution exists, uniform convergence cannot be expected near the boundary as $\epsilon \to 0$. Such regions of nonuniform convergence are called *boundary layers* and are the distinguishing feature of singular perturbation problems [2,3].

Several questions usually arise. (a) Does the solution of $P_\epsilon$ have a limit as $\epsilon \to 0$? (b) If a limiting solution exists, does it satisfy the reduced equation? (c) If the limiting solution satisfies the reduced equation, what boundary conditions will be satisfied by the limiting solution? (d) What are the asymptotic representations of the solution?

The main objective of this survey is to study some of the most popular techniques for solving singularly perturbed systems. In section 2, we present so-called averaging methods, which have been introduced by Bogoliubov and Krylov [12] and which have a lot of applications in the theory of nonlinear oscillations. In.section 3, the basic asymptotic methods for constructing the asymptotic solutions of singularly perturbed systems are described. The now classical theory of asymptotic expansions (as presented in Wasow [25], Hoppensteadt [9,10], Vasil'eva and Butuzov [23], O'Malley [16] and Smith [19]) is primarily due to the work of Tikhonov and Levinson in early 1950s. This theory has been unusually effective in numerous significant applications, for example, in control theory [11], and it has been extended to abstract equations and partial differential equations [4]. In section 4, we introduce one constructive method for solving singularly perturbed problems. It presents the new trend of applied mathematics, characterized by the application of computer mathematics and computational techniques. We deal with converging iterative methods based on algorithmic processes, that converge, in the general sense (Cauchy's), to the exact solution of the initial system in the corresponding domain of variation parameters. In analyzing this method, we have mainly used the system of Lyapunov's majorizing equations [14].

Concerning singular perturbations, various approaches can be found in the literature. In this survey we concentrate on the three most effective methods for solving singular perturbed problems from the point of view of asymptotic analysis and numerical analysis.

## 2. AVERAGING METHODS

Quite difficult nonlinear oscillation problems began to receive attention in the late 1920s. In particular, van der Pol [22] studied relaxation oscillations which described triode circuits and he later analyzed analogous physiological problems [8]. Generalizing Poincare's ideas [15], Krylov and Bogoliubov developed averaging methods to treat problems of fast oscillations [12, 19]. The method of coordinate transformation offers concrete possibilities of obtaining specific results. Applicable to ordinary differential equations, this method may be used, in particular, to transform the initial equations into equations whose analytic structure may be assigned beforehand by the investigator.

Suppose, for example, that we have the ODE system,

$$\frac{dx}{dt} = X(x, t, \mu),\ x(0) = x_0, \tag{1}$$

where $x = (x_1, x_2, \ldots, x_s)$ and $X = (X_1, X_2, \ldots, X_s)$ are $s$-dimensional vectors, $x \in G_s$ ($G_s$ is a certain $s$-dimensional domain in the Euclidean space $R_s$), $t \in I$ ($I$ is a finite or infinite time interval), and $\mu$ is a small nonnegative parameter such that $0 \leq \mu \leq \bar{\mu}$. Consequently, the domain in which the vector function $X(x, t, \mu)$ is defined is given by $G_{s+2} = G_s \times I \times I_{\bar{\mu}}$.

In addition to the system (1), we define another system of ODE,

$$\frac{d\bar{x}}{dt} = \bar{X}(\bar{x}, t, \mu),\ \bar{x}(0) = x_0, \tag{2}$$

called the averaged system for system (1). The vector $\bar{X}(\bar{x}, t, \mu)$ is the averaged function for $X(x, t, \mu)$.

The task is to find a nondegenerate differentiable coordinate transformation $x \to \bar{x}$ which converts system (1) in the old variables into system (2) in the new variables.

Definition 3. The ODE system,

$$\frac{dx}{dt} = \mu X(x, t, \mu),\ x(0) = x_0, \tag{3}$$

where $x$ and $X$ are points in the $s$-dimensional Euclidean space, $x \in G_s$, and $\mu \in [0, \bar{\mu}]$ is a small nonnegative parameter, or simply the *standard system* [1].

We write the averaged system corresponding to (3) in the form:

$$\frac{d\bar{x}}{dt} = \mu\bar{X}(\bar{x}, \mu),\ \bar{x}(0) = x_0. \tag{4}$$

We shall now seek a coordinate transformation for the standard system in the form:

$$x = \bar{x} + \mu u(\bar{x}, t, \mu), \tag{5}$$

which transforms equation (3) into equation (4). We notice that transformation (5) is nondegenerate for sufficiently small values of $\mu$. This is because the determinant of the Jacobian matrix $\partial x/\partial\bar{x}$ is equal to $O(1)$, if the vector function $u(\bar{x}, t, \mu)$ is bounded and continuous first-order partial derivatives with respect to $\bar{x}$ and $t$, that is, if it belongs to the class $C^1_{\bar{x}t}$.

Transformation (5) is called the Krylov-Bogoliubov transformation for standard systems. The averaged function is usually constructed with the help of a certain averaged operator $M$ acting on $X(x, t, \mu)$ in $G_{s+2}$:

$$\overline{X}(x, t, \mu) \equiv M[X(x, t, \mu)]. \tag{6}$$

If we substitute the right-hand side of differential equation (2) into the averaged equation (4), it may be written in the form:

$$\frac{d\overline{x}}{dt} = \overline{X}(\overline{x}, t, \mu) \equiv M[X(\overline{x}, t, \mu)]. \tag{7}$$

It is obvious that we can produce an infinite set of averaged equations of the form (4) for the equation (3), since an infinite set of averaged operators $M$ may be applied to the function $X(x, t, \mu)$. In actual practice, however, the very nature of problems represented by differential and other equations usually suggests the choice of the most optimal smoothing operator.

We have two main problems: (1) the problem of justifying the averaging method; and (2) the problem of constructing approximate solutions.

The problem of mathematical justification for averaging methods is treated as a set of mathematical theorems. These theorems lead to estimates of the differences between the solutions of the exact initial equations and the approximate averaged equations, as well as to an estimate of the interval of time during which the realization of a given estimate is guaranteed for a given difference between solutions.

In the problem of finding approximate solutions, the invaluable tool is the Krylov-Bogoliubov transformation encountered first, while constructing and later, while analyzing the averaged equations.

The following points stand out a comparison between the solutions of equations (3) and (7).

(1) Let any arbitrary positive number $\epsilon > 0$ and the time interval $0 \leq t \leq T$ be given. It is required that we find the conditions which the vector function $X(x, t, \mu)$ must satisfy so that

$$\| x(t, \mu) - \overline{x}(t, \mu) \| < \epsilon.$$

If $T$ is an arbitrarily small positive number, then the theorem about the continuous dependence of the solutions on the small variations of the right-hand sides guarantees the $\epsilon$-estimate for the norm $\| x(t, \mu) - \overline{x}(t, \mu) \|$. If $T$ assumes a finite value, such an estimate is not a trivial matter.

(2) Let the time interval $[0, T]$ be given. It is required that we find the upper bound of the norm $\| x - \overline{x} \|$, that is, it is required that we find

$$sup_{t\in[0,T]} \| x(t, \mu) - \overline{x}(t, \mu) \|.$$

A knowledge of the upper bound helps us to determine the actual deviations of approximate solutions from the exact solutions. This information is very useful in applications of this method.

(3) Let a certain positive number $K$ be given. It is required that we find the intervals of variation of $t$ for which

$$\| x(t, \mu) - \bar{x}(t, \mu) \| < K.$$

The three points above form an important part of the theory of differential equations, and various analytic and quantitative methods have been worked out for their investigation. Mathematical investigations dealing with the first and second points also form the main part of the problem of mathematical justification for averaging methods. This branch of mathematics, which is so interesting and extremely useful, was extensively developed after fundamental research was performed by Bogoliubov of the averaging method for standard systems [1].

Let a standard system of ODE be given such that

$$\frac{dx}{dt} = \mu X(x, t, \mu). \tag{8}$$

Equations of type (8) are used to describe many problems of the theory of oscillations and celestial mechanics; hence their investigation is important not only from a theoretical point of view but also from a practical one.

In addition to system (8), let us consider the corresponding averaged system,

$$\frac{d\bar{x}}{dt} = \mu M_t [X(\bar{x}, t, \mu)] = \mu \bar{X}(\bar{x}, \mu), \tag{9}$$

in which the integral mean with respect to $t$ is taken as the averaged operator,

$$M_t [X(x, t, \mu)] = lim_{T\to\infty} \frac{1}{T} \int_0^T X(x, t, \mu)\, dt, \tag{10}$$

while the integrating vector $x$ is taken to be constant.

If in equation (10) we let $\mu = 0$ and introduce the notation

$$\bar{X}_0 (x) = lim_{T\to\infty} \frac{1}{T} \int_0^T X(x, t, 0)\, dt, \tag{11}$$

to the standard system (8) may be assigned the corresponding averaged system:

$$\frac{d\bar{x}}{dt} = \mu \bar{X}_0 (\bar{x}). \tag{12}$$

In the literature, system (12) is called the averaged system in the first approximation [1,5,12].

Henceforth, we shall be comparing the solutions of initial as well as averaged equations generated by the same initial conditions, that is,

$$x(0, \mu) = \overline{x}\ (0, \mu) = x_0.$$

Averaged systems (9) and (12) are autonomous, unlike (8); hence, it is possible to advance toward investigating and finding their solutions. Of course, averaged equations are of little use if their solutions differ from those of the initial equations. However, Bogoliubov has proved a theory for standard systems, providing a justification for the method of averaging and thus establishing the $\epsilon$-closeness of the solutions.

Before formulating Bogoliubov's theorem, it is necessary to introduce an essential mathematical concept.

Definition 4. The vector function $\overline{X}(x)$ is called the *uniform mean value with respect to* $x$ of the vector function $X(x, t)$ for $x \in G_s$, if for any $\epsilon > 0$, there exists an $x$-independent $T(\epsilon) > 0$, such that the inequality,

$$\| \frac{1}{T} \int_0^{\tau} X(x, t)\, dt - \overline{X}(x) \| < \epsilon, \tag{13}$$

holds for any value of $\tau \geq T(\epsilon)$ and for all $x \in G_s$ [1].

Note that, if the domain $G_s$ is bounded, and if the norm $\| X(x, t) \|$ in $G_s$ is bounded and satisfies, with respect to $x$, the Lipschitz condition with a universal constant not depending on $t$, then the existence of a uniform mean follows from the limit (11) at every point $x \in G_s$.

Theorem 1 *Suppose that (1) the vector function $X\ (x, t)$, defined in the open, connected domain $G_{s+1} = G_s \times I$, is bounded in it by the constant $C$ and satisfies with respect to $x$ the Lipschitz condition with a constant $L$; (2) there exists a uniform integral mean with respect to $x \in G_s$; (3) the averaged system (12) has a solution $\overline{x}\ (t, \mu)$ defined for $t \in [0, \infty)$ and contained in $G_s$ with a certain $\rho$-neighborhood. Then for any $\epsilon > 0$, however small, and for any $A > 0$, however large, there exists a $\mu_0(\in, A) > 0$ such that, for all $\mu$ in the interval $0 \leq \mu < \mu_0$ and for all $t$ in the interval $0 \leq t > A\mu^{-1}$, the following inequality holds.*

$$\| x(t, \mu) - - \overline{x}(t, \mu) \| < \epsilon \tag{14}$$

The proof of this theorem and of its several modifications can be found in many books on the theory of nonlinear oscillations [1,12].

We now will give a generalization of Bogoliubov's theorem, called Filatov's theorem [5].

Theorem 2 *Suppose that conditions (1) and (3) of Theorem 1 are satisfied. Suppose, in addition to this, that (1) there exists an integral mean value $\overline{X}(x)$ at every point $x \in G_s$;*

*and (2) in every finite interval $[t_1, t_2]$ along the solution $\bar{x}(t, \mu)$, $\bar{X}(x)$ satisfies the inequality:*

$$\left\| \int_{t_1}^{t_2} \bar{X}\,(\bar{x}(\tau, \mu))\, d\tau \right\| < K\,(t_2 - t_1),\ K = constant.$$

*Then, for any $\epsilon > 0$ and $A > 0$, there exists a $\mu_0(\epsilon, A, \rho, L, \mu_0) > 0$ such that the following inequality is satisfied for $0 \le \mu < \mu_0$ in the time interval $0 \le t \le A\mu^{-1}$,*

$$\| x\,(t, \mu) - \bar{x}\,(t, \mu) \| < \epsilon$$

*Proof.* We observe first that the vector $\bar{X}(x)$ also satisfies the Lipschitz condition. Next we denote the interval $0 \le t \le A\mu^{-1}$ as $I^*$ and estimate the norm of it.

$$\left\| \mu \int_0^t [X(\bar{x}(\tau, \mu), \tau) - \bar{X}(\bar{x}(\tau, \mu))]\, d\tau \right\| =$$

$$\left\| \mu \int_0^t \varphi(\bar{x}, \tau)\, d\tau \right\|$$

Considering that $\varphi(\bar{x}, \tau) = 0$ for $t < \tau \le A\mu^{-1}$, we extend the integration over the entire interval $I^*$. This gives

$$\left\| \mu \int_0^t \varphi(\bar{x}\,(\tau, \mu), \tau)\, d\tau \right\| =$$

$$\Big\| \mu \sum_{k=0}^{m-1} \int_{t_k}^{t_{k-1}} [\varphi(\bar{x}(\tau, \mu), \tau) - \varphi(\bar{x}_k, \tau)]\, d\tau +$$

$$\mu \sum_{k=0}^{m-1} \int_{t_k}^{t_{k-1}} \varphi(\bar{x}_k, \tau)\, d\tau \Big\| \tag{15}$$

where

$$\bar{x}_k = \bar{x}(t_k, \mu),\ t_0 = 0.$$

From condition (1) of Theorem 1 and condition (2) of Theorem 2, we can obtain the following estimate.

$$\left\| \mu \sum_{k=0}^{m-1} \int_{t_k}^{t_{k-1}} [\varphi(\bar{x}(\tau, \mu), \tau) - \varphi(\bar{x}_k, \tau)]\, d\tau \right\| \le \frac{2LA^2K}{m}$$

Let us estimate the norm of the second term in (15). It follows from condition (1) that there exists a positive function $\alpha(x, t)$ which monotonically tends to zero for every fixed $\bar{x}$ as $t \to +\infty$, and

$$\left\| \int_0^t \varphi(\bar{x}, \tau)\, d\tau \right\| \le t\alpha\,(\bar{x}, t).$$

If we introduce the notations:

$$\alpha_0(\mu, m) = sup_{0\le\tau\le Am^{-1}} \tau\alpha(\overline{x}_0, \tau\mu^{-1}),$$

$$\beta(\mu, m) = sup_{0\le\tau\le \mathrm{Am}^{-1}} \tau\alpha(\overline{x}_1, \tau\mu^{-1}), \text{ and}$$

$$\alpha_k(\mu, m) = A\alpha(\overline{x}_k, A\mu^{-1}m^{-1}), k = 1,2,\ldots, m-1,$$

where $\alpha_k(\mu, m) \to 0$ and $\beta(\mu, m) \to 0$ for $\mu \to 0$, we get

$$\|\mu \sum_{k=0}^{m-1} \int_{t_k}^{t_{k-1}} \varphi(\overline{x}_k, \tau)\, d\tau\| \le \alpha_0 + \alpha_1 + 2\sum_{k=2}^{m-1} \alpha_k + \beta = \gamma(\mu, m).$$

Hence,

$$\|\mu \int_0^t \varphi(\overline{x}(\tau, \mu), \tau)\, d\tau\| \le \frac{2LA^2K}{m} + \gamma(\mu, m). \tag{16}$$

It follows from the estimates given by (16) that, for any $a > 0$, there exist $m(a)$ and $\mu_0(a, x_0)$ such that for $\mu < \mu_0$ the following inequality holds for the time interval.

$$\|\mu \int_0^t \varphi(\overline{x}(\tau, \mu), \tau)\, d\tau\| < a \tag{17}$$

Let us introduce a new unknown variable $u(t, \mu)$ as per the formula:

$$u(t, \mu) = \frac{x(t, \mu) - \overline{x}(t, \mu)}{a}. \tag{18}$$

This gives the integral equation,

$$u(t, \mu) = \mu a^{-1} \int_0^t [X(\overline{x}(\tau, \mu) + au(\tau, \mu), \tau)$$

$$- \overline{X}(\overline{x}(\tau, \mu))]\, d\tau,$$

or

$$u(t, \mu) = \mu a^{-1} \int_0^t [X(\overline{x}(\tau, \mu) + au(\tau, \mu), \tau) -$$

$$X(\overline{x}(\tau, \mu), \tau) + X(\overline{x}(\tau, \mu), \tau) - \overline{X}(\overline{x}(\tau, \mu))]\, d\tau,$$

for the unknown vector function $u(t, \mu)$.

Taking into account condition (1) of Theorem 1, the triangle axiom and estimate (17), we get the following inequality for the norm:

$$\| u(t, \mu) \| \leq \mu L \int_0^t \| u(\tau, \mu) \| \, d\tau + 1. \tag{19}$$

the solution of this inequality is

$$\| u(t, \mu) \| \leq e^{\mu L t}. \tag{20}$$

If we now let $a = e^{-AL} inf\{\epsilon, \rho\}$ we get $x(t, \mu) \in G_s$ for $t \in I$, and equation (18) gives us that

$$\| x(t, \mu) - \bar{x}(t, \mu) \| < \epsilon.$$

Thus the $\epsilon$-closeness of the solutions of equations (8) and (12) is proved.

*Remark 1.* It has been shown [6] that the condition $\varphi(\bar{x}, \tau) = 0$ is not necessary for proving this theorem.

*Remark 2.* While $\mu_0$ in Theorem 1 is independent of the starting point $x_0$, it does depend on $x_0$ in Theorem 2. This dependence should be seen as a sort of compensation for our not requiring the existence of a uniform mean.

We can also apply the partial smoothing operator to the system of standard type, averaging, for example, only some of the equations of the systems or only some individual terms of the right-hand sides. More detailed results are given in [6].

An important result in the asymptotic theory is the Banfi-Filatov theorem [7], which is a generalization of Bogoliubov's theorem for the case of an infinite time interval.

Theorem 3. *Suppose that the conditions of Theorem 1 are satisfied and also that the solution $\bar{X}(t, \mu)$ is uniformly and asymptotically stable with respect to t. Then for any $\epsilon > 0$, however small, there exists a $\mu_0 > 0$ such that, for all $\mu$ in the interval $0 \leq \mu \leq \mu_0$ and for all t, the following inequality holds.*

$$\| x(t, \mu) - \bar{x}(t, \mu) \| < \epsilon$$

## 3. ASYMPTOTIC METHODS

In a recent book review, O'Malley [17] gives a profound outline of the history of singular perturbations, starting with Prandtl's 1964 paper on fluid dynamic boundary layers. The benchmark works of Tikhonov [20, 21] and Levinson [13] were to have a major impact on control applications in the 1960's. Vasil'eva continuation [23] of Tikhonov's work and Wasow's book [25] finally placed singular perturbations within the framework of the analytic theory of differential equations. These texts, along with more recent books by Vasil'eva and Butuzov [23] and O'Malley [16] and a paper by Hoppensteadt [10], remain the most readable sources on asymptotic methods for ordinary differential equations.

### 3.1 Noncritical Case

In this section we will consider the Tikhonov-Levinson theory for the singularly perturbed initial value problem:

$$\frac{dx}{dt} = f(x, y, t, \mu)$$

and

$$\mu \frac{dy}{dt} = g(x, y, t, \mu), \tag{21}$$

where $x$ and $y$ are $m$- and $n$-dimensional vectors, respectively, in the interval $0 \leq t \leq 1$; functions $f$ and $g$ are infinitely differentiable in $x$, $y$ and $t$ and they have asymptotic series expansions in $\mu$. Such problems were first analyzed by Tikhonov and Levinson and their students [25].

The corresponding reduced problem consists of the nonlinear differential algebraic system,

$$\frac{dx_0}{dt} = f(x_0, y_0, t, 0)$$

and

$$0 = g(x_0, y_0, t, 0), \tag{22}$$

together with the initial condition $x_0(0) = x(0)$ [2, 3]. The reduced problem could provide the limiting solution on $0 < t \leq 1$, if the corresponding limiting inner problem,

$$\frac{dz_0}{d\tau} = g(x(0), z_0, 0, 0), \; z_0(0) = y(0), \tag{23}$$

were to have a bounded solution $z_0(\tau)$, for all $\tau > 0$, which matched $y_0$ in the sense that

$$z_0(\infty) = y_0(0).$$

We shall say that a given problem is *boundary layer stable* if a bounded solution $z_0(\tau)$ of this limiting inner problem (23) exists whenever $\tau > 0$ and has a limit at infinity. Occasionally, it happens that limiting solutions do not satisfy the limiting system.

If $g_y$ is not singular alond the solution $(x_0, y_0)$ of the reduced problem, we can differentiate the algebraic constraint $g(x_0, y_0, t, 0) = 0$ with respect to $t$ to obtain $g_x f + g_y \dot{y}_0 + g_t = 0$. Thus, $y_0$ satisfies the initial value problem,

$$y_0 = -g_y^{-1}(x_0, y_0, t, 0)\,[g_x(x_0, y_0, t, 0) f(x_0, y_0, t, 0) + g_t(x_0, y_0, t, 0)], \; y_0(0) = z_0(\infty),$$

which is coupled to the remaining initial value problem,

$$\frac{dx_0}{dt} = f(x_0, y_0, t, 0), x_0(0) = x(0).$$

Not that the initial value problem for $x_0$ and $y_o$ has the same differential order as the original system; but since it is not singularly perturbed, it is not a stiff problem. Assuming the existence of a solution $(x_0, y_0)$ throughout the interval $0 \leq t \leq 1$, we should be able to approximate it numerically without difficulty.

Sometimes we can directly find a solution,

$$y_0 = \Phi(x_0, t),$$

of the algebraic system $g(x_0, y_0, t, 0)$, starting at the point $(x(0), z_0(\infty), 0)$. Then the limiting solution is obtained by our simply solving the reduced-order initial value problem,

$$\frac{dx_0}{dt} = f(x_0, \Phi(x_0, t), t, 0) \equiv F(x_0, t), x_0(0) = x(0),$$

which gives us $x_0(t)$ and thereby $y_0$(t). The implicit function theorem, indeed, guarantees the existence of a locally unique root $\Phi$, as long as $g_y(x_0(t), y_0(t), t, 0)$ remains nonsingular; but it does not provide any simple way of solving $g = 0$ for $y_0$. Note that $g = 0$ may have other solutions. The right one to use, though, is the root selected through the limiting boundary layer stability problem. This is the solution $(x_0, y_0)$ passing through $(x(0), y_0(0), 0)$, where the initial value $y_0(0) = z_0(\infty)$ is obtained by integrating the initial value problem for $z_0(\tau)$ from $\tau = 0$ to $\infty$. If $g_y$ becomes singular during the $t$ integration, our procedure for obtaining the outer limit generally breaks down. Such problems were completely investigated by Vasil'eva [23] and we shall consider the main results in the next section.

Successful integration of the limiting inner problem (23) for $z(\tau)$ relies on stability considerations. Assume that

$$(z - z_0(\infty))' \, g(x(0), z, 0, 0) \leq -k(z - z_0(\infty))'(z - z_0(\infty))$$

holds for all relevant $z$ and for some $k > 0$. We can show that

$$z_0(\tau) = z_0(\infty) + O(e^{-k\tau})$$

holds for all $\tau \geq 0$, since $\eta_0(\tau) \equiv z_0(\tau) - z_0(\infty)$ satisfies the initial value problem: $d\eta_0 / d\tau = \underline{g((x(0)}, \eta_0 + z_0(\infty), 0, 0)$, $\eta_0(0) = y(0) - z_0(\infty)$. Using the inner product norm $\| \eta_0 \| = \sqrt{\eta_0' \eta_0}$, we have

$$\frac{d}{d\tau} \| \eta_0 \|^2 = 2 \frac{d\eta'_0}{d\tau} \eta_0 =$$

$$2\eta' g(x(0), \eta_0 + z_0(\infty), 0, 0) \leq -2k \| \eta_0 \|^2,$$

so, upon integration, the existence of a solution $\eta_0(\tau)$ on $\tau \geq 0$ is guaranteed. This solution satisfies

$$\| \eta_0(\tau) \| \leq e^{-k\tau} \| \eta_0 (0) \|$$

for all $\tau \geq 0$.

Because

$$\eta' g(x(0), \eta + z_0(\infty), 0, 0) = \eta' \left(\int_0^1 g_y (x(0), s\eta + z_0(\infty), 0, 0) ds\right) d\eta,$$

our stability hypothesis would hold under the classical assumption that $g_y(x(0), z, 0, 0)$ remains strictly stable for all $z$. To simply treat the linearized problem, let us instead assume that $g_y(x(0), z_0(\tau), 0, 0)$ remains strictly stable for all $\tau \geq 0$. This will imply the stability of $g_y(x(0), z_0(\infty), 0, 0) = g_y(x_0(0), y_0(0), 0, 0)$.

In an analogous fashion, it will be convenient to guarantee that the limiting outer solution $(x_0(t), y_0(t))$ will remain attractive to small perturbations throughout $0 \leq t \leq 1$. We shall guarantee stability of this limiting outer solution by assuming that

$$g_y(x_0(t), y_0(t), t, 0)$$

remains a strictly stable matrix for $0 \leq t \leq 1$.

Under these hypotheses, Tikhonov's theorem states that system (21), for $\mu \to 0$, has solutions,

$$x - x_0 = O(1) \text{ and } y - y_0 = O(1),$$

uniformly for $t \in [d, T]$, where $d$ is an arbitrary small positive number [20].

There are two proofs of this theorem—one by Hoppensteadt [9] and one by Tikhonov [21]. The proof by Hoppensteadt is based on the construction of Lyapunov functions and is quite different from Tikhonov's original proof.

In the further development of the asymptotic analysis, one would wish to construct approximations valid for $t \in [0, T]$. This has been accomplished by O'Malley [16, 17], following the work of Vasil'eva [23].

Using the above hypotheses with Tikhonov's theorem, we obtain an asymptotic solution to our initial value problem in the form,

$$\begin{aligned} x(t, \mu) &= X(t, \mu) + \mu\xi(\tau, \mu) \text{ and} \\ y(t, \mu) &= Y(t, \mu) + \eta(\tau, \mu), \end{aligned} \tag{24}$$

with an outer expansion

$$X(t, \mu) = \sum_{j=0}^{\infty} X_j(t)\mu^j \text{ and}$$

$$Y(t, \mu) = \sum_{j=0}^{\infty} Y_j(t)\mu^j.$$

There is an initial layer correction $(\mu\xi, \eta)$ satisfying

$$\xi(\tau, \mu) = \sum_{j=0}^{\infty} \xi_j(\tau)\mu^j \text{ and}$$

$$\eta(\tau, \mu) = \sum_{j=0}^{\infty} \eta_j(\tau)\mu^j,$$

whose terms $(\xi_j, \eta_j)$ all decay to zero as $\tau = t/\mu$ tends to infinity. Thus, the outer expansion must be a smooth solution of the system,

$$\dot{X} = f(X, Y, t, \mu) \text{ and}$$
$$\mu\dot{\mathrm{Y}} = g(\mathrm{X}, \mathrm{Y}, t, \mu).$$

Equating coefficients successively, we first find that $(X_0, Y_0)$ must satisfy the nonlinear reduced system,

$$\dot{X} = F(X_0, t), X_0(0) = x(0) \text{ and}$$
$$\mathrm{Y}_0 = \Phi(\mathrm{X}_0, t),$$

for the root $\Phi$ of $g(X_0, \Phi(X_0, t), t, 0) = 0$, obtained by matching the limiting inner solution at $t = 0^+$. Later terms of the form $(X_j, Y_j)$ must likewise satisfy linear differential algebraic systems. These systems must be of the form,

$$\dot{\mathrm{X}} = f_x(\mathrm{X}_0, \mathrm{Y}_0, t, 0)\,\mathrm{X}_j + f_y(\mathrm{X}_0, \mathrm{Y}_0, t, 0)\,\mathrm{Y}_j + \tilde{f}_{j-1}(t) \text{ and}$$
$$0 = g_x(\mathrm{X}_0, \mathrm{Y}_0, t, 0)\,\mathrm{X}_j + g_x(\mathrm{X}_0, \mathrm{Y}_0, t, 0)\,\mathrm{Y}_j + \tilde{g}_{j-1}(t),$$

where $\tilde{f}_{j-1}$ and $\tilde{g}_{j-1}$ are known in terms of $t$ and respectively in terms of the preceding coefficients $X_k$ and $Y_k$ for $k > j$ and their derivatives. The derivative with respect to $X_0$ of $g(X_0, \Phi, t, 0)$ equals 0 and this implies that $\Phi_x = -g_x{}^{-1}g_x$. Hence we can solve the latter algebraic system to obtain

$$Y_j = \Phi_x(X_0, t)\,X_j + \hat{g}_{j-1}(t).$$

This leaves linear system as

$$\dot{X}_j = F_x(X_0, t)\,X_j + \tilde{f}_{j-1}(t)$$

for $X_j$, since differentiation of $F(X, t)$ gives $f(X, \Phi, t, 0)$ and this implies that $F_x = f_x + f_y\Phi_x$. Note that homogeneous system, $dp/dt = F(X_0, t)p$, is the variation equation for $X_0$. It follows that the outer solution $(X(t, \mu), Y(t, \mu))$ can be completely determined asymptotically once the initial value $X(0, \mu)$ is specified. The initial condition for $X_j$ can be determined by the previous term $\xi_{j-1}$ in the initial layer correction, since the $\mu^j$ coefficient in the initial condition $x(0) = X(0, \mu) + \mu\xi(0, \mu)$ implies that

$$X_j(0) = -\xi_{j-1}(0)$$

for each $j > 0$.

The $\mu^j$ coefficient in the other initial condition, $y(0) = Y(0, \mu) + \eta(0, \mu)$, then specifies

$$\eta_j(0) = -Y_j(0),$$

which is already completely specified by $X_j(0)$. The initial value $\eta(0, \mu)$ is adequate for the termwise determination of the initial layer correction.

Differentiation of the assumed form (24) of the asymptotic solution tells us that the initial layer correction $(\mu\xi, \eta)$ must be obtained as a decaying solution of the nonlinear system,

$$\frac{d\xi}{d\tau} = f(X + \mu\xi, Y + \eta, \mu\tau, \mu) - f(X, Y, \mu\tau, \mu) \text{ and}$$

$$\frac{d\eta}{d\tau} = g(X + \mu\xi, Y + \eta, \mu\tau, \mu) - g(X, Y, \mu\tau, \mu),$$

on the semi-infinite interval $\tau \geq 0$. Then the leading terms must satisfy the nonlinear system

$$\frac{d\xi_0}{d\tau} = f(x(0), Y_0 + \eta_0, 0, 0) - f(x(0), y_0(0), 0, 0) \text{ and}$$

$$\frac{d\eta_0}{d\tau} = g(x(0), Y_0(0) + \eta_0, 0, 0) - g(x(0), Y_0(0), 0, 0),$$

on $\tau \geq 0$ and decay to zero as $\tau \to \infty$. Later terms must satisfy a linearized system,

$$\frac{d\xi_k}{d\tau} = f_y(x(0), Y_0(0) + \eta_0(\tau), 0, 0)\eta_k + p_{k-1}(\tau) \text{ and}$$

$$\frac{d\eta_k}{d\tau} = g_y(x(0), Y_0(0) + \eta_0(\tau), 0, 0)\eta_k + q_{k-1}(\tau),$$

where the $p_{k-1}$'s and $q_{k-1}$'s are known successively. We can show that these terms all decay to 0 exponentially as $\tau \to \infty$.

Note that $z_0(\tau) = \eta_0\tau + Y_0(0)$ satisfies the limiting inner problem:

$$\frac{dz_0}{d\tau} = g(x(0), z_0, 0, 0), \ z_0(0) = y(0).$$

Under our boundary layer stability hypothesis, for $\tau \geq 0$, $z_0(\tau)$ has a unique solution which decay exponentially to $Y_0(0)$ as $\tau \to \infty$. Knowing $z_0(\tau)$, we can obtain $\eta_0(\tau) = z_0(\tau) - Y_0(0)$ and, thereby, $d\xi_0/d\tau$. Because $\xi_0(\infty) = 0$, we must take

$$\xi_0(\tau) = -\int_\tau^\infty [f(x(0), z_0(s), 0, 0) - f(x(0), z_0(\infty), 0, 0)]\, ds.$$

Then, we can see that $\xi_0$, $\eta_0$ and their derivatives all decay exponentially to zero as $\tau \to \infty$. Moreover, $\xi_0$ determines $X_1(0) = -\xi_0(0)$ and, thereby, $X_1(t)$ and $Y_1(\mathrm{t})$. It also enables us to specify a linear initial value problem,

$$\frac{d\eta_1}{d\tau} = g_y(x(0), z_0(\tau), 0, 0)\eta_1 + q_0(\tau),\ \eta_1(0) = -Y_1(0),$$

for $\eta_1(\tau)$. Using variation of parameters, $\eta_1$ can be obtained in terms of the fundamental matrix $Q(\tau)$ for the linearized (or variational) system,

$$\frac{dQ}{d\tau} = g_y(x(0), z_0(\tau), 0, 0)\, Q,\ Q(0) = I,$$

that is,

$$\eta_1(\tau) = -Q(\tau)\, Y_1(0) + \int_0^\tau Q(\tau)Q^{-1}(s)q_0(s)\, ds.$$

Our boundary layer stability hypothesis implies that

$$Q(\tau) = O(e^{-k\tau}),$$

as $\tau \to \infty$. The exponential decay of $Q$ implies the same of $\eta_1$. Hence,

$$\xi_1(\tau) = -\int_\tau^\infty [f_y(x(0), z_0(s), 0, 0)\eta_1(s) + p_0(s)]\, ds.$$

Thus we have specified the initial vector $X_2(0) = -\xi_1(0)$ needed to obtain the second-order terms in the outer expansion. Later terms follow in the same manner. This procedure is called the O'Malley-Hoppensteadt construction [17].

## 3.2 Critical Case

The difficulty with the construction of the asymptotic expansions of (21) arises from the fact that the solution of the reduced problem (22) cannot, in general, satisfy all of the supplementary conditions prescribed for (21). All of the problems described in the above section were characterized by the fact that the equation $0 = g(x_0, y_0, t, 0)$ had one or several isolated solutions of the form $x_0$. However, in applications, one frequently encounters a case where this equation has a family of solutions which depends on several arbitrary functions. This case was investigated by Vasil'eva and Butuzov [24] and was called critical case.

Let us consider the differential equation,

$$\mu \frac{dx}{dt} = A(t)x + \mu f(x, t, \mu), \tag{25}$$

where $\mu > 0$ is a small parameter, $x$ and $f$ are $n$-dimensional vector-functions, $A(t)$ is an $n \times n$ matrix, and $0 \leq t \leq T$. A solution of equation (25) should satisfy the initial condition:

$$x(0, \mu) = x^0. \tag{26}$$

If we formally set $\mu = 0$ in (25), then we obtain the reduced equation:

$$A(t)\overline{x} = 0. \tag{27}$$

If $detA(t) \neq 0$ for $0 \leq t \leq T$, equation (27) has the unique solution $\overline{x} \equiv 0$. In [23] it was shown that, if the eigenvalues $\lambda_i(t)$ of $A(t)$ for $0 \leq t \leq T$ satisfy the inequalities

$$Re\lambda_i(t) < 0, \qquad i = 1, \ldots, n,$$

then the solution $x(t, \mu)$ of (25) with initial condition (26) converges, as $\mu \rightarrow 0$, to $\overline{x} \equiv 0$.

Suppose, however, that $detA(t) \equiv 0$ for $0 \leq t \leq T$. Then equation (27) has infinitely many solutions. There arises the question: under what conditions will the solution $x(t, \mu)$ of (25) with initial condition (26) converge as $\mu \rightarrow 0$ to one of these solutions and, in particular, to which one? The present section is concerned with this question as well as with the question of the construction of the asymptotic expansion of $x(t, \mu)$ with respect to $\mu$.

We impose the following four additional conditions on equation (25). They will help us reach an important conclusion.

**I**. Suppose that $A(t)$ and $f(x, t, \mu)$ have continuous partial derivatives of order $(m + 2)$ (with respect to each argument) for $0 \leq t \leq T$ and for $(x, t, \mu)$ in the domain $D(x, t, \mu) = D(x, t) \times [0, \mu_0]$, where $D(x, t)$ is a domain in $(x, t)$-space and $\mu_0$ is a positive constant.

The next two conditions are concerned with each eigenvalue $\lambda_i(t)$ for $i = 1,\ldots, n$ of $A(t)$. Note that the assumption that $detA(t) \equiv 0$ for $0 \leq t \leq T$ implies that at least one $\lambda_i$ $(t)$ is identically zero.

**II**. Suppose that for $0 \leq t \leq T$ the following conditions hold.

$$\lambda_i \equiv 0, \qquad i = 1, \ldots, k; k < n, \tag{28}$$

and

$$Re\lambda_i(t) < 0, \qquad i = k + 1, \ldots, n \tag{29}$$

*Remark 3.* In [23] the initial value problem was studied under the assumption that condition (29) was satisfied for all $i = 1,\dots, n$ (that is, the noncritical case). If at least one $\lambda_i(t)$ has a positive real part then, generally speaking, the solution of the initial value problem is unbounded as $\mu \to 0$.

**III**. Suppose that for each $t$ in $[0, T]$ there are $k$ linearly independent eigenvectors, denoted $e_i(t)$ for $i = 1,\dots, k$ of $A(t)$ corresponding to the $k$ identically zero eigenvalues.

Thus, we are considering cases where the number of linearly independent eigenvectors corresponding to $\lambda \equiv 0$ is equal to the multiplicity of $\lambda \equiv 0$. For the remaining eigenvalues for which $Re\lambda_i < 0$, neither their multiplicity nor the number of eigenvectors corresponding to them is of importance; indeed, both of these quantities can change as $t$ varies.

As we already stated, our goal is the construction of the asymptotic expansion of the solution of the problem (25)–(26). We construct a series formally satisfying equation (25) and condition (26) and having the form,

$$x(t, \mu) = \bar{x}(t, \mu) + \pi x(\tau, \mu). \tag{30}$$

$$\bar{x}(t, \mu) = \bar{x}_0 + \mu \bar{x}_1(t) + \dots + \mu^m \bar{x}_m(t) + \dots \tag{31}$$

is called the regular series, while

$$\pi x(\tau, \mu) = \pi_0 x(\tau) + \mu \pi_1 x(\tau) + \dots + \mu^m \pi_m x(\tau) + \dots \tag{32}$$

is called the boundary series for $\tau = t/\mu$.

The coefficients in the series (31) and (32) are determined by formally substituting (30) into (25) and (26) and equating terms with like powers of $\mu$ according to a definite rule which we state below. Each coefficient $\pi_i x(\tau)$ of the series (32) will be called a boundary function, and we will require that the boundary functions converge to zero as $\tau \to \infty$. Thus the formal algorithm for the construction of the series (31) and (32) requires that

$$\pi_i x(\tau) \to 0 \text{ as } \tau \to \infty. \tag{33}$$

Now we will present the procedure for determining the coefficients in (31) and (32). For this purpose we first represent $f(x, t, \mu)$ in the form,

$$f(\bar{x}(t, \mu) + \pi x(\tau, \mu), t, \mu) = f(\bar{x}(t, \mu), t, \mu) + [f(\bar{x}(\tau\mu, \mu) + \pi x(\tau, \mu), \tau\mu, \mu) -$$
$$f(\bar{x}(\tau\mu, \mu), \tau\mu, \mu)] \equiv \bar{f} + \pi f,$$

where

$$\bar{f}(\bar{x}, t, \mu) = \bar{f}_0(t) + \mu \bar{f}_1(t) + \dots + \mu^m \bar{f}_1(t) + \dots$$

and

$$\pi f = \pi_0 f(\tau) + \mu \pi_1 f(\tau) + \dots + \mu^m \pi_m f(\tau) = \dots .$$

We perform this operation on $A(t)x$:

$$A(t)\,(\overline{x}(t,\mu)+\pi x\,(\tau,\mu)) = A(t)\overline{x}(t,\mu) + A(\tau,\mu)\pi\, x\,(\tau,\mu) \equiv A\overline{x} + \pi(Ax).$$

Taking account of the transformations on $f$ and $Ax$, we now substitute (30) into (25) and (26):

$$\mu\frac{d}{dt}(\overline{x}_0 + \mu\overline{x}_1 + \ldots) + \frac{d}{d\tau}(\pi_0 x + \mu\pi_1 x + \ldots) = A\overline{x} + \pi(Ax) + \mu(\overline{f} + \pi f) \tag{34}$$

and

$$\overline{x}(0,\mu) + \pi\, x\,(0,\mu) = x^0. \tag{35}$$

Next we equate coefficients of like powers of $\mu$ on both sides of equations (34) and (35); and separating those terms depending on $t$ and those depending on $\tau$, we obtain equations and initial conditions for determining all of the coefficients $\overline{x}_i(t)$ and $\pi_i x(\tau)$ of the series (31) and (32).

For $\overline{x}_0(t)$ we obtain a linear homogeneous system of algebraic equations,

$$A(t)\overline{x}_0(t) = 0, \tag{36}$$

which coincides with the reduced equation (27). By virtue of condition III, the general solution of (36) can be written in the form,

$$\overline{x}_0(t) = \sum_{i=1}^{k} \alpha_i(t) e_i(t), \tag{37}$$

where the $e_i(t)$'s for $i = 1,\ldots, k$ are the linearly independent eigenvectors corresponding to the zero eigenvalues of $A(t)$, and the $\alpha_i(t)$'s are arbitrary scalar functions. We can rewrite (37) in the form,

$$\overline{x}_0(t) = e(t)\alpha(t), \tag{38}$$

where $e(t)$ is an $n \times k$ matrix and $\alpha(t)$ is a $k$-dimensional vector function.

For $\pi_0 x(\tau)$ we obtain a linear homogeneous system with constant coefficients for the differential equation:

$$\frac{d}{d\tau}\pi_0 x = A(0)\pi_0 x. \tag{39}$$

The general solution of this system can be written in the form,

$$\pi_0 x(\tau) = \sum_{i=1}^{k} c_i e_i(0) + \sum_{i=k+1}^{n} c_i w_i(\tau)\exp\,(\lambda_i(0)\tau), \tag{40}$$

where each $c_i$ for $i = 1,\ldots, n$ is an arbitrary constant, each $e_i(0)$ for $i = 1,\ldots, k$ is an eigenvector of $A(0)$ corresponding to a zero eigenvalue, and the $w_i(\tau)$'s for $i = k + 1,\ldots, n$ are known vector functions whose components are polynomials in $\tau$.

By virtue of condition (29), the second term on the right-hand side of (40) converges to zero as $\tau \to \infty$. Therefore, in order that condition (33) hold, it is necessary to set $c_i = 0$ for $i = 1,\ldots, k$.

The initial condition for $\pi_0 x(\tau)$ is

$$\pi_0\, x(0) = x^0 - \bar{x}_0(0) = x^0 - \sum_{i=1}^{k} \alpha_i(0) e_i(0);$$

or when $c_i = 0$ for $i = 1,\ldots, k$, we get

$$\sum_{i=1}^{k} \alpha_i(0) e_i(0) + \sum_{i=k+1}^{n} c_i w_i(0) = x^0. \tag{41}$$

System (41) has a unique solution. Thus $\pi_0 x(\tau)$ is completely determined. The function $\bar{x}_0(t)$ is not defined until each function $\alpha_i(t)$ for $i = 1,\ldots, k$, with each initial value $\alpha_i(0)$ found form (41), is first defined. Let us set $\alpha(0) = \alpha^0$.

For $\bar{x}_1(t)$ we obtain the linear nonhomogeneous system of algebraic equations,

$$A(t)\bar{x}_1(t) = -f(\bar{x}_0(t), t, 0) + \frac{d\bar{x}_0(t)}{dt} = \varphi(t). \tag{42}$$

Since $detA(t) \equiv 0$ for $0 \le t \le T$, a necessary and sufficient condition for the solvability of system (42) is that its right-hand side be orthogonal to each of the eigenvectors $g_j(t)$ for $j = 1,\ldots, k$ of the adjoint matrix $A^*(t)$ corresponding to the zero eigenvalues. Thus the solvability condition for (42) can be written as

$$< g_j(t), -f(e(t)\alpha(t), t, 0) + \frac{d}{dt}(e(t)\alpha(t)) > = 0, \qquad j = 1, \ldots, k,$$

where $<a, b>$ is a scalar product of two vectors. This condition can be rewritten as

$$(g(t)e(t))\frac{d\alpha}{dt} = g(t)(f(e(t)\alpha(t), t, 0) - e'(t)\alpha(t)), \tag{43}$$

with

$$\alpha(0) = \alpha_0,$$

or as

$$\frac{d\alpha}{dt} = F_0(\alpha, t). \tag{44}$$

**IV**. Suppose that equation (44) with initial condition $\alpha(0) = \alpha_0$ has a solution $\alpha = \alpha(t)$ for $0 \leq t \leq T$.

Now that $\alpha(t)$ is determined, the solution $\overline{x}_0(t)$ of the reduced system (36) is complete. The general solution of system (42) can be written as

$$\overline{x}_1(t) = \sum_{i=1}^{k} \beta_i(t)e_i(t) + \tilde{x}_1(t) = e(t)\beta(t) + \tilde{x}_1(t), \tag{45}$$

where $\tilde{x}_1(t)$ is a particular solution of (42) and $\beta(t)$ is an arbitrary $k$-dimensional vector function. For $\pi_1 x(\tau)$ we get

$$\frac{d\pi_1 x}{d\tau} = A(0)\pi_1 x + \tau A'(0)\pi_0 x(\tau) + f(\overline{x}_0(0) + \pi_0 x(\tau), 0, 0)$$

$$-f(\overline{x}_0(0), 0, 0)\ \pi_1 x(0) = -\ \overline{x}_1(0). \tag{46}$$

The general solution of (46) can be written as

$$\pi_1 x(\tau) = \sum_{i=1}^{k} d_i e_i(0) + \sum_{i=k+1}^{n} d_i w_i(\tau)\, exp\,(\lambda_i(0)\tau) + \tilde{\pi}_1 x(\tau), \tag{47}$$

where each $d_i$ is an arbitrary constant and $\tilde{\pi}_1 x(\tau)$ is a particular solution of (46). It is not difficult to see that $\tilde{\pi}_1 x(\tau)$ can be chosen so that $\| \tilde{\pi}_1 x(\tau) \|$ satisfies the same inequality as $\| \tilde{\pi}_0 x(\tau) \|$, that is, $\| \tilde{\pi}_1 x(\tau) \| \leq cexp(-k\tau)$, for $\tau \geq 0$.

The determination of the remaining terms in the series (31) and (32) proceeds analogously. At the $i$th stage, an arbitrary vector function $\gamma(t)$ enters the expresion for $x_i(t)$. First, we determine $\gamma(0)$ from:

$$\sum_{i=1}^{k} \gamma_i e_i(0) + \sum_{i=k+1}^{n} d_i w_i(0) = -\tilde{x}_i(0) - \tilde{\pi}_i x(0).$$

Then from the solvability condition for $\overline{x}_{i+1}(t)$, we obatin $\gamma(t)$, through a linear differential equation of the form:

$$\frac{d\gamma}{dt} = B(t)\gamma + F_i(t),$$

for which $\gamma_i(t)$ is finally determined.

The boundary functions $\pi_i x(\tau)$ are constructed like $\pi_1 x(\tau)$ and also satisfy the exponential estimate:

$$\| \pi_i x(\tau) \| \leq cexp\,(-k\tau), \qquad \tau \geq 0.$$

The following theorem is true for the estimate of the remainder term of series (30), if $m$-partial sum is $x_m(t, \mu) = \sum_{i=0}^{m} \mu^i(\bar{x}_i(t) + \pi_i x(\tau))$.

THEOREM 4 *Under conditions I–IV, there exist positive constants, $\mu_0$ and $c$, such that for $0 < \mu \leq \mu_0$ the solution $x(t, \mu)$ of the problem (25)–(26) exists in the interval [0, T], and it is unique and satisfies the inequality*:

$$\| x(t, \mu) - x_m(t, \mu) \| \leq c\mu^{m+1}, \qquad 0 \leq t \leq T.$$

*Remark 4.* (**a**). Techniques for constructing regular and boundary series are described in detail in [23] and so are the methods for estimating the remainder terms; (**b**). In [24] several problems of concrete physical importance in a number of fields (such as kinetics, the theory of semiconductors, numerical difference schemes) are discussed; (**c**). The results of section 3.2 can be extended in a natural way to cases where $A(t)$ is no longer a matrix, but rather a complex linear operator—for example, an integral operator. Thus, we can extend this techniques for integro-differential equations [24].

## 4. CONSTRUCTIVE METHOD

The present state of applied mathematics is characterized by the application of computer mathematics and computational techniques. The approximation methods have essentially become constructive, primarily because of the fact that the computers and the possibility of working out large programs have led to numerical solutions of several problems.

In this section we shall present one constructive method, based on algorithmic processes that converge in the general (Cauchy's) sense. For analyzing this method Lyapunov's majorizing equations technique will be used.

The general idea of this method is to reduce the initial differential equation to an operator system of the type,

$$x = LF(x, t, \mu), \tag{48}$$

where $F(x, t, \mu)$ is a vector function of $x = (x_1, x_2, \ldots, x_n)$, $t$ and small positive parameter $\mu$. $F(x, t, \mu)$ belongs to the classes $C[t]$ and $C[\mu]$ and it is differentiable (or Lipshitzian) on $x$ in the space,

$$G_{n+2} = G_n \times I_T \times I_{\mu 0} \text{ where } G_n : \| x \| \leq R, I_T : 0 \leq t \leq T,$$

and

$$I_{\mu 0} : 0 \leq \mu \leq \mu_0 \tag{49}$$

The operator $L$ is linear and bounded and, therefore, it is continuous in the space $G[I_T \times I_{\mu_0}]$. The following conditions should be satisfied for the function $F$.

$$F(0, t, 0) = 0$$

and

$$\frac{\partial F(0, t, 0)}{\partial x} = 0 \tag{50}$$

An algorithm for construction of the solutions of operator system (48) is the following:
1) Construct inequalities showing the limitation of $L$; for example,

$$(\| L\, \varphi(t) \|) \leq \wedge q \text{ and}$$
$$(\| \varphi(t) \|) \leq q \text{ for } t \in I_T, \tag{51}$$

where the symbol $(\|.\|)$ means a vector with components $\|.\|$ and $\wedge$ is a constant matrix.
2) Find Lyapunov's majorant $\Phi\,(\alpha, \mu)$ for the function $F(x, t, \mu)$ in $G_{n+2}$.
3) Write a system of Lyapunov's majorizing equations of the form,

$$\alpha = \wedge\, \Phi\,(\alpha, \mu), \tag{52}$$

where $\alpha$ is a vector for which

$$(\| x(t, \mu) \|) \leq \alpha(\mu).$$

We construct the required solutions of system (48) in any neighborhood of $\mu$ with the help of the following convergent successive approximations:

$$x_k = LF\,(x_{k-1}, t, \mu), \qquad k = 1,2, \ldots$$

and

$$x_0 \equiv 0. \tag{53}$$

If the functional majorizing Lyapunov's equations of the form (52) are compounded, we can estimate the interval of values of $\mu$ in which the required solutions exist and the iterations (53) are convergent. The following basic theorem is true.

THEOREM 5 *Suppose system (52) has, for $\mu \in [0, \mu_0]$, a solution $\alpha(\mu) \in C\,[0, \mu_0]$, which is positive for $\mu \geq 0$ and which is such that $\alpha(0) = 0$ and $\| \alpha(\mu_0) \| \leq R$. Then successive approximations (53) converge to the solution $x(t, \mu)$ of (48) for $t \in I_T$ and $\mu \in I_{\mu_0}$; and this solution is unique in the class $C(I_T \times I_{\mu_0})$ and it vanishes for $\mu = 0$.*

Let us consider the singularly perturbed system,

$$\epsilon B\dot{x} = Ax + \mu X\,(x, t), \tag{54}$$

where $x$ is an $n$-dimensional vector, $A$ and $B$ are $n \times n$ matrices for which $detB = 0$ and $rankB = n - r$, $X(x, t)$ is a $2\pi$ periodic vector function which is continuous on $t$ and nonlinear and differentiable on $x$ while $\| x \| \leq R$, and $\epsilon$ and $\mu$ are small parameters. The task is to find a $2\pi$ periodic solution $x(t, \mu, \epsilon)$ of (54), which is continuous in $\epsilon$ and $\mu$. We shall consider the case when $detA \neq 0$ and we shall call it the noncritical case.

For $\epsilon = 0$ we obtain a reduced system for system (54),

$$0 = Ax_0 + \mu X(x_0, t). \tag{55}$$

Since $A$ has no zero eigenvalues, then a unique solution $x = x_0(t, \mu)$ of system (55) exists in the interval $\mu \in [0, \mu_0]$. Let us replace in system (54)

$$x = x_0 + Dy, \tag{56}$$

where $D$ is a constant $n \times n$ matrix, such that matrix $D^{-1} BD$ is either a diagonal or a Jordan matrix. Then we get the following system for $y(t, \mu, \epsilon)$.

$$\epsilon D^{-1}BD\dot{y} = D^{-1}ADy + F(y, t, \mu, \epsilon), \tag{57}$$

where

$$F(y, t, \mu, \epsilon) = D^{-1}\left[-\epsilon B\frac{dx_0}{dt} + \mu X(x_0 + Dy, t) - \mu X(x_0, t)\right].$$

Let us denote $D^{-1}AD = C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$, where $C_i$'s for $i = 1, 2, 3, 4$ are $(n - r) \times (n - r), r \times (n - r), (n - r) \times r, r \times r$ matrices. Furthermore $u = (y_1, y_2, \ldots, y_{n-r})$, $v = (y_{n-r+1}, \ldots, y_n)$, $U = (F_1, F_2, \ldots, F_{n-r})$, $V = (F_{n-r+1}, \ldots, F_n)$, and $D^{-1}AD = \begin{bmatrix} \wedge \\ 0 \end{bmatrix}$. Then system (57) can be written in the form:

$$\epsilon \hat{\wedge} \dot{u} = C_1 u + C_2 v + U(u, v, \epsilon, \mu)$$

and

$$0 = C_3 u + C_4 v + V(u, v, \epsilon, \mu). \tag{58}$$

From the second subsystem of (58), we express $v = -C^{-1}_4 (V + C_3 u)$ and substitute it into the first subsystem. Then we get

$$\epsilon \dot{u} = \hat{C}u + \Phi (u, v, t, \mu, \epsilon) + \mu Qu$$

and

$$v = -C_4^{-1} [V(u, v, t, \mu, \epsilon) + C_3 u], \tag{59}$$

where

$$\hat{C} = (\hat{\wedge})^{-1} [C_1 - C_2 C_4^{-1} C_3],$$

$$\Phi (u, v, t, \mu, \epsilon) = \hat{\wedge}^{-1} [U - C_2 C_4^{-1} V] - \mu Qu,$$

and

$$Q = Q(t, \mu) = \frac{\partial U}{\partial x} |_{x=x_0}.$$

The successive approximations $u_k(t, \mu, \epsilon)$ and $v_k(t, \mu, \epsilon)$ for $k = 1, 2, \ldots$ can be determined as $2\pi$-periodic solutions of the system:

$$\epsilon\, \dot{u}_k = \hat{C} u_k + \Phi\,(u_{k-1}, v_{k-1}, t, \mu, \epsilon) + \mu Q u_{k-1}$$

and

$$v_k = C_4^{-1}\,[V(u_{k-1}, v_{k-1}, t\, \mu, \epsilon) + C_3 u_k],$$

with

$$u_0 \equiv 0 \text{ and } v_0 \equiv 0. \tag{60}$$

An auxiliary system which corresponds to the above iterative process is

$$\epsilon \dot{u} = \hat{C} u + \varphi(t, \mu, \epsilon) + \mu Q u$$

and

$$v = -C_4^{-1}\,[\psi(t, \mu, \epsilon) + C_3 u], \tag{61}$$

where $\varphi(t, \mu, \epsilon)$ and $\psi(t, \mu, \epsilon)$ are periodic functions in $t$ which are continuous in all arguments. Since $A$ and, therefore, $C$ have no critical eigenvalues, then this system has unique $2\pi$-periodic solutions, $u(t, \mu, \epsilon)$ and $v(t, \mu, \epsilon)$ in the interval $0 < \mu \leq \mu_1$. We shall consider this solution as a result of action by the operators $L_\epsilon$, $L^1{}_\epsilon$, and $L^2{}_\epsilon$ on the functions $\varphi(t, \mu, \epsilon)$ and $\psi(t, \mu, \epsilon)$.

The following operator system is equivalent to system (61).

$$u = L_\epsilon\,[\varphi(t, \mu, \epsilon) + \mu Q u]$$

and

$$v = L_\epsilon^1\,[\psi(t, \mu, \epsilon)] + L_\epsilon^2\,[\varphi(t, \mu, \epsilon) + \mu Q u], \tag{62}$$

where operators $L_\epsilon$, $L^1_\epsilon$, and $L^2_\epsilon$ are linear and bounded and satisfy the estimates:

$$|||\, L_\epsilon \varphi(t)\, ||| \leq \rho\, |||\, \varphi(t)\, |||,$$

$$|||\, L_\epsilon^1 \psi(t)\, ||| \leq \rho_1\, |||\, \psi(t)\, |||$$

and

$$||| L_\epsilon^2 \varphi(t) ||| \leq \rho_2 ||| \varphi(t) |||. \tag{63}$$

The majorizing equations coming from the operator system (62) and from the estimates of (63) can be written in the form,

$$\alpha = \rho \frac{||| \varphi |||}{1 - \mu\rho a}$$

and

$$\beta = \rho_1 ||| \psi ||| + \rho_2 ||| \varphi ||| \frac{1}{1 - \mu\rho a}, \tag{64}$$

where $\alpha$ and $\beta$ major $u$ and $v$, respectively, and $||| Q(t) ||| \leq a$.

Therefore, it can be granted that $2\pi$-periodic solutions, $u(t, \mu, \epsilon)$ and $v(t, \mu, \epsilon)$ of system (61) exist if $\mu\rho a < 1$. Also, these solutions can be found with successive approximations (60) and with

$$||| u(t, \mu, \epsilon) ||| \leq \alpha = \rho \frac{||| \varphi |||}{1 - \mu\rho a}$$

and

$$||| v(t, \mu, \epsilon) ||| \leq \beta = \rho_1 ||| \psi ||| + \rho_2 \frac{||| \varphi |||}{1 - \mu\rho a}. \tag{65}$$

These estimates allow us to write the majorizing equations for system (58) as

$$\alpha = \frac{\rho}{1 - \mu\rho a} \tilde{\Phi} (\alpha, \beta, \mu, \epsilon)$$

and

$$\beta = \rho_1 \tilde{\Psi} (\alpha, \beta, \mu, \epsilon) + \frac{\rho_2}{1 - \mu\rho a} \tilde{\Phi} (\alpha, \beta, \mu, \epsilon), \tag{66}$$

where $\tilde{\Phi}(\alpha, \beta, \mu, \epsilon)$ and $\tilde{\Psi}(\alpha, \beta, \mu, \epsilon)$ are Lyapunov's majorants for $U(u, v, t, \mu, \epsilon)$ and $v, t, \mu, \epsilon)$, respectively.

On the basis of the properties of Lyapunov's majorizing equations [14], we reach the following conclusion. For every fixed $\mu \in (0, 1/\rho a)$, there exists an interval $(0, \epsilon_*)$ in which system (66) has positive solutions, $\alpha = \alpha(\mu, \epsilon)$ and $\beta = \beta(\mu, \epsilon)$, such that $\alpha(\mu, 0) = \beta(\mu,$

0) = 0. Therefore, in this interval a unique periodic solution, $y(t, \mu, \epsilon)$, of system (57) exists and it vanishes as $\epsilon \to 0$. Also, this solution is a boundary of successive approximations (60). According to substitution (56), we obtain for system (54) the solution,

$$x(t, \mu, \epsilon) = x_0(t, \mu) + Dy(t, \mu, \epsilon), \tag{67}$$

which tends to $x_0(t, \mu)$ as $\epsilon \to 0$. Thus the following theorem has been proved.

THEOREM 6 *Suppose $\mu$ is such that $\mu\rho a < 1$ and $\epsilon \in (0, \epsilon_*]$ while system (66) has positive solutions $\alpha = \alpha(\mu, \epsilon)$ and $\beta = \beta(\mu, \epsilon)$ with $\alpha(\mu, 0) = \beta(\mu, 0) = 0$. The following conclusions hold.*

*i) System (54) has a unique $2\pi$-periodic solution, $x(t, \mu, \epsilon)$, which is continuous in $\epsilon$ and $\mu$ and which tends to the solution $x_0(t, \mu)$ of the reduced system (55) as $\epsilon \to 0$.*

*ii) The difference $x(t, \mu, \epsilon) - x_0(t, \mu)$ is equal to the limit of the successive approximations $\{u_k, v_k\}$ determined with the help of system (60), and*

$$|||\, x(t, \mu, \epsilon) - x_0(t, \mu)\, ||| \leq \gamma(\mu, \epsilon),$$

*where $\gamma(\mu, \epsilon) = col(\alpha, \beta)$.*

## References


1. Bogoliubov, N. N. and Yu. A. Mitropd'skii, *Asymptotic methods in the theory of nonlinear oscillations*, Moscow, 1963 (in Russian).
2. Campbell, S. L., *Singular systems of differential equations I*, Pitman, San Francisco, 1980.
3. Campbell, S. L., *Singular systems of differential equations II*, Pitman, San Francisco, 1982.
4. Eckhaus, W., *Asymptotic analysis of singular perturbations*, North Holland, Amsterdam, 1979.
5. Filatov, A. N., *Asymptotic methods in the theory of differential and integro-differential equations*, FAN Akad. Nauk Uz., Tashkent, 1974.
6. Filatov, A. N. and L. V., Sharova, *Integral inequalities and the theory of nonlinear vibrations*, Nauka, Moscow, 1976 (in Russian).
7. Filatov, A. N., *Presentation at VII International Konferentz uber Nichtlineare Schwingungen*, Band I, 1, Akademie-Verlag, Berlin 1977.
8. Grasman, J., *Asymptotic methods for relaxation oscillations and applications*, Springer-Verlag, New York, 1987.
9. Hoppensteadt, F. C., Singular perturbations on the infinite interval, *Trans. Amer. Math. Soc.* ***123***(1966), 521–535.
10. Hoppensteadt, F. C., Properties of solutions of ordinary differential equations with a Small Parameter, *Comm. Pure Appl. Math.* *24*(1971), 807–840.
11. Kokotovic, P. V., Applications of Singular perturbations techniques to control problems, *SIAM Review*, **26**, (1984), 501–550.
12. Krylov, N. M. and N. N. Bogoliubov, *Introduction to nonlinear mechanics*, Princeton University Press, Princeton, N. J. 1947.
13. Levin, J. J. and N., Levinson, Singular perturbations of nonlinear systems of differential equations and an associated boundary layer equations, *J. Rational Mech. Anal.* *3*(1954), 247–270.
14. Lyapunov, A. M. *General problem about the stability of motion*, Gostekhizdat, Moscow, 1950 (in Russian).
15. Minorsky, N., *Introduction to nonlinear mechanics*, J. W. Edwards, Ann Arbor, Mich., 1947.
16. O'Malley R., Jr., *Introduction to singular perturbations*, Academic Press, New York, 1974.
17. O'Malley R., Jr., *Singularly perturbation methods for ordinary differential equations*, Springer-Verlag, New York, 1991.
18. Slavova, A. D., Constructive method for solving nonlinear singularly perturbed systems in critical case, *JAMSA*, ***6***, NI, (1993) 69–83.
19. Smith, D. R., *Singular perturbation theory* Cambridge University Press, Cambridge, 1985.

20. Tikhonov, A. N., *Mat. Sb.*, **22**(*64), N2, (1984), 193–204.*
21. Tikhonov, A. N., Mat. Sb., 31(*73*), (1952), 575–586.
22. Van der Pol, B. On relaxation oscillations, *Phil. Mag*, **2**, (1926), 978–992.
23. Vasil'eva, A. B. and V. F. Butuzov, *Asymptotic exapansions of solutions singularly perturbed equations*, Nauka, Moscow, 1973 (in Russian).
24. Vasil'eva, A. B. and V. F. Butuzov, *Singularly perturbed equations in the critical case*, Nauka, Moscow, 1978 (in Russian).
25. Wasow, W., *Asymptotic expansions for ordinary Differential equations*, Wiley Interscience, New York, 1965.