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A UNIFIED APPROACH TO FIXED-ORDER CONTROLLER DESIGN VIA LINEAR MATRIX INEQUALITIES

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We consider the design of fixed-order (or low-order) linear controllers which meet certain performance and/or robustness specifications. The following three problems are considered; covariance control as a nominal performance problem, $\mathcal{D}$-stabilization as a robust stabilization problem, and robust L_∞ control problem as a robust performance problem. All three control problems are converted to a single linear algebra problem of solving a linear matrix inequality (LMI) of the type $BGC + (BGC)^T + Q < 0$ for the unknown matrix G. Thus this paper addresses the fixed-order controller design problem in a unified way. Necessary and sufficient conditions for the existence of a fixed-order controller which satisfies the design specifications for each problem are derived, and an explicit controller formula is given. In any case, the resulting problem is shown to be a search for a (structured) positive definite matrix X such that $X \in \mathcal{C}_1$ and $X^{-1} \in \mathcal{C}_2$ where $\mathcal{C}_1$ and $\mathcal{C}_2$ are convex sets defined by LMIs. Computational aspects of the nonconvex LMI problem are discussed.

KEYWORDS: Control theory, robust control, linear systems

1. INTRODUCTION

Fixed-order control design is a very important open problem which takes controller complexity into account. Several *analytical solutions*[1] to the fixed-order controller design problem are available in the literature ([15], [16], [25], [28], [29]). However, the main difficulty in these results is that it is not easy to develop a computational algorithm which guarantees to solve coupled nonlinear matrix *equations* describing the analytical solutions (*see* [26], [27], [36] for some computational approaches).

Linear matrix inequalities (LMIs) have gained much attention in recent years ([5], [10], [11], [20], [31], [45]) as a computational tool which plays a crucial role for solving certain control problems. If an analytical solution for a control problem is obtained in terms of coupled matrix equations, it is not easy to solve them. On the other hand, coupled LMIs can be solved efficiently by convex programming ([2], [4], [13]) if the coupling constraint is convex. For instance, an analytical solution to the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control problem is first

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[1] By analytical solutions, we mean some mathematical characterizations of controllers which satisfy design specifications. It should be noted that computational issues are not necessarily addressed by analytical solutions.

obtained in terms of coupled Riccati equations [3] which are nontrivial to solve. The computational difficulty has been overcome ([20], [23]) by a convex programming approach using LMIs where the controller order is not fixed *a priori* or is equal to the plant order. For the *fixed-order* controller case, however, convexity of the problem is destroyed by a nonconvex coupling constraint [20].

In this paper, we shall present analytical solutions to a variety of fixed order controller design problems in terms of coupled LMIs. The following control problems are considered; disturbance attenuation, robust stabilization, and robust performance problems. As a measure for the disturbance attenuation level in the first problem, we consider the covariance of the error signal subject to the white noise excitation. Such a matrix-valued performance measure can naturally handle multiple objective control problems, and has been studied extensively in the covariance control literature ([15], [41], [42], [47]). The problem we consider is to design a fixed-order stabilizing controller which yields the error covariance bounded above by a specified matrix. Such an upper bound approach results in a convex problem if the controller order is not fixed *a priori*, which is much easier than the problem of solving the exact assignability conditions for the state covariance [43] given by nonlinear coupled matrix *equations*. For the robust stabilization problem, we consider the design of controllers which robustly stabilize a linear time-invariant system with norm-bounded time-varying structured uncertainty. The notion of stability is Q-stability [10]. The concept of Q-stability is developed as a conservative, but computationally tractable upper bound on the structured singular value ([7], [9]). It is defined by a scaled $\mathscr{H}_\infty$ norm condition, and is closely related to quadratic stability ([24], [1], [34], [40]). Finally, as a robust performance problem, we shall consider a design of Q-stabilizing controllers for the given uncertain plant, which guaranteê a bound on the peak value of the error signal subject to unit energy disturbances for all admissible structured perturbations. Such a performance measure, known as the $\mathscr{L}_2$ to $\mathscr{L}_\infty$ gain, has been studied as an operator norm [46] or as a part of the covariance analysis [6]. These analysis results for the "nominal system" have been extended for systems with a norm-bounded time-varying structured uncertainty [17]. Control synthesis techniques for the $\mathscr{L}_2$ to $\mathscr{L}_\infty$ gain specification are available in [49], [37] for the case where there is no uncertainty in the plant and the controller order is equal to the plant order. We shall address the fixed-order controller design for systems with structured uncertainty based on the analysis result given in [17].

In our approach, all three control problems are formulated as the problem of solving an LMI for the controller parameter. Thus we provide a unifying method for obtaining analytical solutions for the fixed-order controller design problems with different specifications. Other control objectives such as the linear quadratic (LQ) cost and guaranteed LQ cost for uncertain systems ([17], [35], [44]) can also be treated with equal ease. Our main result states that there exists a fixed-order controller which meets certain specifications (described above) if and only if there exist (structured) symmetric matrices X and Y such that $X = Y^{-1} > 0$ and $X \in C_1$ and $Y \in C_2$ where C_1 and C_2 are convex sets defined by LMIs. Unfortunately, this problem is not convex, since each of the two sets is convex but the coupling constraint is not. However, we believe that our analytical solutions in terms of LMIs provide a new insight into the problem of fixed-order controller design, and hopefully efficient computational algorithms will be developed utilizing the convexity of C_1 and C_2. We shall discuss a heuristic approach to address the nonconvex LMI problem.

We shall use the following notation. An $n \times m$ matrix with real elements is denoted by $A \in \Re^{n \times m}$. A^T denotes the transpose of A. A^+ is the Moore-Penrose inverse of A. $N(A)$ and $R(A)$ denote the nullspace and the range space of A, respectively. $A^\perp$ denotes a left annihilator of A; $N(A^\perp) = R(A)$ and $A^\perp A^{\perp T} > 0$. Note that $A^\perp$ exists if and only if A has linearly dependent rows. Also note that, for a given A, $A^\perp$ is not unique, but throughout the paper, any choice is acceptable. The norm of a matrix $\|A\|$ is the largest singular value of A. For a symmetric matrix A, $\lambda_{max}(A)$ denotes the largest eigenvalue. For a symmetric nonnegative definite matrix A, $A^{1/2}$ denotes the unique nonnegative definite square root of A. The notation $A > 0$ for a positive definite matrix implies that A is symmetric. For a stable transfer matrix $T(s)$, $\|T\|_\infty$ and $\|T\|_2$ denote the H_∞ and H_2 norms, respectively. $\mathcal{E}[\cdot]$ is the expectation operator for stochastic processes.

2. CONTROL PROBLEMS

2.1. Framework for Control Design

Consider the feedback system depicted in Figure 1 where P is the generalized plant, C is the controller, and Δ is the uncertainty. We shall consider the following control problems;

- *Disturbance Attenuation*: Design a controller C for the nominal plant P ($\Delta \equiv 0$) such that the error signal e is sufficiently "small" in some sense in response to a certain class of disturbance signals d.
- *Robust Stabilization*: Design a controller C such that the closed loop system is internally stable for all perturbations Δ belonging to a known class of uncertainty set $\mathcal{B}\Delta$.
- *Robust Performance*: Design a controller C such that the closed loop system is internally stable and the error signal e is sufficiently attenuated in response to a certain class of disturbance signals d, for all perturbations Δ in the uncertainty set $\mathcal{B}\Delta$.

The above "conceptual" control problems will be made specific in the sequel. To this end, consider the following linear time-invariant continuous-time system with norm-bounded time-varying structured uncertainty;

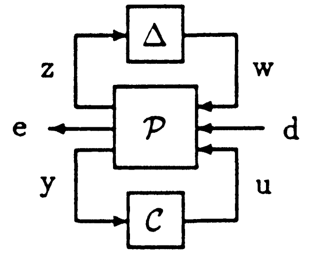

Figure 1 Control system configuration.

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \\ e(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A & B_0 & B_1 & B_2 \\ C_0 & D_{00} & D_{01} & D_{02} \\ C_1 & D_{10} & D_{11} & D_{12} \\ C_2 & D_{20} & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \\ d(t) \\ u(t) \end{bmatrix}, \; w(t) = \Delta(t) z(t)$$

where $x \in \Re^{n_p}$ is the state, $d \in \Re^{n_d}$ is the disturbance, $u \in \Re^{n_u}$ is the control input, $e \in \Re^{n_e}$ is the error signal, $y \in \Re^{n_y}$ is the measured output, and $z \in \Re^{n_z}$ and $w \in \Re^{n_w}$ are the exogenous signals to describe the uncertainty Δ. We shall assume $n_w = n_z$ for simplicity. The uncertainty Δ is known to belong to the following set;

$$\mathcal{B}\boldsymbol{\Delta} := \{\Delta : \Re \to \Re^{n_w \times n_z},\ \|\Delta(t)\| \le 1,\ \Delta(t) \in \boldsymbol{\Delta}\}, \tag{1}$$

where

$$\boldsymbol{\Delta} := \{block\ diag(\delta_1 I_{k_1} \ldots \delta_s I_{k_s} \Delta_1 \ldots \Delta_f) : \delta_i \in \Re, \quad \Delta_i \in \Re^{k_{s+i} \times k_{s+i}}\}.$$

For the static output feedback controller

$$u(t) = Gy(t),$$

the closed loop system can be described by

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \\ e(t) \end{bmatrix} = \begin{bmatrix} \hat{A} & \hat{B}_0 & \hat{B}_1 \\ \hat{C}_0 & \hat{D}_{00} & \hat{D}_{01} \\ \hat{C}_1 & \hat{D}_{10} & \hat{D}_{11} \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \\ d(t) \end{bmatrix} \tag{2}$$

where the closed loop matrices are defined by

$$\begin{bmatrix} \hat{A} & \hat{B}_0 & \hat{B}_1 \\ \hat{C}_0 & \hat{D}_{00} & \hat{D}_{01} \\ \hat{C}_1 & \hat{D}_{10} & \hat{D}_{11} \end{bmatrix} := \begin{bmatrix} A & B_0 & B_1 \\ C_0 & D_{00} & D_{01} \\ C_1 & D_{10} & D_{11} \end{bmatrix} + \begin{bmatrix} B_2 \\ D_{02} \\ D_{12} \end{bmatrix} G \, [C_2 \quad D_{20} \quad D_{21}] \tag{3}$$

where $D_{22} = 0$ is assumed to guarantee well-posedness of the feedback connection. For the dynamic controller of the form

$$\begin{bmatrix} \dot{x}_c(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} A_c & B_c \\ C_c & D_c \end{bmatrix} \begin{bmatrix} x_c(t) \\ y(t) \end{bmatrix}$$

where $x_c \in \Re^{n_c}$ is the controller state, the closed loop system has exactly the same structure as eq. (2) for the static output feedback case, but the plant matrices and the controller parameter must be replaced with the following augmented matrices;

$$\begin{bmatrix} A & B_0 & B_1 & B_2 \\ C_0 & D_{00} & D_{01} & D_{02} \\ C_1 & D_{10} & D_{11} & D_{12} \\ C_2 & D_{20} & D_{21} & G^T \end{bmatrix} \leftarrow \begin{bmatrix} A & 0 & B_0 & B_1 & B_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n_c} \\ C_0 & 0 & D_{00} & D_{01} & D_{02} & 0 \\ C_1 & 0 & D_{10} & D_{11} & D_{12} & 0 \\ C_2 & 0 & D_{20} & D_{21} & D_c^T & B_c^T \\ 0 & I_{n_c} & 0 & 0 & C_c^T & A_c^T \end{bmatrix}$$

where the closed loop state is $[x^T \; x_c^T]^T$. Hence, a fixed-order dynamic controller design problem is a special case of the static output feedback problem. For this reason, we shall mainly discuss the static output feedback problem in the sequel. Note, however, that standard assumptions such as $D_{12}^T D_{12} > 0$ are restrictive in this case since the augmented matrix $[D_{12} \; 0]$ for the dynamic controller will never be of full column rank. Therefore, we shall not impose any such assumptions. For simplicity, we assume that there is no redundant actuator ($B_2^T B_2 > 0$) or sensor ($C_2 C_2^T > 0$). These assumptions can be removed (*see* [20]).

2.2. Problem Formulation

As a disturbance attenuation problem, we shall focus on the covariance control problem. Consider the following plant

$$\begin{bmatrix} \dot{x}(t) \\ e(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & 0 \\ C_2 & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ d(t) \\ u(t) \end{bmatrix} \tag{4}$$

with a static output feedback controller $u(t) = Gy(t)$, where d is a zero-mean white noise input with covariance I. Let a matrix E denote the covariance of the error signal e;

$$E := \lim_{t \to \infty} \mathcal{E}[e(t) e^T(t)].$$

The problem can be stated as follows.

COVARIANCE CONTROL PROBLEM *Let a positive definite matrix $\bar{E} > 0$ be given. Determine if there exists a static output feedback gain G such that $E < \bar{E}$. Provide a formula for such a controller.*

The matrix-valued performance measure E defined in the stochastic setting can be used to define certain deterministic performance measures. For example, as is well-known, the H_2 norm of the transfer matrix $T_{ed}(s)$ from d to e is given by

$$\| T_{ed} \|_2^2 = tr(E).$$

Another example is the $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain ([6], [46]);

$$\sup_{t,d} \{ \| e(t) \|^2 : x(0) = 0, \int_0^\infty d^T(t)d(t)dt \leq 1, t \geq 0\} = \| E \|.$$

The $\mathcal{L}_2$ to $\mathcal{L}_\infty$ gain is the peak value (in the Euclidean norm sense) of the error signal e subject to the worst case disturbance d with unit energy. Thus, the covariance control problem can be slightly modified to address these different problems. The following lemma is useful to solve the covariance control problem.

LEMMA 1 *Let a matrix $\bar{E} > 0$ and a controller G be given. The following statements are equivalent.*

(*i*) *The controller G is stabilizing and yields $E < \bar{E}$.*

(*ii*) *There exists a matrix $P > 0$ such that*

$$\hat{A}P + P\hat{A}^T + \hat{B}_1\hat{B}_1^T < 0, \tag{5}$$

$$\hat{C}_1 P \hat{C}_1^T < \bar{E}.$$

Proof The result is well-known (*e.g.*, [37]), and hence the proof is omitted.

For the robust stabilization problem, consider the following uncertain system

$$\begin{bmatrix} \dot{x}(t) \\ x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A & B_0 & B_2 \\ C_o & D_{00} & D_{02} \\ C_2 & D_{20} & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ W(t) \\ u(t) \end{bmatrix}, \qquad w(t) = \Delta(t)z(t), \tag{6}$$

with a static output feedback controller $u(t) = Gy(t)$, where the uncertainty Δ belongs to the set $\mathcal{B}\Delta$ defined in eq. (1).

To pose a robust stabilization problem, let us introduce the notion of Q-stability, or the state space upper bound μ-test [10], which is defined by a scaled $\mathcal{H}_\infty$ norm condition on the transfer matrix T_{zw} from w to z as follows.

DEFINITION The uncertain linear system, eq. (6) is said to be Q-stable with respect to the uncertainty set $\mathcal{B}\Delta$ if T_{zw} is stable and there exists a matrix $S \in \mathcal{S}$ such that $\|ST_{zw}S^{-1}\|_\infty < 1$, where the scaling set $\mathcal{S}$ corresponding to $\mathcal{B}\Delta$ is defined by

$$\mathcal{S} := \{block\ diag(S_1 \ldots S_s s_1 I_{ks+1} \ldots s_f I_{ks+f}) : S_i \in \Re^{ki \times ki}, s_i \in R, S_i > 0, s_i > 0\}.$$

In general, Q-stability implies quadratic stability ([1], [22], [24], [33]) or, equivalently, the existence of a single quadratic Lyapunov function which can be used to prove stability of the whole family of the uncertain systems. See [30], [38] for detailed discussion on the relation between the two notions. Q-stability is known to be conservative, but its computational tractability motivates us to consider the following problem.

Q-Stabilization Problem *Determine if there exists a static output feedback gain G which Q-stabilizes the uncertain system (6). Give a formula for such a controller.*

Note that the Q-stabilization problem can also be interpreted as a robust disturbance attenuation problem if one of the uncertainty blocks represents the $\mathcal{L}_2$ to $\mathcal{L}_2$ performance block ([7], [8]).

The following lemma characterizes the Q-stability.

Lemma 2 *Let a controller G be given and consider the uncertain system (6). The following statements are equivalent.*

(*i*) *The closed loop system is* Q-stable for $\Delta \in \mathcal{B}\Delta$.

(*ii*) *There exist matrices* $P > 0$ *and* $S \in \mathcal{S}$ *such that*

$$\begin{bmatrix} P\hat{A} + \hat{A}^T P + \hat{B}_0 S \hat{B}_0^T & P\hat{C}_0^T + \hat{B}_0 S \hat{D}_{00}^T \\ \hat{C}_0 P + \hat{D}_{00} S \hat{B}_0^T & \hat{D}_{00} S \hat{D}_{00}^T - S \end{bmatrix} < 0. \tag{7}$$

Proof The result simply follows from the strict bounded real lemma [48] and the definition of Q-stability and the Schur complement formula.

Finally, we shall state a robust performance problem. Consider the following uncertain system

$$\begin{bmatrix} \dot{x}(t) \\ z(t) \\ e(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A & B_0 & B_1 & B_2 \\ C_0 & D_{00} & D_{01} & D_{02} \\ C_1 & 0 & 0 & 0 \\ C_2 & D_{20} & D_{21} & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \\ d(t) \\ u(t) \end{bmatrix}, \quad w(t) = \Delta(t)\, z(t), \tag{8}$$

with a static output feedback controller $u(t) = Gy(t)$, where $\Delta \in \mathcal{B}\Delta$ is the uncertainty. Let an $\mathcal{L}_\infty$ (peak value) performance of the error signal e be defined by

$$\hat{J}(G, \Delta, d) := \| e \|_{\mathcal{L}\infty}^2 \; (= \sup_{t \geq 0} [e^T(t) e(t)]),$$

where e is the error signal of the closed loop system with the controller G in response to a disturbance input d with a zero-initial state $x(0) = 0$ in the presence of the perturbation Δ. We define the worst case $\mathcal{L}_\infty$ performance J as follows;

$$J(G) := \sup_{\Delta, d} \{\hat{J}(G, \Delta, d): \int_0^\infty d^T(t) d(t) dt \leq 1, \Delta \in \mathcal{B}\Delta\}.$$

Now the problem can be stated as follows.

Robust $\mathcal{L}_\infty$ Control Problem *Let a positive scalar* $\gamma > 0$ *be given. Find a* Q-stabilizing static output feedback gain G such that J(G) < γ.

A practical significance of this problem is that the peak value of the error signal e is guaranteed to be less than $\sqrt{\gamma}$ for any disturbance d with a unit energy and for any norm-bounded structured perturbation $\Delta \in \mathcal{B}\Delta$.

The following lemma will be useful to solve the problem.

LEMMA 3 *Let a controller G be given and consider the uncertain system (8). The following statements are equivalent.*

(*i*) *The closed loop system is Q-stable for* $\Delta \in \mathcal{B}\Delta$.

(*ii*) *There exist matrices* $P > 0$ *and* $S \in \mathcal{S}$ *such that*

$$\begin{bmatrix} P\hat{A} + \hat{A}^T P + \hat{B}_0 S \hat{B}^T + \hat{B}_1 \hat{B}_1^T & P\hat{C}_0^T + \hat{B}_0 S \hat{D}_{00}^T + \hat{B}_1 \hat{D}_{01}^T \\ \hat{C}_0 P + \hat{D}_{00} S \hat{B}_0^T + \hat{D}_{01} \hat{B}_1^T & \hat{D}_{00} S D_{00}^T - S + \hat{D}_{01} \hat{D}_{01}^T \end{bmatrix} < 0. \quad (9)$$

In this case, we have

$$J(G) < \bar{J}(G)$$

where

$$\bar{J}(G) := \inf \{ \| \hat{C}_1 P \hat{C}_1^T \| : P > 0 \text{ and } S \in \mathcal{S} \text{ satisfy } (9) \}.$$

Proof See [17].

The robust $\mathcal{L}_\infty$ control problem may be considered as a combination of the covariance control and the Q-stabilization problems, where the robustness is guaranteed by the scaled $\mathcal{H}_\infty$ norm bound and the performance is measured by the maximum singular value of the "error covariance". It can be shown that $\hat{C}_1 P \hat{C}_1^T$ is an upper bound for the error covariance E for the nominal system ($\Delta \equiv 0$). A significance of this matrix here is that its maximum singular value defines a bound on the *worst case* $\mathcal{L}_\infty$ performance for the time-varying uncertain closed loop system. Thus, the robust $\mathcal{L}_\infty$ performance bound $J(G) < \gamma$ can be achieved by imposing the constraint $\bar{J}(G) < \gamma$. Since the performance bound $\bar{J}(G)$ in Lemma 3 is, in general, not tight, due to the conservativeness of Q-stability, the constraint $J(G) < \gamma$ does not imply $\bar{J}(G) < \gamma$. In this paper, however, we shall consider the design of Q-stabilizing controllers such that $\bar{J}(G) < \gamma$. If we let the bound γ approach infinity, then the problem becomes the Q-stabilization problem and the LMI (9) reduces to (7) by letting the performance matrices (the ones with subscript 1) be zero. On the other hand, if there is no uncertainty Δ or, equivalently, all the uncertainty matrices (the ones with subscript 0) are zero, then the LMI (9) reduces to the Lyapunov inequality (5), and the matrix $\hat{C}_1 P \hat{C}_1^T$ becomes the "tight" upper bound on the (nominal) error covariance E.

3. A UNIFIED LMI APPROACH

In this section, we shall show that all the three control problems stated in the previous section reduce to the same mathematical (linear algebra) problem, and can be solved in a unified way based on a linear matrix inequality.

THEOREM 1 *Consider a linear matrix inequality with an unknown matrix G*

$$BGC + (BGC)^T + D < 0. \tag{10}$$

(a) A controller G solves the covariance control problem if and only if there exists $P > 0$ *such that* $C_1 P C_1^T < \bar{E}$ *and (10) holds where*

$$[B\ C^T\ D] := \left[\begin{array}{c|c|cc} B_2 & PC_2^T & AP + PA^T & B_1 \\ 0 & D_{21}^T & B_1^T & -I \end{array}\right]. \tag{11}$$

(b) A controller G solves the Q-stabilization problem if and only if there exist $P > 0$ *and* $S \in \mathcal{S}$ *such that (10) holds where*

$$[B\ C^T\ D] := \left[\begin{array}{c|c|ccc} B_2 & PC_2^T & AP + PA^T & PC^T & B_0 \\ D_{02} & 0 & C_0 P & -S & D_{00} \\ 0 & D_{20}^T & B_0^T & D_{00}^T & -S^{-1} \end{array}\right]. \tag{12}$$

(c) A controller G solves the robust $\mathcal{L}_\infty$ *control problem if there exist* $P > 0$ *and* $S \in \mathcal{S}$ *such that* $\| C_1 P C^T{}_1 \| < \gamma$ *and (10) holds where*

$$[B\ C^T\ D] := \left[\begin{array}{c|c|cccc} B_2 & PC_2^T & AP + PA^T & PC_0^T & B_0 & B_1 \\ D_{02} & 0 & C_0 P & -S & D_{00} & D_{01} \\ 0 & D_{20}^T & B_0^T & D_{00}^T & -S^{-1} & 0 \\ 0 & D_{21}^T & B_1^T & D_{01}^T & 0 & -I \end{array}\right]. \tag{13}$$

Proof We shall prove (c) only. Statements (a) and (b) follow in a similar manner. Completing the square in (9), we have

$$\begin{bmatrix} \hat{A}P + P\hat{A} & P\hat{C}_0^T \\ \hat{C}_0 P & -S \end{bmatrix} + \begin{bmatrix} \hat{B}_0 & \hat{B}_1 \\ \hat{D}_{00} & \hat{D}_{01} \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{B}_0 & \hat{B}_1 \\ \hat{D}_{00} & \hat{D}_{01} \end{bmatrix}^T < 0.$$

Using the Schur complement formula, the above inequality is equivalent to

$$\begin{bmatrix} \hat{A}P + P\hat{A} & P\hat{C}_0^T & \hat{B}_0 & \hat{B}_1 \\ \hat{C}_0 P & -S & \hat{D}_{00} & \hat{D}_{01} \\ \hat{B}_0^T & D_{00}^T & -S^{-1} & 0 \\ \hat{B}_1^T & D_{01}^T & 0 & -I \end{bmatrix} < 0.$$

It is straightforward to verify the equivalence between the above inequality and the LMI (10) with (13) using the definitions for the closed-loop matrices (3).

Theorem 1 shows that all the three control problems stated in the previous section can be reduced to the single problem of solving the LMI (10) for the controller G. In general, solvability conditions will be given in terms of P and S. In this way, we can decouple the parameter search for (P,S,G) into those for G and (P,S), where the search for (P,S) does not involve the controller parameter G, and in fact, the search for G is not necessary since an explicit formula for G will be given in terms of (P,S). The following lemma provides a complete solution to this linear algebra problem.

LEMMA 4 *Let matrices* $B \in \Re^{n\times m}$, $C \in \Re^{k\times n}$ *and* $D = D^T \in \Re^{n\times n}$ *be given. Suppose rank* $(C) = k < n$. *Then there exists a matrix* $G \in \Re^{m\times k}$ *satisfying*

$$BGC + (BGC)^T + D < 0$$

if and only if the matrices B, C *and* D *satisfy*

$$B^{\perp} DB^{\perp T} < 0,\ \ C^{T\perp} DC^{T\perp T} < 0.$$

In this case, one such matrix G *is given by*

$$G = -\rho\, B^T\Phi\, C^T\, (C\, \Phi\, C^T)^{-1} \tag{14}$$

where $\rho > 0$ *is a (sufficiently large) scalar such that* $\Phi > 0$ *where*

$$\Phi := (\rho B\, B^T - D)^{-1}.$$

Proof See [20].

In the context of control problems, the full rank condition on the matrix C in Lemma 4 is satisfied by the assumption that there are no redundant sensors. If $B\,B^T > 0$ and/or $C^T C > 0$ in Lemma 4, then $B^{\perp}$ and/or $C^{T\perp}$ do(es) not exist, in which case, the LMI problem becomes much easier [20]. Clearly, the matrix G corresponds to the controller. Although Lemma 4 only provides one controller, an explicit formula for *all* controllers are available in [20].

4. STATIC OUTPUT FEEDBACK CONTROLLERS

This section presents solutions to the control problems described in section 2.2 in terms of LMIs. The results are obtained by directly applying Lemma 4 to Theorem 1. Since the unified LMI approach described in section 3 allows us to specialize the result for the robust $\mathcal{L}_\infty$ control problem to those for the Q-stabilization problem and the covariance control problem, we shall state the solution to the robust $\mathcal{L}_\infty$ control problem first, then obtain the results for the other two problems as special cases. To state the result, we need the following definitions;

$$\bar{B}_0 := [B_0\, B_1],\quad \bar{D}_{00} := [D_{00}\, D_{01}],\ \bar{D}_{20} := [D_{20}\, D_{21}],$$

$$\bar{S} := \begin{bmatrix} S & 0 \\ 0 & I \end{bmatrix},\ \hat{R} := \begin{bmatrix} R & 0 \\ 0 & I \end{bmatrix}.$$

THEOREM 2 *Let a scalar $\gamma > 0$ be given. The following statements are equivalent.*

(*i*) *There exists a static output feedback controller G which Q-stabilizes the uncertain plant (8) and yields the robust L_∞ performance bound $\bar{J}(G) < \gamma$.*

(*ii*) There exist matrices P, *Q, R and S such that $P = Q^{-1} > 0$ and $R = S^{-1} \in \mathcal{S}$ and*

$$C_l P C_l^T < I, \tag{15}$$

$$\begin{bmatrix} B_2 \\ D_{02} \end{bmatrix}^{\perp} \left(\begin{bmatrix} AP+PA^T & PC_0^T \\ C_0 P & -S \end{bmatrix} + \begin{bmatrix} \bar{B}_0 \\ \bar{D}_{00} \end{bmatrix} \bar{S}\, [\bar{B}_0^T\, \bar{D}_{00}^T] \right) \begin{bmatrix} B_2 \\ D_{02} \end{bmatrix}^{\perp T} < 0, \tag{16}$$

$$\begin{bmatrix} C_2^T \\ \bar{D}_{20}^T \end{bmatrix}^{\perp} \left(\begin{bmatrix} QA + A^T Q & Q\bar{B}_0 \\ \bar{B}_0^T Q & -R \end{bmatrix} + \begin{bmatrix} C_0^T \\ \bar{D}_{00}^T \end{bmatrix} R\, [C_0\, \bar{D}_{00}] \right) \begin{bmatrix} C_2^T \\ \bar{D}_{20}^T \end{bmatrix}^{\perp T} < 0. \tag{17}$$

In this case, one such controller is given by (14) where the matrices B, C and D are given by (13) using any matrices P, S satisfying the above inequalities with some Q and R.

Proof The result follows directly by applying Lemma 4 to statement (c) of Theorem 1, where we used the following choices of the left annihilators;

$$\begin{bmatrix} B_2 \\ D_{02} \\ 0 \end{bmatrix}^{\perp} = \begin{bmatrix} \begin{bmatrix} B_2 \\ D_{02} \end{bmatrix}^{\perp} & 0 \\ 0 & I \end{bmatrix}, \quad \begin{bmatrix} PC_2^T \\ 0 \\ \bar{D}_{20}^T \end{bmatrix}^{\perp} = \begin{bmatrix} \begin{bmatrix} C_2^T \\ \bar{D}_{20}^T \end{bmatrix}^{\perp} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} P^{-1} & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix}$$

The dimensions of the inequalities describing the existence conditions have been reduced by the use of the Schur complement formula.

The computational problem of finding matrices *P*, *Q*, *R*, and *S* satisfying the conditions in statement (ii) of Theorem 2 is not convex due to the coupling constraints $P = Q^{-1}$ and $R = S^{-1}$. Thus it is nontrivial to develop an algorithm which can find such matrices whenever they exist. The computational issue will be discussed in section 5. Here, let us specialize the result to the state feedback case. The following corollary shows that the state feedback problem can be made convex *even with the nonzero D_{00} term* (although many of the currently available state feedback or full information results in the literature (*e.g.*, [32]) assume that $D_{00} = 0$).

COROLLARY 1 *Let a scalar $\gamma < 0$ be given. Suppose the state can be measured without noise*;

$$C_2 = I,\ D_{20} = 0,\ D_{21} = 0.$$

Then the following statements are equivalent.

(*i*) *There exists a static state feedback controller G which Q-stablizes the uncertain plant (8) and yields the robust $\mathcal{L}_\infty$ performance bound $\bar{J}\,(G) < \gamma$.*

(*ii*) *There exist matrices $P > O$ and $S \in \mathcal{S}$ such that*

$$C_1 P C_1^T < \gamma I, \quad S > \bar{D}_{00} \bar{S} \bar{D}_{00}^T, \tag{18}$$

$$\begin{bmatrix} B_2 \\ D_{02} \end{bmatrix}^{\perp} \left(\begin{bmatrix} AP + PA^T & PC_0^T \\ C_0 P & -S \end{bmatrix} + \begin{bmatrix} \bar{B}_0 \\ \bar{D}_{00} \end{bmatrix} \bar{S} [\bar{B}_o^T \bar{D}_{00}^T] \right) \begin{bmatrix} B_2 \\ D_{02} \end{bmatrix}^{\perp T} < 0.$$

Proof Using a choice of left annihilator

$$\begin{bmatrix} C_2 & | & D_{20} & D_{21} \end{bmatrix}^{T\perp} = \begin{bmatrix} 0 & I \end{bmatrix},$$

the LMI (17) reduces to, with $R = S^{-1}$

$$\begin{bmatrix} S^{-1} & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} D_{00}^T \\ D_{01}^T \end{bmatrix} S^{-1} [D_{00}\, D_{01}] > 0. \tag{19}$$

Then it is straightforward to verify the equivalence between the second inequality in (18) and (19) using the Schur complement formula.

Clearly, the set of matrices P and S satisfying the conditions in statement (ii) is convex, and hence we can find a feasible controller for a given performance bound γ via convex programming, or determine that none exists. Note that the conditions also define a convex set with respect to P, S and γ. Thus the bound γ can be minimized by convex programming and an optimal controller can be computed by using the formula given in Lemma 4.

The following theorem provides a solution to the Q-stablization problem.

THEOREM 3 *The following statements are equivalent.*

(*i*) *There exists a static output feedback controller G which Q-stabilizes the uncertain plant (6).*

(*ii*) *There exist matrices P, Q, R and S such that $P = Q^{-1} > 0$, $R = S^{-1} \in \mathcal{S}$ and*

$$\begin{bmatrix} B_2 \\ D_{02} \end{bmatrix}^{\perp} \left(\begin{bmatrix} AP + PA^T & PC_0^T \\ C_0 P & -S \end{bmatrix} + \begin{bmatrix} B_0 \\ D_{00} \end{bmatrix} S [B_0^T D_{00}^T] \right) \begin{bmatrix} B_2 \\ D_{02} \end{bmatrix}^{\perp T} < 0,$$

$$\begin{bmatrix} C_2^T \\ D_{02}^T \end{bmatrix}^{\perp} \left(\begin{bmatrix} QA + A^T Q & QB_0 \\ B_0^T Q & -R \end{bmatrix} + \begin{bmatrix} C_0^T \\ \bar{D}_{00}^T \end{bmatrix} R [C_0\, D_{00}] \right) \begin{bmatrix} C_2^T \\ D_{20}^T \end{bmatrix}^{\perp T} < 0.$$

In this case, one such controller is given by (14) where the matrices B, C and D are given by (12) using any matrices P, S satisfying the above inequalities with some Q and R.

Proof Letting the matrices B_1, C_1, D_{01}, and D_{21} be zero in Theorem 2, the $\mathcal{L}_\infty$ performance bound can be removed and the Q-stability remains as the only control design specification. Thus, we have the result as a special case of Theorem 2.

Recall that Q-stability is equivalent to the $\mathcal{H}_\infty$ norm bound $\|T_{zw}\|_\infty < 1$ if the uncertainty is unstructured. Hence the above result can be specialized (by letting $S = I$) to the result of [20] which solves the general $\mathcal{H}_\infty$ control problem.

If we assume that there is no uncertainty in the plant, then Theorem 2 reduces to the nominal $\mathcal{L}_\infty$ control problem. In this case, the matrix P becomes a tight upper bound for the closed-loop state covariance and hence, Theorem 2 can be specialized to the solution for the covariance control problem with a slight modification of the $\mathcal{L}_\infty$ performance constraint as follows.

THEOREM 4 *Let a matrix $\bar{E} > 0$ be given. The following statements are equivalent.*

(i) There exists a static output feedback controller G which stabilizes the plant (4) and yields the error covariance $E < \bar{E}$.

(ii) There exist matrices P and Q such that $P = Q^{-1} > 0$ and

$$C_1PC_1 < \bar{E},\ B_2^{\perp}(AP+PA^T + B_1B_1^T)B_2^{\perp T} < 0,$$

$$\begin{bmatrix} C_2^T \\ D_{21}^T \end{bmatrix}^{\perp} \begin{bmatrix} QA + A^TQ & QB_1 \\ B_1^TQ & -I \end{bmatrix} + \begin{bmatrix} C_2^T \\ \bar{D}_{21}^T \end{bmatrix}^{\perp T} < 0.$$

In this case, one such controller is given by (14) where the matrices B, C and D are given by (11) using P satisfying the above inequalities with some Q.

Proof Letting matrices B_0, C_0, D_{00}, D_{01}, D_{02} and D_{20} in Theorem 2 be zero, the robustness specification of the robust $\mathcal{L}_\infty$ control problem is removed. Then the result follows from appropriate choices of left annihilators and the use of the Schur complement formula.

For the Q-stabilization and the covariance control problems, the resulting computational problem is nontrivial due to the coupling condition $P = Q^{-1}$ as for the robust $\mathcal{H}_\infty$ control problem. Clearly, the state feedback case reduces to a tractable convex problem in either control problem (*see* Corollary 1). If we consider the dynamic controller with *unspecified order*, it can be shown using the technique similar to [20] that, for all the control problems considered above, the coupling constraint $P = Q^{-1} > 0$ is replaced by a *convex* coupling constraint $P \geq Q^{-1} > 0$, or equivalently,

$$\begin{bmatrix} P & I \\ I & Q \end{bmatrix} \geq 0.$$

Thus the covariance control problem with unspecified order becomes convex. Note, however, that the other two control problems are still nonconvex due to the coupling condition on the scaling, that is, $S = R^{-1} \in \mathcal{S}$.

5. COMPUTATIONAL ASPECTS

For all the control problems, the resulting computational problems are of the same type, and can be stated in a general form as follows; Find structured symmetric matrices X and Y such that $X = Y^{-1} > 0$ and $X \in \mathcal{C}_1$ and $Y \in \mathcal{C}_2$, where $\mathcal{C}_1$ and $\mathcal{C}_2$ are convex subsets of positive definite matrices defined by LMIs. For the robust $\mathcal{H}_\infty$ control problem, for instance, the structured positive definite matrices X and Y are

$$X = \begin{bmatrix} P & 0 \\ 0 & S \end{bmatrix}, \; P > 0, S \in \mathcal{S}, \; Y = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}, \; Q > 0, R \in \mathcal{S},$$

and the set $\mathcal{C}_1$ is defined by (15) and (16), while the set $\mathcal{C}_2$ is defined by (17) in Theorem 2. Clearly, this problem is not convex due to the coupling constraint $X = Y^{-1}$. This type of problem first appeared in the context of the $\mathcal{L}_\infty$ control with constant scalings for discrete-time systems [32], and has recently been derived as necessary and sufficient conditions for stabilizability via static output feedback for both continuous- and discrete-time systems ([19], [20]). Thus, the problem exhibits a typical difficulty in the design of *fixed-order* controllers. On the other hand, our claim is that, *if* we can develop a computational algorithm to solve this type of problem, then *many of* the fixed order controller design problems with different specifications can be solved.

In the rest of this section, we shall discuss a simple heuristic approach to attack this difficult problem. Consider the following minimization problem;

$$\alpha^* := \min \{ \alpha : \frac{1}{\alpha} \le Y^{1/2} X Y^{1/2} \le \alpha I, X \in \mathcal{C}_1, \quad Y \in \mathcal{C}_2 \}.$$

If the sets $\mathcal{C}_1$ and $\mathcal{C}_2$ are nonempty, then the above optimization problem is well-posed, that is, it has a nonempty feasible domain. Clearly, the original problem of finding $X \in \mathcal{C}_1$ and $Y \in \mathcal{C}_2$ such that $X = Y^{-1}$ has a solution if and only if the optimal value of the above problem is $\alpha^* = 1$. Unfortunately, the minimization problem is still nonconvex and it is difficult to compute α^*. Nevertheless, one can compute a local solution $\bar{\alpha}^*$ which is an upper bound of α^* with a reasonable amount of computation. One way to compute $\bar{\alpha}^*$ is to solve the above minimization problem over one variable while fixing the other alternately;

$$(\hat{\alpha}_k, X_k) := \arg\min \{ \alpha : Y_k^{-1} \le \alpha X, X \le \alpha Y_k^{-1}, X \in \mathcal{C}_1 \},$$
$$(\alpha_{k+1}, Y_{k+1}) := \arg\min \{ \alpha : X_k^{-1} \le \alpha Y, Y \le \alpha X_k^{-1}, Y \in \mathcal{C}_2 \}.$$

Note that each problem is a version of the generalized eigenvalue minimization problem [4], which is quasiconvex. With this approach, the value of α is nonincreasing, that is,

$$\alpha_k \ge \hat{\alpha}_k \ge \alpha_{k+1} \ge \dots \ge 1,$$

and hence (local) convergence is guaranteed. This type of algorithm was first proposed for fixed order output feedback stabilization problem [12], and its extensions have been investigated ([21], [18], [39]) utilizing the notion of the analytic center. From our numerical experiences, it often happens that the local solution $\bar{\alpha}^*$ turns out to be global with $\bar{\alpha}^* = 1$. Thus we believe that it is worth trying these heuristic algorithms to solve the fixed order control problems considered in this paper for practical applications, although there is no guarantee for global convergence.

6. CONCLUSIONS

A unified approach based on linear matrix inequalities for fixed-order controller design is proposed. The covariance control problem, the Q-stabilization problem and the robust $\mathscr{L}_\infty$ control problem are all reduced to a single problem of solving an LMI

$$BGC + (BGC)^T + D < 0,$$

for the controller G, where matrices C and D are functions of positive definite matrices $P > 0$ and $S \in S$ (Theorem 1). Necessary and sufficient conditions for the existence of a feasible controller are given in terms of matrix inequalities involving (P, S), and an explicit controller formula is given using the matrix pair (P, S).

The problem of finding a matrix pair (P, S) is shown to be of the same type for all the control problems considered in this paper, and is given by: Find a matrix pair (X, Y) such that

$$X = Y^{-1} > 0, \qquad X \in \mathscr{C}_1, \qquad Y \in \mathscr{C}_2$$

where $\mathscr{C}_1$ and $\mathscr{C}_2$ are finite dimensional convex set defined by LMIs. Such a nonconvex LMI problem has arisen in the literature ([32], [20], [19]), and our results show that many other fixed-order control problems can be reduced to the nonconvex LMI problem in a unified way. In fact, the approach described in this paper may be used to derive analytical solutions to most of the control problems with design specifications given in terms of Riccati and/or Lyapunov-type inequalities, such as the LQ suboptimal control, the robust $\mathscr{H}_2$ control ([17], [44]), the mixed $\mathscr{H}_2/\mathscr{H}_\infty$ control ([3], [23]), and the positive real control [14]. Although the nonconvex LMI problem is a difficult problem to solve, it is possible to develop heuristic algorithms which are of practical value. One such approach is discussed.

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