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Research Article

The Vulnerability of Some Networks including Cycles via Domination Parameters

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Let $G = (V(G), E(G))$ be an undirected simple connected graph. A network is usually represented by an undirected simple graph where vertices represent processors and edges represent links between processors. Finding the vulnerability values of communication networks modeled by graphs is important for network designers. The vulnerability value of a communication network shows the resistance of the network after the disruption of some centers or connection lines until a communication breakdown. The domination number and its variations are the most important vulnerability parameters for network vulnerability. Some variations of domination numbers are the 2-domination number, the bondage number, the reinforcement number, the average lower domination number, the average lower 2-domination number, and so forth. In this paper, we study the vulnerability of cycles and related graphs, namely, fans, k-pyramids, and n-gon books, via domination parameters. Then, exact solutions of the domination parameters are obtained for the above-mentioned graphs.

1. Introduction

Graph theory has become one of the most powerful mathematical tools in the analysis and study of the architecture of a network. Networks are important structures and appear in many different applications and settings. The study of networks has become an important area of multidisciplinary research involving computer science, mathematics, chemistry, social sciences, informatics, and other theoretical and applied sciences [1–3].

It is known that communication systems are often exposed to failures and attacks. So robustness of the network topology is a key aspect in the design of computer networks. The stability of a communication network, composed of processing nodes and communication links, is of prime importance to network designers. As the network begins losing links or nodes, eventually there is a loss in its effectiveness [4]. In the literature, various measures were defined to measure the robustness of network and a variety of graph theoretic parameters have been used to derive formulas to calculate network vulnerability. Graph vulnerability relates to the study of graph when some of its elements (vertices or edges) are removed. The measures of graph vulnerability are usually invariants that measure how the deletion of one or more network elements changes properties of the network [5, 6]. The best known measure of reliability of a graph is its connectivity. The vertex (edge) connectivity is defined to be the minimum number of vertices (edges) whose deletion results in a disconnected or trivial graph [7]. Then the toughness [8], the integrity [9], the domination number [10, 11], the bondage number [12, 13], the 2-domination number [14], and the 2-bondage number [15] have been defined. Moreover, there are many graph theoretical parameters depending upon local damage for the graphs like the average lower independence number [16, 17], the average lower domination number [17, 18], the average connectivity [19], the average lower connectivity [20] and the average lower bondage number [6]. The average parameters have been found to be more useful in some circumstance than the corresponding measures based on worst-case situation [6].

A natural way to model the topology of a communications network is as a graph consisting of vertices and edges. In this paper, we consider simple finite undirected graphs by ignoring any variation in the type of edges. Let $G = (V(G), E(G))$ be a simple undirected graph of order n. We begin by recalling some standard definitions that we need throughout this paper. For any vertex $v \in V(G)$, *the open neighborhood of* v is $N_G(v) = \{u \in V \mid uv \in E(G)\}$ and *closed neighborhood of* v is $N_G[v] = N_G(v) \cup \{v\}$. The *degree of vertex* v in G is denoted by $d_G(v)$, that is, the size of its open neighborhood. The *maximum degree* of G is $\max\{d_G(v) \mid v \in V(G)\}$ and the *minimum degree* of G is $\min\{d_G(v) \mid v \in V(G)\}$. The maximum and minimum degrees of a graph G are denoted by $\Delta(G)$ and $\delta(G)$, respectively [11]. The graph G is called *r-regular graph* if $d_G(v) = r$ for every vertex $v \in V(G)$. The vertex v is called isolated vertex if $d_G(v) = 0$. The *null graph* on n-vertices consists of n-isolated vertices with no edges. The join of graphs G and H, denoted by $G \vee H$, is obtained from the disjoint union $G + H$ by adding the edges $\{vw \mid v \in V(G), w \in V(H)\}$ [21]. We will use $\lfloor x \rfloor$ and $\lceil x \rceil$ for the largest integer not larger than x and smallest integer not less than x, respectively.

Cycles and various related graphs have been studied for many reasons. Fans, wheels, pyramids, bipyramids, and n-cycle books are among such graphs. The definitions of these graphs will be given in Sections 3.2, 3.3, and 3.4.

Our aim in this paper is to consider the computing of the average lower domination number (ALDN) and the average lower 2-domination number (AL2DN) of some networks including cycles. In Section 2, definitions and well-known basic results have been given for ALDN and AL2DN, respectively. In Section 3, ALDN and AL2DN of some networks including cycles, namely, fans, k-pyramids, and n-gon books, have been determined.

2. The Average Lower Domination Number Parameters and Basic Results

A set $S \subseteq V(G)$ is a *dominating set* if every vertex in $V(G)-S$ is adjacent to at least one vertex in S. The minimum cardinality taken over all *dominating sets* of G is called the *domination number* of G and it is denoted by $\gamma(G)$ [10]. Moreover, a *2-dominating set* of a graph G is a set $D \subseteq V(G)$ of vertices of graph G such that every vertex of $V(G) - D$ has at least two neighbors in D. The *2-domination number* of a graph G, denoted by $\gamma_2(G)$, is the minimum cardinality of a *2-dominating set* of the graph G [2, 11, 14].

In 2004, Henning introduced the concept of average domination and average independence [17]. The *average lower domination number* of a graph G, denoted by $\gamma_{\mathrm{av}}(G)$, is defined as $\gamma_{\mathrm{av}}(G) = (1/|V(G)|)\sum_{v\in V(G)} \gamma_v(G)$, where the *lower domination number*, denoted by $\gamma_v(G)$, is the minimum cardinality of a dominating set of the graph G that contains the vertex v [17, 18, 22]. In [23], the *average lower 2-domination number* of a graph G was defined. The AL2DN is defined by $\gamma_{2\mathrm{av}}(G) = (1/|V(G)|)\sum_{v\in V(G)} \gamma_{2v}(G)$, where the *lower 2-domination number*, denoted by $\gamma_{2v}(G)$, of the graph G relative to v is the minimum cardinality of 2-dominating set in the graph G that contains the vertex v. Moreover, Turaci showed that AL2DN is more sensitive than other vulnerability parameters, namely, connectivity, domination number, ALDN, and 2-domination number, in [23].

Theorem 1 (see [17]). *For any graph G of order n with domination number $\gamma(G)$, $\gamma_{av}(G) \le (\gamma(G) + 1) - \gamma(G)/n$, with equality if and only if G has a unique $\gamma(G)$-set.*

Theorem 2 (see [17]). *If $K_{1,n-1}$ is a star graph of order n, where $n \ge 3$, then $\gamma_{av}(K_{1,n-1}) = (2n-1)/n$.*

Theorem 3 (see [17]). *If P_n is a path graph of order n, then*

$$\gamma_{av}(P_n) = \begin{cases} \dfrac{n+2}{3} - \dfrac{2}{3n}, & \text{if } n \equiv 2 \pmod 3; \\ \dfrac{n+2}{3}, & \text{otherwise.} \end{cases} \quad (1)$$

Theorem 4 (see [17]). *If K_n is a complete graph of order n, then $\gamma_{av}(K_n) = 1$.*

Observation 5. If W_n is a wheel graph of order $n+1$, then $\gamma_{\mathrm{av}}(W_n) = (2n+1)/(n+1)$.

Theorem 6 (see [15]). *If K_n is a complete graph of order n, where $n \ge 2$, then $\gamma_2(K_n) = 2$.*

Theorem 7 (see [15]). *If P_n is a path graph of order n, then $\gamma_2(P_n) = \lfloor n/2 \rfloor + 1$.*

Theorem 8 (see [15]). *If C_n is a cycle graph of order n, where $n \ge 3$, $\gamma_2(C_n) = \lfloor (n+1)/2 \rfloor$.*

Theorem 9 (see [15]). *If W_n is a wheel graph of order $n+1$, where $n \ge 5$,*

$$\gamma_2(W_n) = \begin{cases} 2, & \text{if } n = 3, 4; \\ 1 + \left\lceil \dfrac{n}{3} \right\rceil, & \text{otherwise.} \end{cases} \quad (2)$$

Theorem 10 (see [23]). *Let G be any connected graph of order n. If $\gamma_2(G)$-set is unique, then*

$$\gamma_{2av}(G) = (\gamma_2(G) + 1) - \frac{\gamma_2(G)}{n}. \quad (3)$$

Theorem 11 (see [23]). *Let G be any connected graph of order n. If $\delta(G) \ge 2$, then*

$$\gamma_2(G) \le \gamma_{2av}(G) \le (\gamma_2(G) + 1) - \frac{\gamma_2(G)}{n}. \quad (4)$$

Theorem 12 (see [23]). *Let G be any connected graph of order n, where $n \ge 2$. Then,*

$$2 \le \gamma_{2av}(G) \le n - 1 + \frac{1}{n}. \quad (5)$$

Theorem 13 (see [23]). *Let G and H be two connected graphs of order n and m, respectively. If $n \ge 2$ and $m \ge 2$, then $\gamma_{2av}(G) + \gamma_{2av}(H) \ge \gamma_{2av}(G+H)$.*

Theorem 14 (see [23]). *Let T be any connected tree of order n. If T has s support vertices and $(n-s)$ leaf vertices, then $\gamma_{2av}(T) \ge n - s + s/n$.*

Theorem 15 (see [23]). *If P_n is a path graph of order n, then*

$$\gamma_{2av}(P_n) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor + 2 - \frac{\lfloor n/2 \rfloor + 1}{n}, & \text{if } n \text{ is odd;} \\ \left\lfloor \frac{n}{2} \right\rfloor + 1, & \text{if } n \text{ is even.} \end{cases} \quad (6)$$

Theorem 16 (see [23]). *If W_n is a wheel graph of order $n + 1$, where $n \geq 5$, then*

$$\gamma_{2av}(W_n) = 1 + \left\lceil \frac{n}{3} \right\rceil. \quad (7)$$

Theorem 17 (see [23]). *If K_n is a complete graph of order n, where $n \geq 2$, then $\gamma_{2av}(K_n) = 2$.*

Theorem 18 (see [23]). *If $K_{1,n-1}$ is a star graph of order n, where $n \geq 3$, then*

$$\gamma_{2av}(K_{1,n-1}) = n - 1 + \frac{1}{n}. \quad (8)$$

3. Calculation of ALDN and AL2DN of Cycles and Related Networks

3.1. Cycles

Theorem 19 (see [17]). *If C_n is a cycle graph of order n, then $\gamma_{av}(C_n) = 2$.*

Theorem 20 (see [23]). *If C_n is a cycle graph of order n, then $\gamma_{2av}(C_n) = \lfloor (n + 1)/2 \rfloor$.*

3.2. Fans

Definition 21 (see [21]). If one joins a vertex of C_n ($n \geq 3$) to all other vertices, the resulting graph is called a *fan* (also known as a *shell*), denoted by F_n. For $n = 3$, we notice that $F_3 \equiv C_3$. Fans can be described by the join operation $F_n = P_{n-1} + v$, where $n \geq 3$. There is a vertex with $(n - 1)$-degree, namely, u, in the graph F_n.

Theorem 22. *Let F_n be a fan of order n and $n \geq 5$; then $\gamma_2(F_n) = \lceil (n + 2)/3 \rceil$.*

Proof. The 2-dominating set is formed by two ways in F_n.

Case 1. Let D_1 be a 2-dominating set and let D_1 include the vertex u. So, the vertex u dominates vertices of $V(F_n) - \{u\}$ by once. Clearly, these $(n - 1)$-vertices which are dominated once form the path P_{n-1}. Due to the fact that $\gamma(P_{n-1}) = \lceil (n - 1)/3 \rceil$, these $\lceil (n - 1)/3 \rceil$ vertices must be taken to the set D_1. So, $|D_1| = 1 + \lceil (n - 1)/3 \rceil = \lceil (n + 2)/3 \rceil$ is obtained.

Case 2. Let D_2 be a 2-dominating set and let D_2 not include vertex u. So, the set D_2 must include vertices of subgraph path P_{n-1}. By Theorem 7, we have $|D_2| = 1 + \lceil (n - 1)/2 \rceil = \lceil (n + 1)/2 \rceil$.

By Cases 1 and 2, $|D_2| \geq |D_1|$ is obtained for $n \geq 5$. As a result, we get $\gamma_2(F_n) = \lceil (n + 2)/3 \rceil$. □

Theorem 23. *Let F_n be a fan of order n and $n \geq 3$; then $\gamma_{av}(F_n) = (2n - 1)/n$.*

Proof. By the definition of domination number and the structure of F_n, the dominating set of F_n is unique and $\gamma(F_n) = 1$. By Theorem 1, we get $\gamma_{av}(F_n) = 1 + 1 - 1/n = (2n - 1)/n$. □

Theorem 24. *Let F_n be a fan of order n and $n \geq 5$; then*

$$\gamma_{2av}(F_n) = \frac{1}{n}\left[\left\lceil \frac{(n+2)}{3} \right\rceil + (n-1)\left(\gamma_{av}(P_{n-1}) + 1\right)\right]. \quad (9)$$

Proof. When $\gamma_{2v}(F_n)$ is calculated for all vertices in F_n, the vertices in three cases should be examined.

Case 1. For the vertex u, the 2-dominating set must include the vertex u by Theorem 22. The rest of the proof of this case is similar to Case 1 of Theorem 22. So, we get $\gamma_{2u}(F_n) = 1 + \lceil (n - 1)/3 \rceil = \lceil (n + 2)/3 \rceil$.

Case 2. For all vertices $v_i \in V(F_n) - \{u\}$ which forms the graph P_{n-1}. We know that $\sum_{i=1}^{n-1} \gamma_{v_i}(F_n) = (n - 1)\gamma_{\text{av}}(P_{n-1})$ by the definition of ALDN. Furthermore, we know that the vertex u must be in the 2-dominating set. So, we have $\sum_{i=1}^{n-1} \gamma_{2v_i} = (n - 1)\gamma_{\text{av}}(P_{n-1}) + (n - 1)$ for the sum of the lower 2-domination number of all vertices $v_i \in V(F_n) - \{u\}$. As a result, $\sum_{i=1}^{n-1} \gamma_{2v_i}(F_n) = (n - 1)(\gamma_{\text{av}}(P_{n-1}) + 1)$ is obtained.

By Cases 1 and 2, we get

$$\begin{aligned} \gamma_{2\text{av}}(F_n) &= \frac{1}{|V(F_n)|}\left(\gamma_{2u}(F_n) + \sum_{i=1}^{n-1}\gamma_{2v_i}(F_n)\right) \\ &= \frac{1}{n}\left[\left\lceil \frac{(n+2)}{3} \right\rceil + (n-1)\left(\gamma_{\text{av}}(P_{n-1}) + 1\right)\right]. \end{aligned} \quad (10)$$

□

Remark 25. Let F_n be a fan of order n and $n \geq 5$; then

$$\gamma_{2\text{av}}(F_n) = \begin{cases} \frac{(n^2 + 3n - 3)/3 + \lceil (n-1)/3 \rceil}{n}, & \text{if } n \equiv 0 \pmod 3; \\ \frac{(n^2 + 3n - 1)/3 + \lceil (n-1)/3 \rceil}{n}, & \text{otherwise.} \end{cases} \quad (11)$$

Proof. This is clear from Theorems 3 and 24. □

3.3. k-Pyramids

Definition 26 (see [21]). The join graph $C_n \vee N_k$ ($n \geq 3$, $k \geq 1$), where N_k is the null graph of order k, is called a k-pyramid and is denoted by $kP(n)$. The 2-pyramid $C_n \vee N_2$ is called bipyramid and is denoted by $\text{BP}(n)$. The 1-pyramid $C_n \vee N_1$ is the wheel graph W_n.

Theorem 27. *Let $BP(n)$ be a bipyramid of order $n+2$ and $n \geq 7$; then $\gamma(BP(n)) = 2$.*

Proof. Because $\Delta(\mathrm{BP}(n)) < n+1$, the domination number $\gamma(\mathrm{BP}(n))$ is greater than 1. Let v_1 and v_2 be vertices whose degrees are n, and let D be a dominating set. The dominating set D is formed by 3 cases.

Case 1. Let $v_1, v_2 \in D$. Due to $N_{\mathrm{BP}(n)}[v_1, v_2] = V(\mathrm{BP}(n))$, the set D is a dominating set.

Case 2. Let $v_1 \in D$ and let u_i be any vertex of C_n. Due to the fact that $N_{\mathrm{BP}(n)}[v_1, u_i] = V(\mathrm{BP}(n))$, the set D is a dominating set.

Case 3. Let $v_1, v_2 \notin D$. Then, the set D includes only vertices of C_n. So, we have $|D| \geq \lceil n/3 \rceil$.

By Cases 1, 2, and 3, $\gamma(\mathrm{BP}(n)) = 2$ is obtained. □

Theorem 28. *Let $BP(n)$ be a bipyramid of order $n+2$ and $n \geq 7$; then*

$$\gamma_{av}(BP(n)) = 2. \tag{12}$$

Proof. When $\gamma_v(\mathrm{BP}(n))$ is calculated for all vertices in the graph $\mathrm{BP}(n)$, the vertices in two cases should be examined. Let v_1 and v_2 be vertices whose degrees are n.

Case 1. For the vertices v_1 and v_2, we know that the set $\{v_1, v_2\}$ dominates all vertices of $V(\mathrm{BP}(n)) - \{v_1, v_2\}$. So, we get $\gamma_{v_1}(\mathrm{BP}(n)) = \gamma_{v_2}(\mathrm{BP}(n)) = 2$.

Case 2. For all vertices $v_i \in V(\mathrm{BP}(n)) - \{v_1, v_2\}$, by the definition of lower domination number and Case 2 of Theorem 27, we have $\gamma_{v_i}(\mathrm{BP}(n)) = 2$ for all vertices $v_i \in V(\mathrm{BP}(n)) - \{v_1, v_2\}$.

By Cases 1 and 2, we get $\gamma_{\mathrm{av}}(\mathrm{BP}(n)) = 2$. □

Theorem 29. *Let $BP(n)$ be a bipyramid of order $n+2$ and $n \geq 3$; then $\gamma_2(BP(n)) = 2$. Furthermore, the 2-dominating set of $BP(n)$ is unique.*

Proof. Let u_i be the vertices of C_n, where $1 \leq i \leq n$, and let v_j be vertices of the graph $\overline{K_2}$, where $1 \leq j \leq 2$. There are 3 cases while forming 2-dominating set.

Case 1. Let D_1 be a 2-dominating set of $\mathrm{BP}(n)$. Furthermore, the set D_1 contains vertices v_1 and v_2. Clearly, $N_{\mathrm{BP}(n)}[v_1, v_2] = V(\mathrm{BP}(n))$. So, $|D_1| = 2$ is obtained.

Case 2. Let D_2 be any 2-dominating set, and the set D_2 contains v_1 or v_2. So, the set D_2 1-dominates all vertices of C_n. By the definition of $\gamma(C_n)$, if we add $\lceil n/3 \rceil$ vertices to the set D_2, then all of C_n graph's vertices are 2-dominated. So, $\lceil n/3 \rceil \geq 2$ for $n \geq 5$. Hence, $|D_2| = 1 + \lceil n/3 \rceil$ is obtained.

Case 3. Let D_3 be any 2-dominating set. Moreover, the set D_3 contains only vertices u_i of C_n. It is clear that the set D_3 is the 2-dominating set of C_n. By Theorem 8, we have $|D_3| = \lfloor (n+1)/2 \rfloor$.

By Cases 1, 2, and 3 we get $|D_1| = 2$, $|D_2| \geq 3$, and $|D_3| \geq 3$ for $n \geq 5$. As a result, by the definition of 2-domination number, we have $\gamma_2(\mathrm{BP}(n)) = 2$ and this 2-dominating set is unique. □

Theorem 30. *Let $BP(n)$ be a bipyramid of order $n+2$ and $n \geq 3$; then*

$$\gamma_{2av}(BP(n)) = \frac{3n+4}{n+2}. \tag{13}$$

Proof. Since the 2-dominating set is unique, we have $\gamma_{2\mathrm{av}}(\mathrm{BP}(n)) = 2 + 1 - 2/(n+2) = (3n+4)/(n+2)$ by Theorem 10. □

Theorem 31. *For the 3-pyramid $3P(n)$ with $n \geq 7$, $\gamma_2(3P(n)) = 3$.*

Proof. There are 3 vertices whose degrees are n in the graph $3P(n)$ and they are shown by u_1, u_2, and u_3. We have four cases while 2-dominating set is forming.

Case 1. Let D_1 be any 2-dominating set of $3P(n)$ and let D_1 contain vertices u_1, u_2, and u_3. It is clear that $N_{3P(n)}[u_1, u_2, u_3] = V(3P(n))$ is obtained. So, the set D_1 is 2-dominating set of $3P(n)$. Thus, $|D_1| = 3$ is obtained.

Case 2. Let D_2 be any 2-dominating set of $3P(n)$ and let set D_2 include any two vertices of the vertices u_1, u_2, and u_3. These two vertices are 2-dominated by the vertices of C_n. If the remaining vertex is added to D_2 set, it is similar to set D_1 and all vertices of $3P(n)$ are 2-dominated. If the remaining vertex is not added to D_2 set, two vertices of $V(3P(n)) - \{u_1, u_2, u_3\}$ must be added to the set D_2. Thus, $|D_2| \geq 3$ is obtained.

Case 3. Let D_3 be any 2-dominating set of $3P(n)$ and let the set D_3 include any one vertex of u_1, u_2, and u_3. So, the vertices of C_n are dominated once by the set D_3. Since the set D_3 does not include the remaining two vertices of the vertices u_1, u_2, and u_3, $\lceil n/3 \rceil$ vertices of C_n must be taken to set D_3 to 2-dominate all vertices of C_n and the remaining two vertices of the vertices u_1, u_2, and u_3. Because $n \geq 7$, we have $|D_3| = \lceil n/3 \rceil + 1$.

Case 4. Let D_4 be any 2-dominating set of $3P(n)$ and let the set D_4 not include any vertices of u_1, u_2, and u_3. So, we must add vertices of the graph C_n to set D_4. We know that $\gamma_2(C_n) = \lfloor (n+1)/2 \rfloor$ by Theorem 8. Thus, $|D_4| \geq \lfloor (n+1)/2 \rfloor$ is obtained.

By Cases 1, 2, 3, and 4 we get $|D_1| = 3$, $|D_2| \geq 3$, $|D_3| \geq \lceil n/3 \rceil + 1$, and $|D_4| \geq \lfloor (n+1)/2 \rfloor$ for $n \geq 7$. As a result we have $\gamma_2(3P(n)) = 3$ by the definition of 2-domination number and this 2-dominating set is unique. □

Theorem 32. *For the 3-pyramid $3P(n)$ with $n \geq 7$, $\gamma_{2av}(3P(n)) = (4n+9)/(n+3)$.*

Proof. Since the 2-dominating set is unique, we have $\gamma_{2av}(3P(n)) = 3 + 1 - 3/(n+3) = (4n+9)/(n+3)$ by Theorem 10. □

Theorem 33. *Let $kP(n)$ be a k-pyramid with $k \geq 4$ and $n \geq 7$; then $\gamma_2(kP(n)) = 4$.*

Proof. The vertices of C_n are denoted by v_i, where $i \in \{1,\ldots,n\}$, and the remaining k-vertices are denoted by u_j, where $j \in \{1,\ldots,k\}$. We have three cases while 2-dominating set is forming.

Case 1. Let D_1 be any 2-dominating set of the graph $kP(n)$ and let D_1 contain only vertices u_j, where $j \in \{1,\ldots,k\}$. It is clear that set $\{u_1, u_2 \ldots, u_j\}$ 2-dominated all vertices of $kP(n)$. So, $|D_1| = k$ and $|D_1| \geq 4$ are obtained.

Case 2. Let D_2 be any 2-dominating set of $kP(n)$ and let D_2 contain only vertices of v_i, where $i \in \{1,\ldots,n\}$. By Theorem 8, we have $|D_2| = \lfloor (n+1)/2 \rfloor$. As a result, $|D_2| \geq 4$ is obtained.

Case 3. Let D_3 be any 2-dominating set of $kP(n)$ and let D_3 contain any vertex of u_j and v_i, where $j \in \{1,\ldots,k\}$ and $i \in \{1,\ldots,n\}$. Let $D_3 = \{u_1, v_1\}$. Let $= \{u_j \mid 1 \leq j \leq k\}$. The vertex u_1 dominates vertices of $V(C_n) - \{u_1\}$ by once and the vertex v_1 dominates vertices of $U - \{v_1\}$ by once. If any vertex of v_i and any vertex of u_j are added to set D_3, then D_3 will be 2-dominating set of the graph $kP(n)$. Hence, $|D_3| = 4$ is obtained.

By Cases 1, 2, and 3 we get $|D_1| \geq 4$, $|D_2| \geq 4$, and $|D_3| = 4$ for $n \geq 7$. As a result, by the definition of 2-domination number, we have $\gamma_2(kP(n)) = 4$. □

Theorem 34. *Let $kP(n)$ be a k-pyramid with $k \geq 4$ and $n \geq 7$; then $\gamma_{2av}(kP(n)) = 4$.*

Proof. By Theorem 33, we have the lower 2-domination number as 4 for all vertices of $kP(n)$. Thus, $\gamma_{2av}(kP(n)) = 4$ is obtained. □

3.4. n-Gon Books

Definition 35 (see [21]). When k copies of C_n $(n \geq 3)$ share a common edge, they will form an *n-gon book of k pages* and are denoted by $B(n,k)$. The degree set of $B(n,k)$ is $\{2, k+1\}$. Therefore, the vertices of $B(n,k)$ are of two kinds: vertices of degree 2, which will be referred to as *minor* vertices, and vertices of degree $k+1$, which will be referred to as *major* vertices. The minor vertices of $B(n,k)$ are labeled $v_{11}, v_{12} \ldots, v_{1j}, v_{21}, v_{22}, \ldots, v_{2j}, \ldots, v_{k1}, v_{k2}, \ldots, v_{kj}$, that is v_{ij}, where $i \in \{1,\ldots,k\}$ and $j \in \{1,\ldots,n-2\}$.

Theorem 36. *Let $B(n,k)$ be an n-gon book and $n \geq 5$; then*

$$\gamma(B(n,k)) = \begin{cases} k\left\lceil \dfrac{n-4}{3} \right\rceil + 2, & n \equiv 1 \pmod 3; \\ k\left\lceil \dfrac{n-3}{3} \right\rceil + 1, & \textit{otherwise}. \end{cases} \quad (14)$$

Proof. Let u_1 and u_2 be two major vertices. We have three cases depending on the dominating set of $B(n,k)$ that are including major vertices or not.

Case 1. Let D_1 be a dominating set, and let D_1 include two major vertices. The set $\{u_1, u_2\}$ dominates $2k$-vertices from all C_n. So, there are $(n-4)$-vertices which are not dominated from each C_n. Then the k-distinct path P_{n-4} is obtained by these minor vertices. Thus, $|D_1| = k\lceil (n-4)/3 \rceil + 2$ is obtained.

Case 2. Let D_2 be a dominating set, and let $u_1 \in D_2$ (or $u_2 \in D_2$). The vertex u_1 dominates the vertex u_2 and k-vertices which are adjacent to the vertex u_1 from all C_n. Then the remaining $(n-3)$-vertices are not dominated from each C_n. So, the k-distinct path P_{n-3} is obtained by these minor vertices. As a result, $|D_2| = k\lceil (n-4)/3 \rceil + 1$ is obtained.

Case 3. Let D_3 be a dominating set, and let $\{u_1, u_2\} \notin D_3$. Since the set D_3 includes only minor vertices, we have two subcases depending on n.

Case 3.1. Let $n \equiv 0, 1 \pmod 3$. Due to the structure of $B(n,k)$, the set D_3 must include $\lceil (n-2)/3 \rceil$-vertices from each C_n. So, the whole vertices of $B(n,k)$ are dominated. As a result we have $|D_3| = k\lceil (n-2)/3 \rceil$.

Case 3.2. Let $n \equiv 2 \pmod 3$. Due to the structure of $B(n,k)$, two minor vertices which are neighbors to the major vertices must be taken to the set D_3 from any graph C_n. Furthermore, the remaining $(k-1)(n-2)$-vertices are not dominated in the graph $B(n,k)$. Since $(k-1)\lceil (n-2)/3 \rceil$-vertices must be taken to the set D_3, $|D_3| = 2 + \lceil (n-6)/3 \rceil + (k-1)\lceil (n-2)/3 \rceil$ is obtained.

By Cases 3.1 and 3.2, we get

$$|D_3| = \begin{cases} 2 + \left\lceil \dfrac{n-6}{3} \right\rceil + (k-1) \cdot \left\lceil \dfrac{n-2}{3} \right\rceil, & n \equiv 2 \pmod 3; \\ k \cdot \left\lceil \dfrac{n-2}{3} \right\rceil, & \text{otherwise}. \end{cases} \quad (15)$$

Clearly, if $n \equiv 0 \pmod 3$, then we have $|D_3| > |D_1| > |D_2|$, if $n \equiv 1 \pmod 3$, then we have $|D_2| > |D_3| > |D_1|$, and if $n \equiv 2 \pmod 3$, then we have $|D_1| > |D_3| = |D_2|$. As a result, we get

$$\gamma B(n,k) = \begin{cases} k\left\lceil \dfrac{n-4}{3} \right\rceil + 2, & n \equiv 1 \pmod 3; \\ k\left\lceil \dfrac{n-3}{3} \right\rceil + 1, & \text{otherwise}. \end{cases} \quad (16)$$

Thus the proof is completed. □

Theorem 37. *Let $B(n,k)$ be an n-gon book and $n \geq 5$; then*

$$\gamma_{av}(B(n,k)) = \begin{cases} k\left\lceil \frac{n-3}{3} \right\rceil + 1, & if\ n \equiv 2 \pmod 3; \\ 3 + k\left\lceil \frac{n-4}{3} \right\rceil - \frac{2 + k\lceil (n-4)/3 \rceil}{k(n-2)+2}, & if\ n \equiv 1 \pmod 3; \\ \frac{1}{k(n-2)+2}\left(\left[(n-2)(k+2) - k\left\lceil \frac{n}{3} \right\rceil\right]\left(1 + k\left\lceil \frac{n-3}{3} \right\rceil\right) + k\left\lceil \frac{n}{3} \right\rceil\left(2 + k\left\lceil \frac{n-3}{3} \right\rceil\right)\right), & if\ n \equiv 0 \pmod 3. \end{cases} \quad (17)$$

Proof. While finding lower domination number of all vertices of $B(n,k)$, we have three cases depending on n.

Case 1. Let $n \equiv 1 \pmod 3$, and let D be a dominating set. Clearly, the set D is a unique dominating set. By Theorem 10, we have

$$\gamma_{\rm av}(B(n,k)) = 3 + k\left\lceil \frac{n-4}{3} \right\rceil - \frac{2 + k\lceil (n-4)/3 \rceil}{k(n-2)+2}. \quad (18)$$

Case 2. Let $n \equiv 2 \pmod 3$, and let D be a dominating set. By Theorem 36, the set D is formed by two ways. Clearly, $\gamma_{v_i}(B(n,k)) = \lceil (n-3)/3 \rceil + 1$ is obtained for every $v_i \in V(B(n,k))$. As a result, we have $\gamma_{\rm av}(B(n,k)) = 1 + k\lceil (n-3)/3 \rceil$.

Case 3. Let $n \equiv 0 \pmod 3$, and let D be a dominating set. By Case 2 of Theorem 36, we know that the set D includes either the vertex u_1 or the vertex u_2. Let $S = \{v_{ij} \mid i \in \{1,\ldots,k\} \wedge j \in \{1,4,\ldots,n-2\}\}$. Clearly, the set D contains vertices of $V(B(n,k)) - S$ by Theorem 36. Thus, we have $\gamma_{v_i}(B(n,k)) = 1 + k\lceil (n-3)/3 \rceil$ for every $v_i \in V(B(n,k)) - S$. Furthermore, we have $\gamma_{v_i^*}(B(n,k)) = 2 + k\lceil (n-3)/3 \rceil$ for every $v_i^* \in S$. Thus,

$$\begin{aligned} \gamma_{\rm av}(B(n,k)) &= \frac{1}{|V(B(n,k))|}\left(\sum_{v_i \in V(B(n,k))-S} \gamma_{v_i}(B(n,k)) + \sum_{v_i^* \in S} \gamma_{v_i^*}(B(n,k))\right) \\ &= \frac{1}{k(n-2)+2}\left(\left[(n-2)(k+2) - k\left\lceil \frac{n}{3} \right\rceil\right] \cdot \left(1 + k\left\lceil \frac{n-3}{3} \right\rceil\right) + k\left\lceil \frac{n}{3} \right\rceil\left(2 + k\left\lceil \frac{n-3}{3} \right\rceil\right)\right). \end{aligned} \quad (19)$$

By Cases 1, 2, and 3 the proof is completed. □

Theorem 38. *Let $B(n,k)$ be an n-gon book and $n \geq 5$; then*

$$\gamma_2(B(n,k)) = \begin{cases} k\left(\left\lfloor \frac{n+1}{2} \right\rfloor - 1\right) + 1, & if\ n\ is\ even; \\ k\left(\left\lfloor \frac{n-4}{2} \right\rfloor + 1\right) + 2, & if\ n\ is\ odd. \end{cases} \quad (20)$$

Proof. Let u_1 and u_2 be two major vertices. We have two cases depending on n.

Case 1. Let n be even, and let D be 2-dominating set. We know that the set D contains either the vertex u_1 or the vertex u_2 by the definition of $B(n,k)$ and Theorem 8. Thus, $(\lfloor (n+1)/2 \rfloor - 1)$-vertices of each C_n must be taken to the set D. Since there is k-graph C_n, we have $|D| = k(\lfloor (n+1)/2 \rfloor - 1) + 1$. So, $\gamma_2(B(n,k)) = k(\lfloor (n+1)/2 \rfloor - 1) + 1$ is obtained.

Case 2. Let n be odd. We have two subcases depending on the 2-dominating set of $B(n,k)$ that are including major vertices or not.

Case 2.1. Let D_1 be 2-dominating set, and let D_1 include any major vertex. Clearly, the set D must include two major vertices in this subcase by Theorem 8. So, the set $\{u_1, u_2\}$ dominates $2k$-vertices from all C_n. Then, $(n-4)$-vertices are not dominated from each C_n. Thus, the k-distinct path P_{n-4} is obtained by these minor vertices. As a result, $|D_1| = k(\lfloor (n-4)/2 \rfloor + 1) + 2$ is obtained.

Case 2.2. Let D_2 be 2-dominating set, and let $\{u_1, u_2\} \notin D_2$. Clearly, the set D_2 includes $(\lfloor (n-2)/2 \rfloor + 1)$-vertices of $V(C_n) - \{u_1, u_2\}$ of each C_n by Theorem 7. Thus, $|D_2| = k(\lfloor (n-2)/2 \rfloor + 1)$ is obtained.

By Cases 2.1 and 2.2 $|D_1| = |D_2|$ for $k = 2$ and $n \geq 5$. Then we have $|D_1| < |D_2|$ for $k \geq 3$ and $n \geq 5$. Furthermore, 2-dominating set is unique for $k \geq 3$ and $n \geq 5$. As a result, $\gamma_2(B(n,k)) = k(\lfloor (n-4)/2 \rfloor + 1) + 2$ is obtained.

By Cases 1 and 2, the proof is completed. □

Theorem 39. *Let $B(n,k)$ be an n-gon book, where $n \geq 3$ and $k \geq 3$; then*

$$\gamma_{2av}(B(n,k)) = \begin{cases} k\left(\left\lfloor \frac{n+1}{2} \right\rfloor - 1\right) + 1, & if\ n\ is\ even; \\ k\left(\left\lfloor \frac{n-4}{2} \right\rfloor + 1\right) + 3 - \frac{k(\lfloor (n-4)/2 \rfloor + 1) + 2}{k(n-2)+2}, & if\ n\ is\ odd. \end{cases} \quad (21)$$

Proof. Let u_1 and u_2 be two major vertices. We have two cases depending on n.

Case 1. Let n be even. We have that the 2-dominating set including the vertex u_1 is 2-dominating set of any C_n. Since the vertex u_1 is common vertex of each C_n and by Theorem 8, we get $\gamma_{2u_1}(B(n,k)) = k(\lfloor (n+1)/2 \rfloor - 1) + 1$. Similarly, we get $\gamma_{2u_2}(B(n,k)) = k(\lfloor (n+1)/2 \rfloor - 1) + 1$. Let v_{i1} be minor vertex which is neighbor to the vertex u_1 of any C_n, and let $v_{i(n-2)}$ be minor vertex which is neighbor to the vertex u_2 in the same graph C_n. It is easy to see that the 2-dominating set including the vertex v_{i1} is the 2-dominating set including the vertex u_2 by Theorem 8. Similarly, the 2-dominating set including the vertex $v_{i(n-2)}$ is the 2-dominating set including the vertex u_1. Thus, $\gamma_{2v_{i1}}(B(n,k)) = \gamma_{2v_{i(n-2)}}(B(n,k)) = k(\lfloor (n+1)/2 \rfloor - 1) + 1$ is obtained. Furthermore, we get $\gamma_{2v_{ij}}(B(n,k)) = k(\lfloor (n+1)/2 \rfloor - 1) + 1$ for every $v_{ij} \in V(B(n,k)) - \{u_1, u_2\}$, where $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n-2\}$. As a result, $\gamma_{2\mathrm{av}}(B(n,k)) = k(\lfloor (n+1)/2 \rfloor - 1) + 1$ is obtained.

Case 2. Let n be odd. We have two subcases depending on k.

Case 2.1. Let $k = 2$. We know that 2-dominating set is formed by two ways in Case 2 of Theorem 38. Clearly, $\gamma_{2v}(B(n,k)) = k(\lfloor (n-4)/2 \rfloor + 1) + 2$ is obtained for every vertex $v \in V(B(n,k))$. As a result, $\gamma_{2\mathrm{av}}(B(n,k)) = k(\lfloor (n-4)/2 \rfloor + 1) + 2$ is obtained.

Case 2.2. Let $k \geq 3$. We know that 2-dominating set is unique by Case 2 of Theorem 38. So, $\gamma_{2\mathrm{av}}(B(n,k)) = k(\lfloor (n-4)/2 \rfloor + 1) + 3 - ((k(\lfloor (n-4)/2 \rfloor + 1) + 2)/k(n-2) + 2)$ is obtained by Theorem 10.

By Cases 1 and 2 the proof is completed. □

Theorem 40. *Let $B(n,k)$ be an n-gon book, where $n \geq 3$ and $k = 2$; then*

$$\gamma_{2av}(B(n,k)) = \begin{cases} k\left(\left\lfloor \dfrac{n+1}{2} \right\rfloor - 1\right) + 1, & \text{if } n \text{ is even;} \\ k\left(\left\lfloor \dfrac{n-4}{2} \right\rfloor + 1\right) + 2, & \text{if } n \text{ is odd.} \end{cases} \quad (22)$$

Proof. The proof directly comes from Theorem 39. □

4. Conclusion

In this study, a new defined graph theoretical parameter, namely, the average lower 2-domination number and the average lower domination number, has been studied for the network vulnerability. Additionally, the stability of popular interconnection networks including cycles has been studied and their domination numbers, 2-domination numbers, average lower domination numbers, and average lower domination 2-numbers, have been computed. These networks have been modeled with the fans, the k-pyramids, and the n-gon books.

Competing Interests

The authors declare that they have no competing interests.

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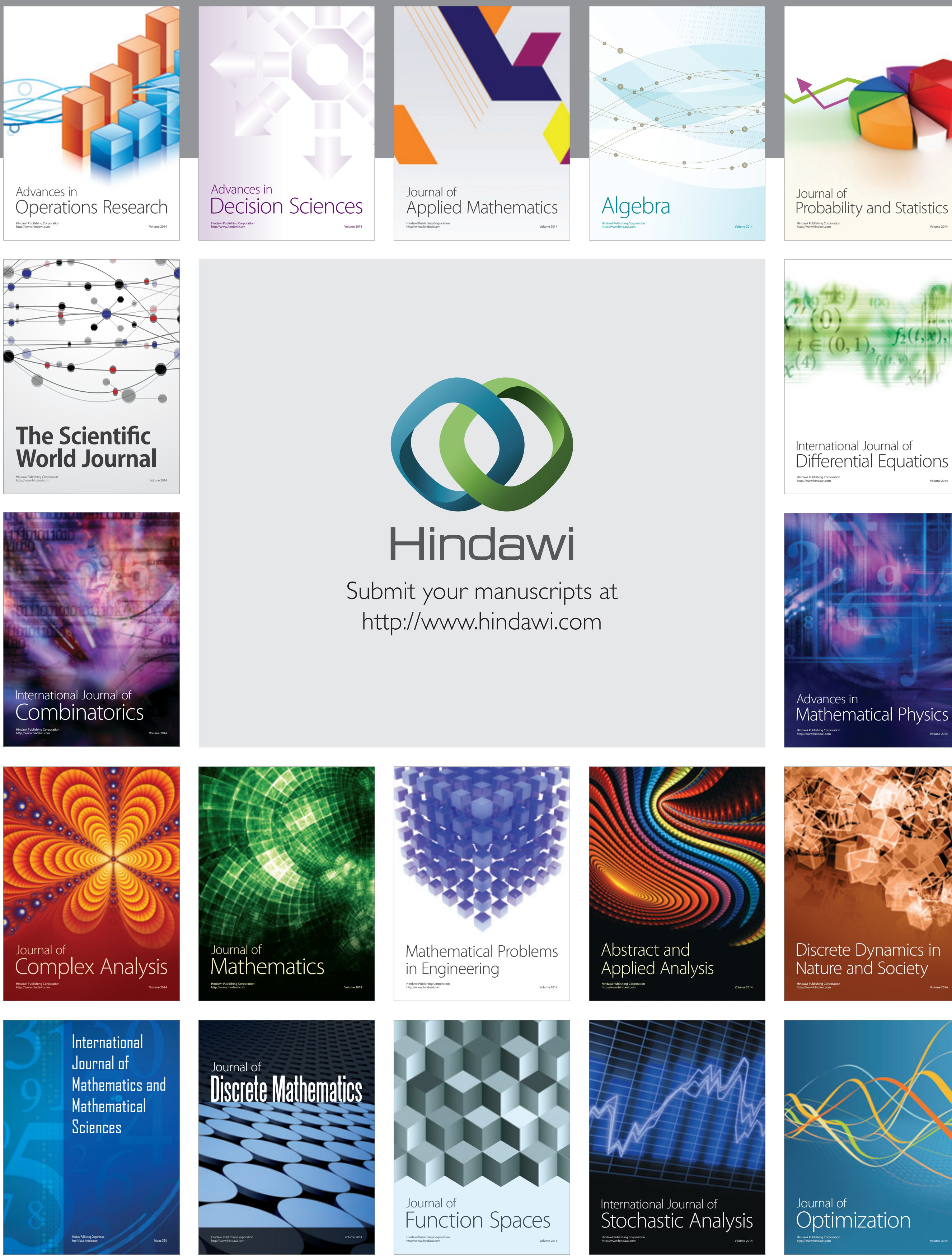
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