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Research Article

Characterizing Jordan Derivable Maps on Triangular Rings by Local Actions

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Suppose that $\mathcal{T} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ is a 2-torsion free triangular ring, and $\mathfrak{S} = \{(A, B) \mid AB = 0, A, B \in \mathcal{T}\} \cup \{(A, X) \mid A \in \mathcal{T}, X \in \{P, Q\}\}$, where P is the standard idempotent of $\mathcal{T}$ and $Q = I - P$. Let $\delta\colon \mathcal{T} \longrightarrow \mathcal{T}$ be a mapping (not necessarily additive) satisfying, $(A, B) \in \mathfrak{S} \Rightarrow \delta(A \circ B) = A^\circ \delta(B) + \delta(A)^\circ B$, where $A \circ B = AB + BA$ is the Jordan product of $\mathcal{T}$. We obtain various equivalent conditions for δ, specifically, we show that δ is an additive derivation. Our result generalizes various results in these directions for triangular rings. As an application, δ on nest algebras are determined.

1. Introduction

Let $\mathcal{R}$ be a ring and $\delta\colon \mathcal{R} \longrightarrow \mathcal{R}$ be a mapping (not necessarily additive). δ is called a derivable map if $\delta(ab) = a\delta(b) + \delta(a)b$ for all $a, b \in \mathcal{R}$. Moreover, δ is called a Jordan derivable map if $\delta(a \circ b) = a \circ \delta(b) + \delta(a) \circ b$ for all $a, b \in \mathcal{R}$, where $a \circ b = ab + ba$ is the Jordan product of $\mathcal{R}$. An additive derivable mapping δ is called additive derivation. If δ is an additive Jordan derivable mapping, then it is called an additive Jordan derivation. These defined maps are important classes of maps on the rings and there has been many studies on them from different directions, and here we mention some of these study routes which is interesting for us.

One of the interesting issues is the study of relationship between the additive and multiplicative structure of maps on the rings. In this line of investigation, first Martinadle [1] considered some conditions on a ring $\mathcal{R}$, proved that any multiplicative bijection map of $\mathcal{R}$ is additive. Then the question, what maps on a ring $\mathcal{R}$ are automatically additive was considered and different results were obtained in this line, we refer the reader to [2, 3] and references therein for more details. Especially, it has been proved on special rings that every derivable map or Jordan derivable map is additive, for instance, see [4–6].

Obviously, any additive derivation is an additive Jordan derivation, but the converse may not hold in general (see [7]). Another interesting study routes on derivations and Jordan derivations is: on what rings (algebras) is any (linear) additive Jordan derivation is (linear) additive derivation? The first result in this way was obtained by Herstein [8], which proved that on 2-torsion-free prime rings, any additive Jordan derivation is an additive derivation. Later in [9], this result was generalized for 2-torsion free semiprime rings, and after that, this result was proved for other various rings (algebras) or the structure of additive (linear) Jordan derivations on some rings characterized in terms of additive (linear) derivations (see [7, 10–13] and references therein).

Another way to study derivable maps (additive derivations) and Jordan derivable maps (additive Jordan derivations) is to study them according to local conditions. One of these local conditions is studying the maps which operate on special pairs of elements of a ring $\mathcal{R}$ like special maps. More precisely, assume that $\mathfrak{S} \subseteq \mathcal{R} \times \mathcal{R}$, in this line of investigation, considering those maps that for all $(a, b) \in \mathfrak{S}$, operate like derivable maps (additive derivations) or Jordan derivable maps (additive Jordan derivations). First Brešar in [14] proved that if δ is an additive map on a unital ring $\mathcal{R}$, that contains a nontrivial idempotent and $a\delta(b) + \delta(a)b = 0$ for all $a, b \in \mathcal{R}$ with $ab = 0$, then $\delta(a) = \tau(a) + ca$, where τ

is an additive derivation on $\mathscr{R}$ and c belongs to the center of $\mathscr{R}$. Following this line of investigation, derivations and Jordan derivations (additive or nonadditive) at zero products or another special pairs of several rings or algebras has been studied and considerable results has been achieved. For instance, see [12, 15–21] and the references therein.

In the research lines mentioned above, some considerable results on prime rings or semiprime rings have been achieved. Of course this study routes have been established on some non-semiprime rings or operator algebras (especially nest algebras), of which we can mention triangular rings as one of the most important ones. In the following we introduce this ring and hint at some results on it. Let $\mathscr{A}$ and $\mathscr{B}$ be unital rings and $\mathscr{M}$ be a unital $(\mathscr{A},\mathscr{B})$-bimodule, which is faithful as a left $\mathscr{A}$-module and also as a right $\mathscr{B}$ -module. The triangular ring Tri $(\mathscr{A},\mathscr{M},\mathscr{B})$ is as follows:

$$\mathrm{Tri}(\mathscr{A},\mathscr{M},\mathscr{B}) = \left\{\begin{bmatrix} a & m \\ 0 & b \end{bmatrix} : a \in \mathscr{A}, m \in \mathscr{M}, \quad b \in \mathscr{B}\right\}, \tag{1}$$

under the usual matrix operations. Tri $(\mathscr{A},\mathscr{M},\mathscr{B})$ is a unital ring with identity $I = \begin{bmatrix} 1_{\mathscr{A}} & 0 \\ 0 & 1_{\mathscr{B}} \end{bmatrix}$, where $1_{\mathscr{A}}$, $1_{\mathscr{B}}$ are identities of $\mathscr{A}$ and $\mathscr{B}$, respectively. This ring contains important class of rings like upper triangular block matrices over a unital ring $\mathscr{R}$, especially the ring of upper triangular matrices over a unital ring $\mathscr{R}$, some nest algebras on Banach spaces, especially nest algebras on Hilbert spaces. Note that, if $\mathscr{A}$ and $\mathscr{B}$ be unital algebras over a commutative ring $\mathscr{C}$ and $\mathscr{M}$ be a unital faithful $(\mathscr{A},\mathscr{B})$-bimodule, then Tri $(\mathscr{A},\mathscr{M},\mathscr{B})$ is an algebra over $\mathscr{C}$. Zhang in [22] proved that any linear Jordan derivation on a 2-torsion free triangular algebra Tri $(\mathscr{A},\mathscr{M},\mathscr{B})$ is a linear derivation and in [18] this result is obtained for additive Jordan derivations on 2-torsion free triangular rings. In [23] has been shown that any Jordan drivable map on a 2-torsion free triangular algebra is an additive derivation, which is a generalization of result of [22]. In [24] it has been proved that a linear map δ on $T_n(\mathbb{C})$ (all $n \times n$ upper triangular matrices over the complex field $\mathbb{C}$), satisfying the following equation:

$$\begin{aligned} A, B \in T_n(\mathbb{C}),\ AB = 0 \Rightarrow \delta(A \circ B) \\ = \delta(A)^{\circ} B + A^{\circ} \delta(B), \end{aligned} \tag{2}$$

is a linear derivation. In [21] it has been shown that a mapping δ (not necessarily additive) on $T_n(\mathbb{F})$ ($\mathbb{F}$ is a field and $n \geq 3$) satisfying the following equation:

$$\begin{aligned} A, B \in T_n(\mathbb{F}),\ AB = 0 \Rightarrow \delta(AB) \\ = \delta(A)B + A\delta(B), \end{aligned} \tag{3}$$

is an additive derivation. Let $\mathscr{T} = \mathrm{Tri}(\mathscr{A},\mathscr{M},\mathscr{B})$ be a triangular ring and consider the subset S of $\mathscr{T} \times \mathscr{T}$ as follows:

$$\mathfrak{S} = \{(A,B) \mid AB = 0, A, B \in \mathscr{T}\} \cup \{(A,X) \mid A \in \mathscr{T},\ X \in \{P,Q\}\}, \tag{4}$$

where$P = \begin{bmatrix} 1_{\mathscr{A}} & 0 \\ 0 & 0 \end{bmatrix}$ is the standard idempotent in $\mathscr{T}$ and $Q = I - P$. In this paper we consider a mapping δ (not necessarily additive) on $\mathscr{T}$ which satisfies the following condition:

$$(A,B) \in \mathfrak{S} \Rightarrow \delta(A \circ B) = A^{\circ} \delta(B) + \delta(A)^{\circ} B, \tag{5}$$

and prove that if $\mathscr{T}$ is 2-torsion free, then δ is an additive derivation. Note that if the mapping δ on $\mathscr{T}$ is derivable, Jordan derivable, additive Jordan derivation or δ is an additive map on $\mathscr{T}$ satisfying

$$\begin{aligned} A, B \in \mathscr{T},\ AB = 0 \Rightarrow \delta(A \circ B) \\ = A^{\circ} \delta(B) + \delta(A)^{\circ} B, \end{aligned} \tag{6}$$

then δ satisfies (49) (see proof of Theorem 1). So our result generalizes various results in these directions for triangular rings, especially each of the results of [21, 23], Theorem 4.2 (for $G = 0$), [22]. Next theorem is the main result of our paper.

Theorem 1. *Suppose that $\mathscr{T} = Tri(\mathscr{A},\mathscr{M},\mathscr{B})$ is a 2-torsion free triangular ring and $\delta: \mathscr{T} \longrightarrow \mathscr{T}$ is a mapping (not necessarily additive). Let $\mathfrak{S} \subseteq \mathscr{T} \times \mathscr{T}$ be as follows:*

$$\mathfrak{S} = \{(A,B) \mid AB = 0, A, B \in \mathscr{T}\} \cup \{(A,X) \mid A \in \mathscr{T},\ X \in \{P,Q\}\}, \tag{7}$$

where $P \in \mathscr{T}$ is the standard idempotent and $Q = I - P$. Then the following are equivalent:

(i) $(A,B) \in \mathfrak{S} \Rightarrow \delta(A \circ B) = A^{\circ} \delta(B) + \delta(A)^{\circ} B$;

(ii) $A, B \in \mathscr{T} \Rightarrow \delta(A \circ B) = A^{\circ} \delta(B) + \delta(A)^{\circ} B$;

(iii) δ is additive and $A, B \in \mathscr{T}, AB = 0 \Rightarrow \delta(A^{\circ}B) = A^{\circ} \delta(B) + \delta(A)^{\circ} B$;

(iv) δ is an additive Jordan derivation;

(v) δ is a derivable map;

(vi) δ is an additive derivation.

The proof of this theorem will be given in Section 3. In the above theorem, we have considered the 2-torsion free condition. The necessity of this condition can be a question of interest.

Let x be a fixed element of the ring $\mathscr{R}$. The mapping $I_x: \mathscr{R} \longrightarrow \mathscr{R}$ defined by $I_x(a) = ax - xa (a \in \mathscr{R})$ is an additive derivation which is called inner derivation. On nest algebras, derivations can be characterized in terms of inner derivations. According to this point, Theorem 1 can be obtained in more specific on nest algebras. We present this result in Section 2 as an application of Theorem 1 on nest algebras.

2. Application to Nest Algebras

Let $\mathcal{X}$ be a (real or complex) Banach space, let $B(\mathcal{X})$ be the Banach algebra of all bounded linear operators on $\mathcal{X}$. A nest $\mathcal{N}$ on $\mathcal{X}$ is a chain of closed (under norm topology) subspaces of $\mathcal{X}$ with $\{0\}$ and $\mathcal{X}$ in $\mathcal{N}$ such that for every family $\{N_\alpha\}$ of elements of $\mathcal{N}$, both $\cap N_\alpha$ and $\vee N_\alpha$ (closed linear span of $\{N_\alpha\}$) belong to $\mathcal{N}$. The nest algebra associated to the nest $\mathcal{N}$, denoted by alg $\mathcal{N}$ is as follows:

$$\text{alg}\mathcal{N} = \{T \in \mathcal{B}(X) \colon T(N) \subseteq N \text{ for any } N \in \mathcal{N}\}. \tag{8}$$

We say that $\mathcal{N}$ is nontrivial whenever $\mathcal{N} \neq \{\{0\}, \mathcal{X}\}$. If $\mathcal{N}$ is trivial, then alg $\mathcal{N} = \mathcal{B}(X)$.

Remark 1. Let $\mathcal{N}$ is a nontrivial nest on $\mathcal{X}$ and $N \in \mathcal{N}$ with $N \neq \{0\}$ and $N \neq \mathcal{X}$, is complemented. Then there exists an idempotent $P \in \text{alg}\mathcal{N}$ such that $\text{ran}P = N$, and the nest algebra alg $\mathcal{N}$ has a representation as the following triangular algebra.

$$\text{alg}\mathcal{N} = \text{Tri}(P\text{alg}\mathcal{N}P,\ P\text{alg}\mathcal{N}(I-P),\ (I-P)\text{alg}\mathcal{N}(I-P)), \tag{9}$$

where I is the identity operator. $P\text{alg}\mathcal{N}P$ and $(I-P)\text{alg}\mathcal{N}(I-P)$ are unital algebras with unities P and $I-P$, respectively, and $P\text{alg}\mathcal{N}(I-P)$ is a faithful unital $(P\text{alg}\mathcal{N}P, (I-P)\text{alg}\mathcal{N}(I-P))$-bimodule.

Since any closed (under norm topology) subspace of a Hilbert space is complemented,it follows that for any nontrivial nest $\mathcal{N}$ on a Hilbert space $\mathcal{H}$, each $N \in \mathcal{N}$ with $N \neq \{0\}$ and $N \neq \mathcal{H}$, is complemented. Thus, every nontrivial nest algebra on a Hilbert space satisfies the conclusion in Remark 1.

We have the following result on nest algebras.

Theorem 2. *Let $\mathcal{N}$ be a nest on a Banach space $\mathcal{X}$, and there exists a nontrivial element N in $\mathcal{N}$ which is complemented in $\mathcal{X}$. Suppose that δ: $\text{alg}\mathcal{N} \longrightarrow \text{alg}\mathcal{N}$ is a mapping (not necessarily additive), and $\mathfrak{S} \subseteq \text{alg}\mathcal{N} \times \text{alg}\mathcal{N}$ is as follows:*

$$\mathfrak{S} = \{(A,B) \mid AB = 0, A, B \in \text{alg}\mathcal{N}\} \cup \{(A,X) \mid A \in \text{alg}\mathcal{N},\ X \in \{P,Q\}\}, \tag{10}$$

where $P \in \text{alg}\mathcal{N}$ is the idempotent with $\text{ran}P = N$ and $Q = I - P$. Then the following are equivalent:

(i) $(A,B) \in \mathfrak{S} \Rightarrow \delta(A \circ B) = A^\circ \delta(B) + \delta(A)^\circ B$;

(ii) $(A,B) \in \mathfrak{S} \Rightarrow \delta(A \circ B) = A^\circ \delta(B) + \delta(A)^\circ B$;

(iii) δ is additive and $A, B \in \text{alg}\mathcal{N}, AB = 0 \Rightarrow \delta(A \circ B) = A^\circ \delta(B) + \delta(A)^\circ B$;

(iv) δ is an additive Jordan derivation;

(v) δ is a derivable map;

(vi) δ is an additive derivation.

Suppose, further, that $\mathcal{X}$ is an infinite-dimensional Banach space. Then the above conditions are also equivalent to:

(vii) δ is an inner derivation.

Proof. From Remark 1, alg $\mathcal{N}$ is a triangular algebra, and all the assumptions of Theorem 1 hold. So all cases (i) to (vi) are equal. If $\mathcal{X}$ is an infinite-dimensional Banach space, then by [25] every additive derivation of alg $\mathcal{N}$ is linear. From [26], Theorem 2 any linear derivation of a nest algebra on a Banach space is continuous and by [27] all continuous linear derivations of a nest algebra on a Banach space are inner derivations (see also [28], Theorem 2.3). Given this, it is proved that if $\mathcal{X}$ is an infinite-dimensional Banach space, condition (vii) is equivalent to condition (vi). The proof is complete. □

By Theorem 2, we have the following corollary.

Corollary 1. *Let $\mathcal{N}$ be a nontrivial nest on a Hilbert space $\mathcal{H}$. Suppose that δ: $\text{alg}\mathcal{N} \longrightarrow \text{alg}\mathcal{N}$ is a mapping (not necessarily additive), and $\mathfrak{S} \subseteq \text{alg}\mathcal{N} \times \text{alg}\mathcal{N}$ is as follows*

$$\mathfrak{S} = \{(A,B) \mid AB = 0, A, B \in \text{alg}\mathcal{N}\} \cup \{(A,X) \mid A \in \text{alg}\mathcal{N},\ X \in \{P,Q\}\}, \tag{11}$$

where $P \in \text{alg}\mathcal{N}$ is the is the orthogonal projection on a nontrivial element $N \in \mathcal{N}$, and $Q = I - P$. Then the following are equivalent:

(i) $(A,B) \in \mathfrak{S} \Rightarrow \delta(A \circ B) = A^\circ \delta(B) + \delta(A)^\circ B$;

(ii) $A, B \in \text{alg}\mathcal{N} \Rightarrow \delta(A \circ B) = A^\circ \delta(B) + \delta(A)^\circ B$;

(iii) δ is additive and $A, B \in \text{alg}\mathcal{N}, AB = 0 \Rightarrow \delta(A \circ B) = A^\circ \delta(B) + \delta(A)^\circ B$;

(iv) δ is an additive Jordan derivation;

(v) δ is a derivable map;

(vi) δ is an additive derivation.

Suppose, further, that $\mathscr{H}$ is an infinite-dimensional Banach space. Then the above conditions are also equivalent to:

(vii) δ is an inner derivation.

Note that if $\mathscr{H}$ is a Hilbert space, and $\dim\mathscr{H} < \infty$, then there exist additive derivations of the nest algebra which are not inner (see [29]).

3. Proof of Theorem 1

In this section weassume that $\mathscr{T} = \mathrm{Tri}(\mathscr{A}, \mathscr{M}, \mathscr{B})$ is a 2-torsion free triangular ring, and $P = \begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix}$ is the standard idempotent of $\mathscr{T}$ and $Q = I - P = \begin{bmatrix} 0 & 0 \\ 0 & 1_B \end{bmatrix}$ which is also an idempotent. Also we put

$$\mathscr{T}_{11} = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} | a \in \mathscr{A} \right\},\quad \mathscr{T}_{12} = \left\{ \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} | m \in \mathscr{M} \right\},\quad \mathscr{T}_{22} = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} | b \in \mathscr{B} \right\}. \tag{12}$$

It is obvious that $\mathscr{T} = \mathscr{T}_{11} \oplus \mathscr{T}_{12} \oplus \mathscr{T}_{22}$ and for any $A \in \mathscr{T}$ we have

$$A = PAP + PAQ + QAQ, QAP = 0. \tag{13}$$

Proof of Theorem 1:

The following statements are clear: $(vi) \Rightarrow (i)$, $(vi) \Rightarrow (ii)$, $(vi) \Rightarrow (iii)$, $(vi) \Rightarrow (iv)$, $(vi) \Rightarrow (v)$, $(ii) \Rightarrow (i)$ and $(iv) \Rightarrow (i)$. We just prove the next items and so that the proof is complete.

(iii) $\Rightarrow$ (i): It is enough to prove that for all $A \in \mathscr{T}$ we have

$$\begin{aligned} \delta(A \circ P) &= A^\circ \delta(P) + \delta(A)^\circ P, \\ \delta(A \circ Q) &= A^\circ \delta(Q) + \delta(A)^\circ Q. \end{aligned} \tag{14}$$

Define the additive mapping $\tau: \mathscr{T} \longrightarrow \mathscr{T}$ by $\tau(A) = \delta(A) - I_{\delta(P)}(A)$. Given that $I_{\delta(P)}$ is an additive derivation, it is easily checked that

$$\begin{aligned} A, B \in \mathscr{T}, \quad AB = 0 \Rightarrow \tau(A \circ B) \\ = A^\circ \tau(B) + \tau(A)^\circ B. \end{aligned} \tag{15}$$

Also, $P\tau(P)Q = 0$.

Since $PQ = QP = 0$, thus

$$\begin{aligned} 0 &= \tau(P \circ Q) \\ &= \tau(P) \circ Q + P \circ \tau(Q). \end{aligned} \tag{16}$$

Multiplying both sides of (16) by Q, we arrive at $2Q\tau(P)Q = 0$, and hence $Q\tau(P)Q = 0$. Since $(PAQ)P = 0$ for all $A \in \mathscr{T}$, it follows that

$$\begin{aligned} \tau(PAQ) &= \tau((PAQ) \circ P) \\ &= \tau(PAQ) \circ P + (PAQ) \circ \tau(P). \end{aligned} \tag{17}$$

By (17) and the fact that $Q\tau(P)Q = 0$ we get

$$\tau(PAQ) = \tau(P)PAQ + P \circ \tau(PAQ). \tag{18}$$

Multiplying both sides of (18) by P, we conclude that $P\tau(PAQ)P = 0$. Now multiplying (18) from the left by P, it yields $P\tau(P)PAQ = 0$ for all $A \in \mathscr{T}$. Since M is faithful as a left A-module, we conclude that $P\tau(P)P = 0$. By using the results obtained, $\tau(P) = 0$. In view of (16) and the fact that $\tau(P) = 0$, we get $P \circ \tau(Q) = 0$, thus $\tau(Q) = Q\tau(Q)Q$. Given that $Q(PAQ) = 0$ and with the same argument above, we can prove that $Q\tau(Q)Q = 0$ and hence $\tau(Q) = 0$.

For all $A \in \mathscr{T}$ we have $(PAP)Q = Q(PAP) = 0$, and hence

$$\begin{aligned} 0 &= \tau((PAP) \circ Q) \\ &= \tau(PAP) \circ Q + PAP \circ \tau(Q). \end{aligned} \tag{19}$$

It follows from $\tau(Q) = 0$ that $\tau(PAP) \circ Q = 0$ for all $A \in \mathscr{T}$. Thus

$$\begin{aligned} \tau(PAP) \circ P + PAP \circ \tau(P) = \tau(PAP) \circ P \\ &= \tau(PAP) \circ P + \tau(PAP) \circ Q \\ &= 2\tau(PAP) \\ &= \tau((PAP) \circ P), \end{aligned} \tag{20}$$

for all $A \in \mathscr{T}$. Now according to the results we have

$$\begin{aligned} \tau(A \circ P) &= \tau((PAP) \circ P) + \tau((PAQ) \circ P) + \tau((QAQ) \circ P) \\ &= \tau(PAP) \circ P + PAP \circ \tau(P) \\ &\quad + \tau(PAQ) \circ P + PAQ^\circ \tau(P) \\ &\quad + \tau(QAQ) \circ P + QAQ^\circ \tau(P) \\ &= \tau(A)^\circ P + A^\circ \tau(P), \end{aligned} \tag{21}$$

for all $A \in \mathscr{T}$. By a similar argument as given above, we can prove that

$$\tau(A \circ Q) = \tau(A)^\circ Q + A^\circ \tau(Q), \tag{22}$$

for all $A \in \mathscr{T}$. Since $I_{\delta(P)}$ is an additive derivation, the condition (i) is obtained for δ.

(v) $\Rightarrow$ (i): Since δ is derivable, so

$$\delta(0) = \delta(0)0 + 0\delta(0), \tag{23}$$

and therefore for all $A, B \in \mathscr{T}$ with $AB = 0$, we have

$$\begin{aligned} 0 &= \delta(AB) \\ &= \delta(A)B + A\delta(B), \\ \delta(BA) &= \delta(B)A + B\delta(A). \end{aligned} \tag{24}$$

Thus

$$\begin{aligned} \delta(A \circ B) &= \delta(BA) \\ &= \delta(AB) + \delta(BA) \\ &= \delta(A)^\circ B + A^\circ \delta(B). \end{aligned} \tag{25}$$

Define $\tau: \mathscr{T} \longrightarrow \mathscr{T}$ by $\tau(A) = \delta(A) - I_{\delta(P)}(A)$. So that τ is a derivable map and $P\tau(P)Q = 0$. It follows from $\tau(P) = \tau(P^2) = \tau(P)P + P\tau(P)$ that $P\tau(P)P = 0$ and $Q\tau(P)Q = 0$. Hence $\tau(P) = 0$. Since τ is derivable, then $\tau(0) = 0$ and $\tau(I) = 0$. So $\tau(Q) = 0$.

For all $A \in \mathcal{T}$, we have

$$\begin{aligned}\tau(PAQ) &= \tau(P)PAQ + P\tau(PAQ)\\ &= P\tau(PAQ),\end{aligned} \tag{26}$$

so $Q\tau(PAQ)Q = 0$. Also

$$\begin{aligned}\tau(PAQ) &= \tau(PAQ)Q + PAQ\tau(Q)\\ &= \tau(PAQ)Q,\end{aligned} \tag{27}$$

thus $P\tau(PAQ)P = 0$. Therefore

$$\tau(PAQ) = P\tau(PAQ)Q, \tag{28}$$

for all $A \in \mathcal{T}$. Further

$$\begin{aligned}\tau(PAQ) &= \tau((P + PAQ)Q)\\ &= \tau(P + PAQ)Q + (P + PAQ)\tau(Q)\\ &= \tau(P + PAQ)Q,\\ 0 &= \tau(P)\\ &= \tau((P + PAQ)P)\\ &= \tau(P + PAQ)P + (P + PAQ)\tau(P)\\ &= \tau(P + PAQ)P.\end{aligned} \tag{29}$$

By adding two recent statements, we arrive at the following equation:

$$\tau(P + PAQ) = \tau(PAQ), \tag{30}$$

for all $A \in \mathcal{T}$.

So for all $A, B \in \mathcal{T}$, we have

$$\begin{aligned}\tau(PAQ + PBQ) &= \tau((P + PAQ)(PBQ + Q))\\ &= \tau(P + PAQ)(PBQ + Q) + (P + PAQ)\tau(PBQ + Q)\\ &= \tau(PAQ)(PBQ + Q) + (P + PAQ)\tau(PBQ).\end{aligned} \tag{31}$$

From the above equality and (28) we find that

$$\tau(PAQ + PBQ) = \tau(PAQ) + \tau(PBQ), \tag{32}$$

for all $A, B \in \mathcal{T}$.

We have

$$\begin{aligned}0 &= \tau((PAP)Q)\\ &= \tau(PAP)Q + (PAP)\tau(Q)\\ &= \tau(PAP)Q,\end{aligned} \tag{33}$$

so that

$$\tau(PAP) = P\tau(PAP)P, \tag{34}$$

for all $A \in \mathcal{T}$.

It follows from (32) that

$$\begin{aligned}\tau((PAP + PBP)PCQ) &= \tau(PAPCQ + PBPCQ)\\ &= \tau(PAPCQ) + \tau(PBPCQ)\\ &= \tau(PAP)PCQ + PAP\tau(PCQ)\\ &\quad + \tau(PBP)PCQ + PBP\tau(PCQ).\end{aligned} \tag{35}$$

On the other hand

$$\tau((PAP + PBP)PCQ) = \tau(PAP + PBP)PCQ + (PAP + PBP)\tau(PCQ). \tag{36}$$

Comparing recent two statements we get

$$[\tau(PAP + PBP) - \tau(PAP) - \tau(PBP)]PCQ = 0, \tag{37}$$

for all $A, B, C \in \mathcal{T}$. From the above statement, (34) and faithfulness of $\mathcal{M}$ as a left A-module, we conclude that

$$\tau(PAP + PBP) = \tau(PAP) + \tau(PBP), \tag{38}$$

for all $A, B \in \mathcal{T}$.

By (38), we have

$$\begin{aligned}\tau(A \circ P)P - \tau(AP)P - \tau(PA)P &= \tau((A \circ P)P) - A^{\circ} P\tau(P)\\ &\quad - \tau(AP) + AP\tau(P) - \tau(PAP) + PA\tau(P)\\ &= \tau((2PAP) - \tau(PAP)) - \tau(PAP)\\ &= 0,\end{aligned} \tag{39}$$

for all $A \in \mathcal{T}$. Beside

$$\tau(A\circ P)Q-\tau(AP)Q-\tau(PA)Q=\tau((A\circ P)Q)-(A\circ P)\tau(Q)+AP\tau(Q)-\tau(PAQ)+PA\tau(Q)=0. \tag{40}$$

By adding two recent statements, we obtain

$$\tau(A\circ P)=\tau(AP)+\tau(PA), \tag{41}$$

for all $A\in\mathscr{T}$. Now by the above identity and the fact that τ is a derivable map, we arrive at

$$\tau(A\circ P)=\tau(A)\circ P+A\circ\tau(P), \tag{42}$$

for all $A\in\mathscr{T}$. According to this statement and that $I_{\delta(P)}$ is a derivation, we conclude that

$$\delta(A\circ P)=\delta(A)\circ P+A\circ\delta(P), \tag{43}$$

for all $A\in\mathscr{T}$. By the simillar argument as above, we can prove that

$$\delta(A\circ Q)=\delta(A)\circ Q+A\circ\delta(Q), \tag{44}$$

for all $A\in\mathscr{T}$. Therefore δ satisfies (i).

(i) $\Rightarrow$ (vi): Define the mapping $\tau\colon\mathscr{T}\longrightarrow\mathscr{T}$ by $\tau(A)=\delta(A)-I_{\delta(P)}(A)$ and easily can be seen that τ satisfies

$$(A,B)\in S\Rightarrow\tau(A\circ B)=\tau(A)\circ B+A\circ\tau(B). \tag{45}$$

Also $P\tau(P)Q=0$.

We prove that τ is an additive derivation through the following steps.

Step 1. $\tau(0)=0$.

Proof. Since $0\circ 0=0$ then

$$\begin{aligned}\tau(0)&=\tau(0)\circ 0+\tau(0)\circ 0\\&=0.\end{aligned} \tag{46}$$

□

Step 2. For all $A\in\mathscr{T}$

$$\tau(PAQ)=P\tau(PAQ)Q. \tag{47}$$

Proof. For any $A\in\mathscr{T}$ we have $(PAQ,P)\in\mathfrak{S}$. So that

%

$$\begin{aligned}\tau(PAQ)&=\tau((PAQ)\circ P)\\&=\tau(PAQ)\circ P+PAQ^{\circ}\tau(P)\\&=\tau(PAQ)P+P\tau(PAQ)+PAQ\tau(P)+\tau(P)PAQ.\end{aligned} \tag{48}$$

If we multiplying both sides of (48) by P, we arrive at $P\tau(PAQ)P=0$ and if we multiplying both sides of (48) by Q, we obtain $Q\tau(PAQ)Q=0$. Therefore $\tau(PAQ)=P\tau(PAQ)Q$ for all $A\in\mathscr{T}$. □

Step 3. $\tau(P)=0$ and $\tau(Q)=0$.

Proof. Since $(Q,P)\in\mathfrak{S}$, it follows that

$$\begin{aligned}0&=\tau(Q\circ P)\\&=\tau(Q)\circ P+Q\circ\tau(P).\end{aligned} \tag{49}$$

So

$$\tau(Q)P+P\tau(Q)+Q\tau(P)+\tau(P)Q=0. \tag{50}$$

By multiplying both sides of (50) by Q, we have $Q\tau(P)Q=0$. Now multiplying (50) from the left by P and from the right by Q, we see that $P\tau(P)PAQ=0$ for all $A\in\mathscr{T}$. Faithfulness of $\mathscr{M}$ implies that $P\tau(P)P=0$. By the results obtained and that $P\tau(P)Q=0$, we conclude that $\tau(P)=0$. From this result and (50), we have $\tau(Q)P+P\tau(Q)=0$ and so $P\tau(Q)P=0$ and $P\tau(Q)Q=0$. Since for all $A\in\mathscr{T}$, $(Q,PAQ)\in\mathfrak{S}$, so we have

$$\begin{aligned}\tau(PAQ)&=\tau(Q\circ PAQ)\\&=\tau(Q)\circ(PAQ)+Q\circ\tau(PAQ).\end{aligned} \tag{51}$$

Multiplying both sides of the above identity, we arrive at $PAQ\tau(Q)Q=0$, for all $A\in\mathscr{T}$. By faithfulness of $\mathscr{M}$, we have $Q\tau(Q)Q=0$, so $\tau(Q)=0$. □

Step 4. For all $A\in\mathscr{T}$

$$\tau(PAP)=P\tau(PAP)P,\ \tau(QAQ)=Q\tau(QAQ)Q. \tag{52}$$

Proof. Since $(PAP,Q)\in\mathfrak{S}$ for all $A\in\mathscr{T}$, from Step 3, we have

$$\begin{aligned}0&=\tau((PAP)\circ Q)=\tau(PAP)\circ Q+PAP\circ\tau(Q)\\&=\tau(PAP)Q+Q\tau(PAP).\end{aligned} \tag{53}$$

Hence $Q\tau(PAP)Q=0$ and $P\tau(PAP)Q=0$. So

$$\tau(PAP)=P\tau(PAP)P, \tag{54}$$

for all $A\in\mathscr{T}$. Given that $(QAQ,P)\in\mathfrak{S}$ for all $A\in\mathscr{T}$, using Step 3 and same argument as above, we can prove that

$$\tau(QAQ)=Q\tau(QAQ)Q, \tag{55}$$

for all $A\in\mathscr{T}$. □

Step 5. For all $A,B\in\mathscr{T}$

$$\begin{aligned}&Q\tau(PAP+PBQ)Q=0,\\&P\tau(PAQ+QBQ)P=0.\end{aligned} \tag{56}$$

Proof. Since $(Q,PAP+PBQ)\in\mathfrak{S}$ and $(PAQ+QBQ,P)\in\mathfrak{S}$, for all $A,B\in\mathscr{T}$, from Step 3 we obtain

$$\begin{aligned}\tau(PBQ)&=\tau(Q\circ(PAP+PBQ))\\&=Q\tau(PAP+PBQ)+\tau(PAP+PBQ)Q,\\\tau(PAQ)&=\tau((PAQ+QBQ)\circ P)\\&=\tau(PAQ+QBQ)P+P\tau(PAQ+QBQ).\end{aligned} \tag{57}$$

In the above, multiplying both sides of first statement by Q and multiplying both sides of second statement by P and according to Step 2, the desired result is obtained. □

Step 6. For all $A, B \in \mathcal{T}$

$$\begin{aligned} \tau(PAPBQ) &= \tau(PAP)PBQ + PAP\tau(PBQ), \\ \tau(PAQBQ) &= \tau(PAQ)QBQ + PAQ\tau(QBQ). \end{aligned} \tag{58}$$

Proof. Since $(PBQ, PAP) \in \mathfrak{S}$ for all $A, B \in \mathcal{T}$, so that

$$\begin{aligned} \tau(PAPBQ) &= \tau((PBQ) \circ (PAP)) \\ &= \tau(PBQ)PAP + PAP\tau(PBQ) \\ &\quad + PBQ\tau(PAP) + \tau(PAP)PBQ. \end{aligned} \tag{59}$$

From Steps 2 and 4, we find that $\tau(PBQ)P = 0$ and $Q\tau(PAP) = 0$, hence

$$\tau(PAPBQ) = \tau(PAP)PBQ + PAP\tau(PBQ), \tag{60}$$

for all $A, B \in \mathcal{T}$.

Since $(QBQ, PAQ) \in \mathfrak{S}$ for all $A, B \in \mathcal{T}$, from Steps 2, 4 and similar argument as above, we conclude that

$$\tau(PAQBQ) = \tau(PAQ)QBQ + PAQ\tau(QBQ), \tag{61}$$

for all $A, B \in \mathcal{T}$. □

Step 7. For all $A, B \in \mathcal{T}$

$$\begin{aligned} \tau(PAPBP) &= \tau(PAP)PBP + PAP\tau(PBP), \\ \tau(QAQBQ) &= \tau(QAQ)QBQ + QAQ\tau(QBQ). \end{aligned} \tag{62}$$

Proof. According to Step 6, for all $A, B \in \mathcal{T}$ we have

$$\tau(PAPBPCQ) = \tau(PAPBP)PCQ + PAPBP\tau(PCQ). \tag{63}$$

On the other hand

$$\begin{aligned} \tau(PAPBPCQ) &= PAP\tau(PBPCQ) + \tau(PAP)PBPCQ \\ &= PAP\tau(PBP)PCQ + PAPBP\tau(PCQ) \\ &\quad + \tau(PAP)PBPCQ. \end{aligned} \tag{64}$$

Comparing above statements and according to Step 4, we get

$$P[\tau(PAPBP) - \tau(PAP)PBP - PAP\tau(PBP)]PCQ = 0. \tag{65}$$

Now by the faithfulness of $\mathcal{M}$, we have

$$P[\tau(PAPBP) - \tau(PAP)PBP - PAP\tau(PBP)]P = 0. \tag{66}$$

So

$$[\tau(PAPBP) - \tau(PAP)PBP - PAP\tau(PBP)]P = 0, \tag{67}$$

for all $A, B \in \mathcal{T}$. It follows from Step 4 that

$$[\tau(PAPBP) - \tau(PAP)PBP - PAP\tau(PBP)]Q = 0, \tag{68}$$

for all $A, B \in \mathcal{T}$. By adding two recent statements we arrive at

$$\tau(PAPBP) = \tau(PAP)PBP + PAP\tau(PBP), \tag{69}$$

for all $A, B \in \mathcal{T}$. Using Steps 4, 6 and similar arguments as above, we can show that

$$\tau(QAQBQ) = \tau(QAQ)QBQ + QAQ\tau(QBQ), \tag{70}$$

for all $A, B \in \mathcal{T}$. □

Step 8. For all $A, B \in \mathcal{T}$

$$\begin{aligned} \tau(PAP + PBQ) &= \tau(PAP) + \tau(PBQ), \\ \tau(PAQ + QBQ) &= \tau(PAQ) + \tau(QBQ). \end{aligned} \tag{71}$$

Proof. For all $A, B \in \mathcal{T}$ we have $(PCQ, PAP + PBQ) \in \mathfrak{S}$. So

$$\tau(PCQ \circ (PAP + PBQ)) = \tau(PCQ) \circ (PAP + PBQ) + PCQ \circ \tau(PAP + PBQ). \tag{72}$$

From Step 2, we get

$$\tau(PCQ \circ (PAP + PBQ)) = PAP\tau(PCQ) + PCQ\tau(PAP + PBQ) + \tau(PAP + PBQ)PCQ. \tag{73}$$

Beside by Step 6, we have

$$\begin{aligned}\tau(PCQ\circ(PAP+PBQ)) &= \tau(PAPCQ)\\ &= \tau(PAP)PCQ+PAP\tau(PCQ).\end{aligned} \tag{74}$$

Comparing two recent statements and multiplying outcome relation from the left by P and from the right by Q, and using Step 5, we arrive at

$$P\tau(PAP+PBQ)PCQ=\tau(PAP)PCQ, \tag{75}$$

for all $A,B\in\mathcal{T}$. By faithfulness of $\mathcal{M}$, we conclude that

$$P\tau(PAP+PBQ)P=P\tau(PAP)P. \tag{76}$$

From the above statement, Steps 4 and 5, we obtain

$$\tau(PAP+PBQ)-\tau(PAP)=P[\tau(PAP+PBQ)-\tau(PAP)]Q. \tag{77}$$

So that

$$\begin{aligned}\tau(PAP+PBQ)-\tau(PAP) &= [\tau(PAP+PBQ)-\tau(PAP)]^\circ Q\\ &= \tau(PAP+PBQ)^\circ Q,\end{aligned} \tag{78}$$

for all $A,B\in\mathcal{T}$. Since $(Q,PAP+PBQ)\in\mathfrak{S}$ for all $A,B\in\mathcal{T}$, we have

$$\begin{aligned}\tau(PBQ) &= \tau(Q^\circ(PAP+PBQ))\\ &= \tau(Q)^\circ(PAP+PBQ)+Q^\circ\tau(PAP+PBQ)\\ &= Q^\circ\tau(PAP+PBQ).\end{aligned} \tag{79}$$

Comparing the above statements, we get

$$\tau(PAP+PBQ)=\tau(PAP)+\tau(PBQ), \tag{80}$$

for all $A,B\in\mathcal{T}$.

Since $(PAQ+QBQ,PCQ)\in\mathfrak{S}$ and $(PAQ+QBQ,P)\in\mathfrak{S}$ for all $A,B\in\mathcal{T}$, by the same computation as above, we arrive at

$$\tau(PAQ+QBQ)=\tau(PAQ)+\tau(QBQ), \tag{81}$$

for all $A,B\in\mathcal{T}$. □

Step 9. τ on $\mathcal{T}_{11}$, $\mathcal{T}_{12}$, $\mathcal{T}_{22}$ is additive.

Proof. For all $A,B\in\mathcal{T}$, we have $(PBQ+Q,PAQ+P)\in S$. So that

$$\begin{aligned}\tau(PAQ+PBQ) &= \tau(PBQ+Q)^\circ(PAQ+P)\\ &= \tau(PBQ+Q)^\circ(PAQ+P)+(PBQ+Q)^\circ\tau(PAQ+P).\end{aligned} \tag{82}$$

Now from Steps 2, 3 and 8, we get

$$\tau(PAQ+PBQ)=\tau(PAQ)+\tau(PBQ), \tag{83}$$

for all $A,B\in\mathcal{T}$. Therefore τ is additive on $\mathcal{T}_{12}$.

From Step 6, we have

$$\tau((PAP+PBP)PCQ)=\tau(PAP+PBP)PCQ+(PAP+PBP)\tau(PCQ). \tag{84}$$

On the other hand, according to additivity of τ on $\mathcal{T}_{12}$ and from Step 6, we have

$$\begin{aligned}\tau((PAP+PBP)PCQ) &= \tau(PAPCQ+PBPCQ)\\ &= \tau(PAPCQ)+\tau(PBPCQ)\\ &= \tau(PAP)PCQ+PAP\tau(PCQ)\\ &\quad+\tau(PBP)PCQ+PBP\tau(PCQ).\end{aligned} \tag{85}$$

By comparing the above statements and faithfulness of $\mathcal{M}$, we conclude that

$$P\tau(PAP+PBP)P=P\tau(PAP)P+P\tau(PBP)P, \tag{86}$$

for all $A,B\in\mathcal{T}$. Now Step 4 implies that τ is additive on $\mathcal{T}_{11}$.

By the similar argument as above we conclude that τ is also additive on $\mathcal{T}_{22}$. □

Step 10. For all $A\in\mathcal{T}$.

$$\tau(A)=\tau(PAP)+\tau(PAQ)+\tau(QAQ). \tag{87}$$

Proof. For all $A\in\mathcal{T}$ we have $(A,P)\in\mathfrak{S}$. So that

$$\tau(A^\circ P)=\tau(A)^\circ P+A^\circ\tau(P)=\tau(A)^\circ P. \tag{88}$$

On the other hand

$$\begin{aligned}\tau(A^\circ P) &= \tau((PAP)^\circ P+(PAQ)^\circ P+(QAQ)^\circ P)\\ &= \tau((PAP)^\circ P+(PAQ)^\circ P).\end{aligned} \tag{89}$$

Since $(PAP,P)\in\mathfrak{S}$, $(PAQ,P)\in\mathfrak{S}$ and from Steps 3, 4 and 8, we conclude that

$$\begin{aligned}\tau(A^\circ P) &= \tau((PAP)^\circ P+\tau(PAQ)^\circ P)\\ &= \tau(PAP)^\circ P+\tau(PAQ)^\circ P\\ &= \tau(PAP)^\circ P+\tau(PAQ)^\circ P+\tau(QAQ)^\circ P.\end{aligned} \tag{90}$$

Thus

$$[\tau(A)-\tau(PAP)-\tau(PAQ)-\tau(QAQ)]^\circ P=0. \tag{91}$$

Now for all $A \in \mathcal{T}$, since $(A, Q) \in \mathfrak{S}$ and by the similar argument as above we prove that

$$[\tau(A) - \tau(PAP) - \tau(PAQ) - \tau(QAQ)]\circ Q = 0. \tag{92}$$

Therefore we get the required result. □

Step 11. δ is an additive derivation.

Proof. From Steps 9, 10, for all $A, B \in \mathcal{T}$ we have

$$\begin{aligned}\tau(A+B) &= \tau((PAP+PBP)+(PAQ+PBQ)+(QAQ+QBQ))\\ &= \tau(PAP+PBP)+\tau(PAQ+PBQ)+\tau(QAQ+QBQ)\\ &= \tau(PAP)+\tau(PBP)+\tau(PAQ)+\tau(PBQ)+\tau(QAQ)+\tau(QBQ)\\ &= \tau(A)+\tau(B).\end{aligned} \tag{93}$$

Therefore τ is additive on $\mathcal{T}$.

It follows from Steps 2, 4, 6 and 7 that

$$\tau(AB) = \tau(A)B + A\tau(B), \tag{94}$$

for all $A, B \in \mathcal{T}$. Hence τ is an additive derivation. Since $I_{\delta(P)}$ is also an additive derivation, so $\delta = \tau + I_{\delta(P)}$ is an additive derivation and the proof is complete. □

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Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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