Hindawi
Journal of Mathematics
Volume 2022, Article ID 7403502, 11 pages
https://doi.org/10.1155/2022/7403502

Research Article

Some Characterizations of w-Noetherian Rings and SM Rings

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Received 12 November 2022; Revised 28 November 2022; Accepted 29 November 2022; Published 30 December 2022

Academic Editor: Marco Fontana

In this paper, we characterize w-Noetherian rings and SM rings. More precisely, in terms of the u-operation on a commutative ring R, we prove that R is w-Noetherian if and only if the direct limit of rGV-torsion-free injective R-modules is injective and that R is SM, which can be regarded as a regular w-Noetherian ring, if and only if the direct limit of GV-torsion-free (or rGV-torsion-free) reg-injective R-modules is reg-injective. As a by-product of the proof of the second statement, we also obtain that the direct and inverse limits of u-modules are both u-modules and that SM rings are regular w-coherent.

1. Introduction

In this paper, we assume that R is a commutative ring with identity with total quotient ring $T(R)$.

Generalizing the properties of the integral domain to commutative rings makes it natural to consider regular ideals of rings, and many new kinds of rings that are defined by taking regular ideals emerge. Examples include the Krull ring that was introduced by Kennedy [1] and the Prüfer v-multiplication ring (PvMR) that was presented by Matsuda [2]. These can be referred to [3–5]. The w-operation can be used as a research tool for commutative rings, but it is inaccurate for the above kind of rings. For example, Wang and Kim introduced the concept of the w-semi-hereditary ring in [6], which is also a PvMD-like ring, where a connected ring R is w-semi-hereditary if and only if R is a Prüfer v-multiplication domain (PvMD). This tells us that a direct description of rings defined by regularity using the w-operation is still lacking. Recently, Zhang [7] introduced the regular w-flat module to provide some homological characterizations of the PvMR, which is a good attempt.

In [8], Chang and Oh introduced the notion of general Krull rings, which are also defined by regularity. Also, they asked the following question. Is there a star operation $*$ on a ring so that a general Krull ring can be characterized as a ring in which each proper principal ideal can be written as a finite $*$-product of prime ideals [8, Question 0.2]? In order to answer the above question, they introduced a new star operation u on a ring R, and they showed that R is a general Krull ring if and only if each proper principal ideal of R is written as a finite u-product of prime ideals [8, Theorem 5.6].

Coincidentally, the u-operation on a ring was also introduced by Tao [9], called the w^*-operation in her master thesis independently. Unlike Chang and Oh who studied the u-operation from a ring theory perspective, Tao studied this star operation from a module theory perspective.

In more detail, Noetherian rings can be characterized by injectivity, which is known as the Cartan–Eilenberg–Bass theorem, i.e., a ring R is Noetherian if and only if the direct sum of any number of injective R-modules is injective, if and only if the direct sum of any countable number of injective R-modules is injective, if and only if every injective R-module is Σ-injective, if and only if the direct limit of injective R-modules is injective. An injective R-module E is said to be Σ-*injective* if every direct sum of copies of E is injective. In [10], Zhang et al. proved the w-theoretic analog of the Cartan–Eilenberg–Bass theorem for

Noetherian rings. The w-Noetherian ring is defined to be the ring which satisfies the ascending chain condition (ACC) on w-ideals by Yin et al. [11]. Also, a ring R is w-Noetherian if and only if R satisfies the ACC on its u-ideals (see [8, Theorem 6.9] or [9, Proposition 4.1.9]).

In [12], SM rings are introduced by Wang and Liao. A ring R is called *SM* if R satisfies the ACC on its regular w-ideals [12, Definition 5.10]. SM rings can be regarded as regular w-Noetherian ones loosely. Tao also proved that a ring R is SM if and only if R satisfies the ACC on its regular u-ideals [9, Theorem 4.2.1]. Thus, a natural question arises as to whether SM rings have both w-theoretic and u-theoretic analogs corresponding to the Cartan–Eilenberg–Bass theorem for Noetherian rings. Actually for this purpose, the notions of reg-injective and $\sum$-reg-injective R-modules are introduced in [12]. An R-module E is *reg-injective* if $Ext_R^1(R/I, E) = 0$ for any regular ideal I of R [12, Definition 5.2]. A reg-injective R-module E is $\sum$*-reg-injective* if every direct sum of its copies is reg-injective [12, Definition 6.4]. In [12, Theorem 6.10] and [9, Theorem 4.2.5], it is proved that a ring R is SM if and only if the direct sum of any number of GV-torsion-free (or rGV-torsion-free) reg-injective R-modules is reg-injective, if and only if the direct sum of any countable number of GV-torsion-free (or rGV-torsion-free) reg-injective R-modules is reg-injective, if and only if every GV-torsion-free (or rGV-torsion-free) reg-injective R-module is Σ-reg-injective. However, what about the direct limit of GV-torsion-free (or rGV-torsion-free) reg-injective R-modules for SM rings? Motivated by this question, we first show that the u-operation can induce a torsion theory, denoted by τ_u, by a Gabriel topology

$$\mathfrak{I} = \{I \mid I \text{ is an ideal of } R \text{ with } I_u = R\}. \quad (1)$$

Also, for w-Noetherian rings, we complete the u-theoretic analog of the Cartan–Eilenberg–Bass theorem for Noetherian rings in terms of our existing knowledge of general torsion theory. Then, for SM rings, we complete both the w and u-theoretic analogs of the Cartan–Eilenberg–Bass theorem for Noetherian rings. In the process, the discussion of the direct limit of u-modules is also necessary.

2. Preliminaries

Now we introduce some notations and results needed in this paper from [9, 13]. Let J be a finitely generated ideal of R. If the natural homomorphism $\varphi: R \longrightarrow J^* = \mathrm{Hom}_R(J, R)$ is an isomorphism, then J is called a GV*-ideal*, denoted by $J \in \mathrm{GV}(R)$. Let M be an R-module. Define

$$\mathrm{tor}_{\mathrm{GV}}(M) = \{x \in M \mid Jx = 0 \text{ for some } J \in \mathrm{GV}(R)\}. \quad (2)$$

Thus, $\mathrm{tor}_{\mathrm{GV}}(M)$ is a submodule of M. Also, M is said to be GV*-torsion* (resp., GV*-torsion-free*) if $\mathrm{tor}_{\mathrm{GV}}(M) = M$ (resp., $\mathrm{tor}_{\mathrm{GV}}(M) = 0$). Clearly R is a GV-torsion-free R -module [11, Corollary 1.5]. A GV-torsion-free module M is called a *w-module* if $\mathrm{Ext}_R^1(R/J, M) = 0$ for any $J \in \mathrm{GV}(R)$. The w-envelope of a GV-torsion-free module M is the set given by

$$M_w = \{x \in E(M) \mid Jx \subseteq M \text{ for some } J \in \mathrm{GV}(R)\}, \quad (3)$$

where $E(M)$ is the injective hull of M. It is easy to see that M is a w-module if and only if $M_w = M$. A nonzero ideal P of R is said to be *a prime*w*-ideal* if P is both a prime ideal and a w-ideal and *a maximalw-ideal* if P is maximal in the set of all proper w-ideals of R. Note that each maximal w-ideal is prime [13, Theorem 6.2.14]. A GV-torsion-free module M is *of finite type* if $M_w = N_w$ for some finitely generated submodule N of M [13, Proposition 6.4.2]. A sequence $A \longrightarrow B \longrightarrow C$ of R-modules and homomorphisms is said to be *w-exact* if the sequence $A_m \longrightarrow B_m \longrightarrow C_m$ is exact for any maximal w-ideal m of R. An R-module M is said to be of finitely presented type if there is a w-exact sequence $F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$, where F_1 and F_0 are finitely generated and free modules [13, Definition 6.4.9].

An ideal I of R is *regular* if I contains a nonzero divisor. An ideal J of R is called *anr*GV*-ideal* if J is a regular GV-ideal. Let $r\mathrm{GV}(R)$ denote the set of all *rGV*-ideals of R. Then, *rGV* (R) is a multiplicative system of ideals of R, i.e., $r\mathrm{GV}(R)$ satisfies that (i) $R \in r\mathrm{GV}(R)$ and (ii) if $J_1, J_2 \in r\mathrm{GV}(R)$; then, $J_1 J_2 \in r\mathrm{GV}(R)$. It is clear that $r\mathrm{GV}(R) \subseteq \mathrm{GV}(R)$. But the converse does not hold in general.

Example 1 (see [9, Example 2.1.1]). Let F be a field, and let $D = F[y, z]$ and $K = F(y, z)$, where y and z are indeterminates over F. Then, for the trivial extension $R = D \propto (K/D)_w$, we have that $T(R) = R$. Then, $r\mathrm{GV}(R) = \{R\}$. By [14, Theorem 4.7], R is not a DW ring, i.e., the ring satisfies that every ideal of R is a w-ideal. Then, $GV(R) \neq \{R\}$ by [13, Theorem 6.3.12]. Thus, $GV(R) \nsubseteq rGV(R)$.

Let M be an R-module. Define

$$\mathrm{tor}_{r\mathrm{GV}}(M) = \{x \in M \mid Jx = 0 \text{ for some } J \in r\mathrm{GV}(R)\}. \quad (4)$$

Thus, $\mathrm{tor}_{r\mathrm{GV}}(M)$ is a submodule of M. Also, M is said to be *rGV-torsion* (resp., *rGV-torsion-free*) if $\mathrm{tor}_{r\mathrm{GV}}(M) = M$ (resp., $\mathrm{tor}_{r\mathrm{GV}}(M) = 0$). It is clear that any GV-torsion-free R-module is *rGV*-torsion-free, while any *rGV*-torsion R-module is *GV*-torsion.

Example 2 (see [9, Example 2.2.4]). *Let $J \in GV(R)$, but $J \notin rGV(R)$. Then, R/J is* GV*-torsion, but not rGV-torsion.*

Proposition 1 (see [9, Proposition 2.2.6 and Proposition 2.2.7])

(1) An R-module M is rGV-torsion if and only if $Hom_R(M, N) = 0$ for any rGV-torsion-free R-module N.

(2) An R-module N is rGV-torsion-free if and only if $Hom_R(M, N) = 0$ for any rGV-torsion R-module M.

(3) Let $\{M_i\}$ be a family of R-modules. Then, $\prod M_i$ is rGV-torsion-free if and only if each M_i is rGV-torsion-free.

(4) If M is an rGV-torsion-free R-module, then $E(M)$ is also rGV-torsion-free.

An $rGV(R)$-torsion-free R-module M is called *au-module* if $\mathrm{Ext}_R^1(R/J, M) = 0$ for any $J \in r\mathrm{GV}(R)$[9, Definition 3.1.1]. In [9, Definition 3.2.1], the u-envelope of an rGV-torsion-free R-module M is the set given by

$$M_u = \{x \in E(M) | Jx \subseteq M \text{ for some } J \in r\mathrm{GV}(R)\}. \quad (5)$$

It is clear that $(M_u)_u = M_u$ and that $J \in r\mathrm{GV}(R)$ if and only if $J_u = R$[9, Proposition 3.2.5].

Proposition 2 (see [9, Theorem 3.1.7 and Theorem 3.2.12]). *The following statements are equivalent for anrGV-torsion-free R-module.*

(1) *M is a u-module.*

(2) *$M_u = M$.*

(3) *If $0 \longrightarrow M \longrightarrow N \longrightarrow C \longrightarrow 0$ is an exact sequence in which N is a u-module, then C is rGV-torsion-free.*

(4) *There exists an exact sequence $0 \longrightarrow M \longrightarrow N \longrightarrow C \longrightarrow 0$ such that N is a u-module and C is rGV-torsion-free.*

(5) *$\mathrm{Ext}_R^1(C, M) = 0$ for any rGV-torsion R-module C.*

(6) *$\mathrm{Ext}_R^1(A_u/A, M) = 0$ for any rGV-torsion R-module A.*

An ideal I of R is called a u-ideal if I is a u-module. It is clear that w-ideals of R are u-ideals, but the converse does not hold. One can refer to [8, Example 4.9]. A nonzero ideal p of R is said to be a prime u-ideal if p is both a prime ideal and a u-ideal, denoted by $p \in u\text{-}\mathrm{Spec}(R)$, and a maximal u-ideal if p is maximal in the set of all proper u-ideals of R, denoted by $p \in u\text{-}\mathrm{Max}(R)$. Note that each maximal u-ideal is prime [9, Theorem 3.3.4].

Proposition 3 (see [9, Theorem 3.3.5, Theorem 3.3.6, and Theorem 3.3.7])

(1) *An R-module M is rGV-torsion if and only if $M_m = 0$ for any maximalu-idealm.*

(2) *Let M be an rGV-torsion-free R-module. Then, $M_p = (M_u)_p$.*

(3) *Let M be an rGV-torsion-free R-module and let A and B be submodules of M. Then, $A_u = B_u$if and only if $A_m = B_m$ for any maximalu-idealm of R.*

An R-module M is said to be *u-finitely generated* if there exists some finitely generated submodule N of M such that (M/N) is an rGV-torsion R-module [9, Definition 3.4.1]. An rGV-torsion-free R-module M is u-finitely generated if and only if $M_u = B_u$ for some finitely generated submodule B of M[9, Proposition 3.4.3].

Proposition 4 (see [9, Proposition 3.4.5]). *Let $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ be an exact sequence of R-modules.*

(1) *If A and C are u-finitely generated, then B is u-finitely generated.*

(2) *If B is u-finitely generated, then C is u-finitely generated.*

3. Injective Modules over w-Noetherian Rings

Next, as in [15], we show that the u-operation can induce a torsion theory, denoted by τ_u, by a Gabriel topology

$$\mathfrak{I} = \{I \mid I \text{ is an ideal of } R \text{ with } I_u = R\}. \quad (6)$$

By the proof of [16, Proposition 4.6], the class of all τ_u--torsion R-modules, denoted by $\mathcal{T}_{\tau_u}(R)$, is the set $\{M | M$ is an R-module and $(0:{}_R m)_u = R$ for each nonzero element $m \in M$ $\}$. Let $\mathcal{T}_{rGV}(R)$ denote the set of all rGV-torsion R-modules. The following proposition shows that $\mathcal{T}_{rGV}(R)$ and $\mathcal{T}_{\tau_u}(R)$ coincide. Thus, τ_u-torsion-free (resp., τ_u-torsion) modules and rGV-torsion-free (resp., rGV-torsion) modules coincide. The proof of the following proposition is very similar to that of [15, Proposition 2.10]; however, we give a proof for completeness.

Proposition 5. *For a ring R, $\mathcal{T}_{rGV}(R) = \mathcal{T}_{\tau_u}(R)$.*

Proof. Note that M is an rGV-torsion R-module if and only if for any nonzero element $m \in M$ there exists some $J \in r\mathrm{GV}(R)$ such that $mJ = 0$; if and only if $(0:{}_R m)_u = R$ for any nonzero element $m \in M$; and if and only if M is a τ_u-torsion R-module.

Now we recall some terminology in [17], which is similar to that in [1515]. Let M be an R-module. A submodule N of M is called *τ_u-pure* (resp., *τ_u-dense*) in M if M/N is rGV-torsion-free (resp., rGV-torsion). Obviously if N is a τ_u-dense submodule of an rGV-torsion-free R-module M, then $N_u = M_u$. Set $C_{\tau_u}^M(N) := \{x \in M | (N:{}_R x)_u = R\}$, which is called the *$\tau_u$-closure* of N in M. Then, N is called *τ_u-closed* in M if $C_{\tau_u}^M(N) = N$. It is easy to verify that if M is rGV-torsion-free, then $C_{\tau_u}^M(N) = N_u \cap M$; N is τ_u-dense in M if and only if $C_{\tau_u}^M(N) = M$; and τ_u-closed submodules of M and its τ_u-pure submodules coincide. □

Lemma 1. *Let M be an rGV-torsion-free R-module. If M is a u-module, then τ_u-pure submodules and u-submodules of M coincide.*

Proof. Let N be a submodule of M. Then, the sequence

$$\mathrm{Hom}_R\left(\frac{R}{J}, \frac{M}{N}\right) \longrightarrow \mathrm{Ext}_R^1\left(\frac{R}{J}, N\right) \longrightarrow \mathrm{Ext}_R^1\left(\frac{R}{J}, M\right), \quad (7)$$

is exact for any $J \in rGV(R)$. Note that if N is a τ_u-pure submodule of M, then M/N is rGV-torsion-free. So, $\mathrm{Hom}_R(R/J, M/N) = 0$. Obviously, $\mathrm{Ext}_R^1(R/J, M) = 0$ because M is a u-module. Thus, $\mathrm{Ext}_R^1(R/J, N) = 0$, and so N is a u-module. Conversely, if N is a u-module, then it is easy to prove that M/N is rGV-torsion-free. Thus, N is a τ_u-pure submodule of M.

Recall that an R-module M is said to be *τ_u-Noetherian* if M satisfies the ACC on its τ_u-pure submodules [16, p. 175]. Thus, an rGV-torsion-free R-module M is τ_u-Noetherian if M satisfies the ACC on its u-submodules by Lemma 1. Thus, an rGV-torsion-free τ_u-Noetherian R-module is also called *u-Noetherian* in [9]. A ring R is τ_u*-Noetherian* if R is a τ_u-Noetherian R-module. Note that R is rGV-torsion-free over R. Then, R is τ_u-Noetherian if and only if R satisfies ACC on its u-ideals.

In [9] or [8], it is proved that τ_u-Noetherian rings coincide with w-Noetherian ones. Also, in terms of u-operations, some characterizations of w-Noetherian rings are provided in [9]. □

Proposition 6 (see [9, Proposition 4.1.9 and Theorem 4.1.10]). *The following statements are equivalent for a ringR.*

(1) R is a w-Noetherian ring.

(2) R is τ_u*-Noetherian.*

(3) Every ideal is u-finitely generated, i.e., for each ideal I of R, there exists some finitely generated subideal I_0 *of I such that* $I_u = (I_0)_u$.

(4) Every u-ideal is u-finitely generated.

(5) Every nonempty set ofu -ideals of R has a maximal element.

(6) Every primeu-ideal of R is u-finitely generated over R.

(7) The direct sum of any number of rGV-torsion-free injective R-modules is injective.

(8) The direct sum of any countable number of rGV-torsion-free injective R-modules is injective.

(9) Every rGV-torsion-free injective R-module is $\sum$*-injective.*

Remark

(1) In fact, with the help of the language of torsion theory, the equivalences of (2)–(6) of Proposition 6 can be obtained directly by [16, Proposition 20.1] or [17, Proposition 2.3.3], while the equivalences of (2) and (7)–(9) of Proposition 6 can be obtained directly by [16, Proposition 20.17].

(2) Although τ_u-Noetherian rings coincide with w-Noetherian ones, w-Noetherian R-modules are not necessarily τ_u-Noetherian. An R-module M is *w-Noetherian* if M satisfies ACC on its w-submodules. If N is a w--submodule of M, then N is GV-torsion-free, which implies that N is rGV-torsion-free. Since $N_w = N$, we can get that $N_u = N$. Then, N is a τ_u-pure submodule of M. So, τ_u-Noetherian modules are w-Noetherian by their definitions. But the converse does not hold by [9, Example 4.1.7]. In more detail, let $J \in GV(R)$ and $J \notin r\mathrm{GV}(R)$. Then, R/J is GV-torsion, not rGV-torsion over R. Set $L(R/J) := \{\alpha \in R/J | \text{if } J\alpha = 0 \text{ where } J \in r\mathrm{GV}(R), \text{ then } \alpha = 0\}$. Thus, $L(R/J)$ is an rGV-torsion-free submodule of R/J and $L := (L(R/J))_u \cap (R/J)$ is a τ_u-pure submodule of R/J. Let M be a direct sum of countably infinite number of R/J. Then, M is a w--Noetherian R-module since M is GV-torsion. Note that the chain $L \subseteq L \oplus L \subseteq L \oplus L \oplus L \subseteq \ldots$ of τ_u-pure submodules of M is not stationary. Then, M is not a τ_u-Noetherian R-module.

Next, with the help of the language of torsion theory, we can get more characterizations of w-Noetherian rings in terms of u-operations. For this purpose, first we need some notions. An R-module M is τ_u*-finitely generated* if M has a finitely generated τ_u-dense submodule [16, p. 157]. Thus, an rGV-torsion-free R-module M is τ_u-finitely generated (also u-finitely generated in this case) if there exists a finitely generated submodule N of M such that $N_u \cap M = M$, equivalently $N_u = M_u$. An R-module M is τ_u*-finitely presented* if it is isomorphic to F/K, where F is a finitely generated free R-module and K is a τ_u-finitely generated submodule of F[16, p. 164]. A ring R is τ_u*-coherent* if every finitely generated ideal of R is τ_u-finitely presented [18, Definition 1.2]. From [18, Theorem 3.3], we can get that R is a τ_u-coherent ring if and only if the direct limit of rGV-torsion-free FP-injective R-modules is FP-injective. Recall that an R-module M is said to be *FP-injective* if $\mathrm{Ext}_R^1(N, M) = 0$ for all finitely presented R-modules N. It can be also called an *absolutely pureR-module*. For more details, one can refer to [16].

Now, based on Proposition 6, we complete the u-version of the Cartan–Eilenberg–Bass theorem for Noetherian rings. For this, we need the following.

Lemma 2. *Let* $\{M_i | i \in \Gamma\}$ *be a family of rGV-torsion-free R-modules. Then,* $\varinjlim M_i$ *and* $\varprojlim M_i$*are also rGV-torsion-free.*

Proof. Let $0 \longrightarrow J \longrightarrow R \longrightarrow R/J \longrightarrow 0$ be an exact sequence, where $J \in rGV(R)$. Then, R/J is finitely presented. Thus, $\mathrm{Hom}_R(R/J, \varinjlim M_i) \cong \varinjlim \mathrm{Hom}_R(R/J, M_i)$ for any $J \in rGV(R)$ by [10, Lemma 2.1]. So, $\varinjlim M_i$ is rGV-torsion-free. Since $\varprojlim M_i$ is a submodule of $\prod_{i\in\Gamma} M_i$, which is rGV-torsion-free by Proposition 1 (3), it is clear that $\varprojlim M_i$ is rGV-torsion-free. □

Theorem 1. *The following statements are equivalent for a ringR.*

(1) R is aw-Noetherian ring.

(2) Every finitely generated R-module is τ_u*-finitely presented.*

(3) Every finitely generated R-module is τ_u*-Noetherian.*

(4) The direct limit of rGV-torsion-free injective R-modules is injective.

(5) rGV-torsion-free FP-injective R-modules and rGV-torsion-free injective ones coincide.

Proof

(1)⇒(2) Let M be a finitely generated R-module. Then, there exists an exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$, where F is a finitely generated free R-module. Since w-Noetherian rings and τ_u-Noetherian ones coincide, it follows that F is a τ_u-Noetherian R-module by [16, Proposition 20.4]. Then, K is τ_u-finitely generated [16, Proposition 20.1]. Thus, M is τ_u-finitely presented by [16, Proposition 19.3].

(2)⇒(1) Let I be an ideal of R. Since R/I is finitely generated over R, it is τ_u-finitely presented by (2). Then, I is τ_u-finitely generated by [16, Proposition 19.3]. Thus, R is a τ_u-Noetherian ring again by [16, Proposition 20.1] or [17, Proposition 2.3.3]. Then, R is w-Noetherian.

(1)⇒(3) Let M be a finitely generated R-module. Then, there exists an exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$, where F is a finitely generated free R-module. By [16, Proposition 20.4], F is a τ_u-Noetherian R-module. Again by [16, Proposition 20.4], M is also a τ_u-Noetherian R-module.

(3)⇒(1) It is clear.

(1)⇒(5) The proof of this implication is very similar to that of (1)⇒(6) in [15, Theorem 3.17]; however, we give a proof for completeness. Assume that R is a w-Noetherian ring. For any ideal I of R, I is u-finitely generated over R by Proposition 6. Thus, there exists a finitely generated subideal I_1 of I such that $I_u = (I_1)_u$. So, I/I_1 is an rGV-torsion R-module. Let M be an rGV-torsion-free FP-injective R-module. Then, for the exact sequence $0 \longrightarrow I/I_1 \longrightarrow R/I_1 \longrightarrow R/I \longrightarrow 0$, we can get the following exactsequence:

$$\mathrm{Hom}_R(I/I_1, M) \longrightarrow \mathrm{Ext}_R^1\left(\frac{R}{I}, M\right) \longrightarrow \mathrm{Ext}_R^1\left(\frac{R}{I_1}, M\right). \tag{8}$$

Since I/I_1 is rGV-torsion over R and M is rGV-torsion-free over R, it follows that $\mathrm{Hom}_R(I/I_1, M) = 0$ by Proposition 1 (1). Also, since M is FP-injective, we can get that $\mathrm{Ext}_R^1(R/I_1, M) = 0$. Then, $\mathrm{Ext}_R^1(R/I, M) = 0$. Thus, M is an injective R-module.

(5)⇒(1)Note that the direct sum of FP-injective R-modules is FP-injective by [19, p. 564]. Then, (1) holds by the equivalence of (1) and (7) in Proposition 6.

(2) + (5)⇒(4) By (2), R is τ_u-coherent. Then, we can get that the direct limit of rGV-torsion-free FP-injective R-modules is FP-injective by [4, Theorem 3.3]. By (6) and Lemma 2, the direct limit of rGV-torsion-free injective R-modules is injective.

(4)⇒(1)Since GV-torsion-free R-modules are rGV-torsion-free, we can get that the direct limit of GV-torsion-free injective modules is injective by (4). Then, R is w-Noetherian by [10, Theorem 2.9]. □

4. The Direct and Inverse Limits of u-Modules

One main purpose of this paper is to generalize the Cartan–Eilenberg–Bass theorem for Noetherian rings to SM rings. For this, the discussion of the direct limit of u-modules is necessary [20]. First we show that any rGV-ideal of R is τ_u-finitely presented.

Let M be an R-module, $M^* = \mathrm{Hom}_R(M, R)$, and $\mathrm{End}_R(M) = \mathrm{Hom}_R(M, M)$. Then, we have the natural homomorphism η: $M \otimes_R M^* \longrightarrow \mathrm{End}_R(M)$ by

$$\eta(x \otimes f)(y) = f(y)x, \quad x, y \in M,\ f \in M^*. \tag{9}$$

Lemma 3 (see [13, Theorem 2.6.17]). *Let M be an R-module. Then, M is finitely generated projective if and only if η is an isomorphism.*

Lemma 4. *Let M be an rGV-torsion-free R-module and let η be as in the above. If η_m is an isomorphism for any $m \in u\text{-}Max(R)$, then M is u-finitely generated.*

Proof. Since M is rGV-torsion-free, it is easy to see that $\mathrm{End}_R(M)$ is also rGV-torsion-free. Note that $(\mathrm{Im}(\eta))_m = (\mathrm{End}_R(M))_m$ by considering that η_m is an isomorphism for any $m \in u\text{-}Max(R)$. Then, $(\mathrm{Im}(\eta))_u = (\mathrm{End}_R(M))_u$ by Proposition 3. Thus, there exists some $J \in rGV(R)$ such that $J1_M \subseteq \mathrm{Im}(f)$, where 1_M denotes the identity map on M. Set $J\colon = (a_1, a_2, \ldots, a_n)$. Then, for any $i = 1, 2, \ldots, n$, there are finite sets $\{x_{i1}, x_{i2}, \ldots, x_{it_i}\} \subseteq M$ and $\{f_{i1}, f_{i2}, \ldots, f_{it_i}\} \subseteq M^*$ such that $a_i 1_M = \eta(\sum_{j=1}^{t_i}(x_{ij} \otimes f_{ij}))$. Let B be the submodule of M generated by $\{x_{ij} | i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, t_i\}$. Then, for any $x \in M$, we have $a_i x = a_i 1_M(x) = \eta(\sum_{j=1}^{t_i}(x_{ij} \otimes f_{ij}))(x) = \sum_{j=1}^{t_i} f_{ij}(x) x_{ij} \in B$. Thus, $JM \subseteq B$, which implies that $M \subseteq B_u$. It is clear that $B_u \subseteq M_u$. Then, $M_u = B_u$. Therefore, M is u-finitely generated. □

Lemma 5. *Let $J \in rGV(R)$. Then, $Ext_R^1(J, N)$ is rGV-torsion for any R-module N.*

Proof. Let $J \in rGV(R)$. For the exact sequence $0 \longrightarrow J \longrightarrow R \longrightarrow R/J \longrightarrow 0$, we can get an exact sequence

$$0 = Ext_R^1(R, N) \longrightarrow Ext_R^1(J, N) \longrightarrow Ext_R^2\left(\frac{R}{J}, N\right), \tag{10}$$

where N is an R-module. Note that $\mathrm{Ext}_R^2(R/J, N)$ is an R/J-module and so an rGV-torsion R-module. Then, $\mathrm{Ext}_R^1(J, N)$ is rGV-torsion over R.

Let Sbe a multiplicatively closed set of R. An R-module M is said to be *S-torsion* if $M_S = 0$, and that M is said to be *S-torsion-free* if $sx = 0$, for $s \in S$ and $x \in M$, implies $x = 0$. □

Lemma 6. *Let $J \in rGV(R)$ and $m \in u\text{-}Max(R)$. Then, $Hom_R(J, N)_m \cong Hom_m(J_m, N_m)$ for any R-moduleN.*

Proof. Set $S := R\backslash m$ and set $C := \{x \in N | sx = 0 \text{ for some } s \in S\}$. Then, C is S-torsion and (N/C) is S-torsion-free. For the exact sequence $0 \longrightarrow C \longrightarrow N \longrightarrow N/C \longrightarrow 0$, we can get the following exact sequence:

$$0 \longrightarrow \mathrm{Hom}_R(J, C) \longrightarrow \mathrm{Hom}_R(J, N) \longrightarrow \mathrm{Hom}_R\left(J, \frac{N}{C}\right) \longrightarrow \mathrm{Ext}_R^1(J, C). \tag{11}$$

By Lemma 1, $\mathrm{Ext}_R^1(J, C)_m = 0$. Note that $\mathrm{Hom}_R(J, C)$ is S-torsion. Then, $\mathrm{Hom}_R(J, C)_m = 0$. Thus, $\mathrm{Hom}_R(J, N)_m \cong \mathrm{Hom}_R(J, N/C)_m$. By [21, Lemma 1.7], $\mathrm{Hom}_R(J, N/C)_m \cong \mathrm{Hom}_{R_m}(J_m, (N/C)_m)$. Note that $(N/C)_m = N_m/C_m = N_m$. Then, $\mathrm{Hom}_R(J, N)_m \cong \mathrm{Hom}_{R_m}(J_m, N_m)$. □

Theorem 2. *Any rGV-ideal of R is τ_u-finitely presented.*

Proof. Let $J \in rGV(R)$. Then, there exits an exact sequence $0 \longrightarrow A \longrightarrow F \longrightarrow J \longrightarrow 0$, where$F$ is finitely generated free R module. Let m be a maximal u-ideal of R. Then, $\mathrm{Ext}_R^1(J, N)_m = 0$ for any R-module N by Lemma 1. Note that $J_m = R_m$. Then, $\mathrm{Ext}_{R_m}^1(J_m, N_m) = 0$. Thus, we have the following commutative diagram (12) with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_R(J,N)_{\mathfrak{m}} & \longrightarrow & \mathrm{Hom}_R(F,N)_{\mathfrak{m}} & \longrightarrow & \mathrm{Hom}_R(A,N)_{\mathfrak{m}} & \longrightarrow & 0 \\ & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \\ 0 & \longrightarrow & \mathrm{Hom}_{R_{\mathfrak{m}}}(J_{\mathfrak{m}},N_{\mathfrak{m}}) & \longrightarrow & \mathrm{Hom}_{R_{\mathfrak{m}}}(F_{\mathfrak{m}},N_{\mathfrak{m}}) & \longrightarrow & \mathrm{Hom}_{R_{\mathfrak{m}}}(A_{\mathfrak{m}},N_{\mathfrak{m}}) & \longrightarrow & 0. \end{array} \tag{12}$$

Note that f_1 is an isomorphism by Lemma 5 and f_2 is an isomorphism by [13, Theorem 2.6.16 (1)]. Then, $\mathrm{Hom}_R(A, N)_m \cong \mathrm{Hom}_{R_m}(A_m, N_m)$. In particular, $(A^*)_m \cong (A_m)^*$ and $(\mathrm{End}_R(A))_m \cong \mathrm{End}_{R_m} A_m$. For the exact sequence $0 \longrightarrow A_m \longrightarrow F_m \longrightarrow J_m \longrightarrow 0$, since F_m and J_m are finitely generated and free over R_m, we can get that A_m is also finitely generated free over R_m.

Consider the following commutative diagram:

$$\begin{array}{ccc} A_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} (A^*)_{\mathfrak{m}} & \xrightarrow{\eta_{\mathfrak{m}}} & (\mathrm{End}_R(A))_{\mathfrak{m}} \\ \downarrow & & \downarrow \\ A_{\mathfrak{m}} \otimes_{R_{\mathfrak{m}}} (A_{\mathfrak{m}})^* & \xrightarrow{\eta} & \mathrm{End}_{R_{\mathfrak{m}}}(A_{\mathfrak{m}}). \end{array} \tag{13}$$

By Lemma 3, the arrow in the bottom row is an isomorphism. Note that the vertical arrows are isomorphisms by the above. Then, the arrow in the top row is an isomorphism. So, η: $A \otimes_R A^* \longrightarrow \mathrm{End}_R(A)$ is a u-isomorphism. Note that A is an rGV-torsion-free R-module. Then, A is u-finitely generated over R by Lemma 4. Thus, J is τ_u-finitely presented over R.

Next, we show that the direct and inverse limits of u-modules are also u-modules. To do so, we need the following. □

Lemma 7. *Let $\{M_i\}$ be a family of rGV-torsion-free R-modules. Then, $\varinjlim \mathrm{Hom}_R(J, M_i) \cong \mathrm{Hom}_R(J, \varinjlim M_i)$ for each $J \in rGV(R)$.*

Proof. Let $J \in rGV(R)$. Since J is finitely generated, there is a finitely generated free module F such that $0 \longrightarrow K \longrightarrow F \longrightarrow J \longrightarrow 0$ is an exact sequence of R-modules. Note that J is τ_u-finitely presented by Theorem 2-, and so K is u-finitely generated. Since K is rGV-torsion-free, we can get that there exists a finitely generated submodule K_1 of K such that $K_u = (K_1)_u$. Then, there is an exact sequence $F_1 \stackrel{f}{\longrightarrow} K_1 \longrightarrow 0$, where F_1 is a finitely generated free R-module. Thus, we can get an exact sequence $F_1 \stackrel{f}{\longrightarrow} K \longrightarrow K/\mathrm{Im}(f) \longrightarrow 0$. For any maximal u-ideal m, $(K/\mathrm{Im}(f))_m = (K_m/\mathrm{Im}(f)_m) = ((K_1)_m/\mathrm{Im}(f)_m) = 0$. So, $K/\mathrm{Im}(f)$ is rGV-torsion. Then, we have the following commutative diagram (14) with exact rows:

$$\begin{array}{ccccc} \varinjlim \mathrm{Hom}_R(K/\mathrm{Im}(f), M_i) & \longrightarrow & \varinjlim \mathrm{Hom}_R(K, M_i) & \longrightarrow & \varinjlim \mathrm{Hom}_R(F_1, M_i) \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma \\ \mathrm{Hom}_R(K/\mathrm{Im}(f), \varinjlim M_i) & \longrightarrow & \mathrm{Hom}_R(K, \varinjlim M_i) & \longrightarrow & \mathrm{Hom}_R(F_1, \varinjlim M_i). \end{array} \tag{14}$$

By Lemma 2, $\varinjlim M_i$ is rGV-torsion-free. Then, it follows from Proposition 1 (1) that

$$\begin{aligned} \mathrm{Hom}_R\left(\frac{K}{\mathrm{Im}(f)}, \varinjlim M_i\right) &= 0, \\ \mathrm{Hom}_R\left(\frac{K}{\mathrm{Im}(f)}, M_i\right) &= 0. \end{aligned} \tag{15}$$

By [10, Lemma 2.1], γ is an isomorphism. Then, β is a monomorphism by the Five Lemma.

Consider the following commutative diagram (16):

$$\begin{array}{ccccccc} 0 \longrightarrow & \varinjlim \mathrm{Hom}_R(J, M_i) & \longrightarrow & \varinjlim \mathrm{Hom}_R(F, M_i) & \longrightarrow & \varinjlim \mathrm{Hom}_R(K, M_i) \\ & \downarrow f_1 & & \downarrow f_2 & & \downarrow \beta \\ 0 \longrightarrow & \mathrm{Hom}_R(J, \varinjlim M_i) & \longrightarrow & \mathrm{Hom}_R(F, \varinjlim M_i) & \longrightarrow & \mathrm{Hom}_R(K, \varinjlim M_i) \end{array} \tag{16}$$

By the above proof, β is a monomorphism. Again by [10, Lemma 2.1], f_2 is an isomorphism. Hence, f_1 is an epimorphism by the Five Lemma and f_1 is a monomorphism also by the above proof. Therefore, f_1 is an isomorphism. □

Corollary 1. *Let $\{M_i\}$ be a family of rGV-torsion-free R-modules. If K is rGV-torsion-free and u-finitely generated, then*

$$\beta\colon \varinjlim \mathrm{Hom}_R(K, M_i) \cong \mathrm{Hom}_R\left(K, \varinjlim M_i\right), \tag{17}$$

is a monomorphism.

Theorem 3. *Let $\{M_i\}$ be a family of u-modules over R. Then, $\varinjlim M_i$ is a u-module.*

Proof. Let $J \in rGV(R)$. Then, $0 \longrightarrow J \longrightarrow R \longrightarrow R/J \longrightarrow 0$ is an exact sequence. Consider the following commutative diagram:

$$\begin{array}{cccccccc} \varinjlim \mathrm{Hom}_R(R, M_i) & \longrightarrow & \varinjlim \mathrm{Hom}_R(J, M_i) & \longrightarrow & \varinjlim \mathrm{Ext}^1_R(R/J, M_i) & \longrightarrow & 0 \\ \cong \downarrow & & \downarrow f_1 & & \downarrow f_2 & & \\ \mathrm{Hom}_R(R, \varinjlim M_i) & \longrightarrow & \mathrm{Hom}_R(J, \varinjlim M_i) & \longrightarrow & \mathrm{Ext}^1_R(R/J, \varinjlim M_i) & \longrightarrow & 0 \end{array} \tag{18}$$

By Lemma 7, f_1 is an isomorphism. Then, f_2 is an isomorphism by the Five Lemma. Since each M_i is a u-module, $\mathrm{Ext}^1_R(R/J, \varinjlim M_i) \cong \varinjlim \mathrm{Ext}^1_R(R/J, M_i) = 0$. Hence, $\varinjlim M_i$ is a u-module. □

Lemma 8 (see [22, Theorem 2.22]). *The inverse limit of an inverse system $\{M_i, \psi_i^j\}$ of modules exists and $\varprojlim M_i = \{(a_i) \in \prod M_i a_i = \psi_i^j a_j, \text{whenever } i \le j\}$.*

Theorem 4. *Let $\{M_i, \psi_i^j\}$ be an inverse system of u-modules over R. Then, $\varprojlim M_i$ is a u-module.*

Proof. It suffices to show that every R-homomorphism $f\colon J \longrightarrow \varprojlim M_i$ can be extended to R for each $J \in rGV(R)$.

Let $\lambda\colon \varprojlim M_i \longrightarrow \prod M_i$ be an embedding and let $\lambda_1\colon J \longrightarrow R$ be an embedding. Consider the following diagram:

$$\begin{array}{ccccc} & & \prod M_i & & \\ & & \uparrow \lambda & \nwarrow g & \\ & & \varprojlim M_i & & \\ & & \uparrow f & & \\ 0 & \longrightarrow & J & \xrightarrow{\lambda_1} & R \end{array} \tag{19}$$

Since $\prod M_i$ is a u-module, there is a homomorphism $g\colon R \longrightarrow \prod M_i$ such that $g\lambda_1 = \lambda f$. Set $g(1) := (a_i) \in \prod M_i$. For each $x \in J$, $f(x) = g(x) = x \cdot (a_i) = (xa_i) \in \varprojlim M_i$. Then, $xa_i = \psi_i^j(xa_j) = x\psi_i^j(a_j)$ whenever $i \le j$ by Lemma 8. Hence, $J(a_i - \psi_i^j(a_j)) = 0$. Because M_i is rGV-torsion-free, we can get that $a_i - \psi_i^j(a_j) = 0$, which implies $a_i = \psi_i^j(a_j)$ whenever $i \le j$. Thus, $g(1) = (a_i) \in \varprojlim M_i$ again by Lemma 8. Therefore, $\varprojlim M_i$ is a u-module. □

5. Reg-Injective Modules over SM Rings

In this section, for SM rings, we complete both the w and the u-theoretic analogs of the Cartan–Eilenberg–Bass theorem for Noetherian rings. In [9], some characterizations are given in terms of u-operations.

Proposition 7 (see [9, Theorem 4.2.1]). *The following statements are equivalent for a ringR.*

(1) R is an SM ring.

(2) R satisfies ACC on its regularu-ideals.

(3) Every regularu-ideal of R is u-finitely generated.

(4) Every nonempty set of regular u-ideals of R has a maximal element.

(5) Every regular primeu-ideal of R is u-finitely generated.

(6) Every regular ideal of R is u-finitely generated.

Theorem 5. *Let R be an SM ring. Then, every finitely generated regular ideal of R is τ_u-finitely presented.*

Proof. Let $I = (a_0, a_1, \ldots, a_n)$ be a finitely generated regular ideal of R, where a_0 is a regular element in R. We prove this by induction on n. If $n = 0$, then $I \cong R$, which implies that I is τ_u-finitely presented. Now assume that $I_k = (a_0, a_1, \ldots, a_k)$ is τ_u-finitely presented, where $k < n$. Then, we have the following commutative diagram (20) with exact rows:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_k & \longrightarrow & R^{k+1} & \xrightarrow{f_1} & I_k & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L_{k+1} & \longrightarrow & R^{k+2} & \xrightarrow{f_2} & I_{k+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (I_k :_R Ra_{k+1}) & \longrightarrow & R & \longrightarrow & R/(I_k :_R Ra_{k+1}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & ,
\end{array}
\qquad (20)
$$

where $f_1(r_0, r_1, \ldots, r_k) = r_0a_0 + r_1a_1 + \ldots + r_ka_k$ for any $(r_0, r_1, \ldots, r_k) \in R^{k+1}$ and $f_2(r_0, r_1, \ldots, r_{k+1}) = r_0a_0 + r_1a_1 + \ldots + r_{k+1}a_{k+1}$ for any $(r_0, r_1, \ldots, r_{k+1}) \in R^{k+2}$.

Now we explain why $(I_k\colon {}_R Ra_{k+1}) \cong L_{k+1}/L_k$. Define $g\colon L_{k+1} \longrightarrow (I_k\colon {}_R Ra_{k+1})$ by $g(r_0, r_1, \ldots, r_{k+1}) = r_{k+1}$ for any $(r_0, r_1, \ldots, r_{k+1}) \in L_{k+1}$. Then, it is easy to verify that g is well defined. For any $r \in (I_k\colon {}_R Ra_{k+1})$, $ra_{k+1} \in I_k$. Then, there exists some elements $r_0, r_1, \ldots, r_k$ such that $ra_{k+1} = r_0a_1 + r_1a_1 + \ldots + r_ka_k$. Thus, $(-r_0, -r_1, \ldots, -r_k, r) \in L_{k+1}$ and $g(-r_0, -r_1, \ldots, -r_k, r) = r$, which implies that g is an epimorphism. It is clear that $L_k \subseteq \mathrm{Ker}(g)$. For any $(r_0, r_1, \ldots, r_{k+1}) \in \mathrm{Ker}(g)$, $g(r_0, r_1, \ldots, r_{k+1}) = r_{k+1} = 0$. Then, $(r_0, r_1, \ldots, r_k) \in L_k$. Then, $\mathrm{Ker}(f) \subseteq L_k$. Thus, $\mathrm{Ker}(f) = L_k$. So, $(I_k\colon {}_R Ra_{k+1}) \cong L_{k+1}/L_k$.

Note that $I_k \subseteq (I_k\colon {}_R Ra_{k+1})$. Then, $(I_k\colon {}_R Ra_{k+1})$ is a regular ideal of R. Since R is an SM ring, we can get that $(I_k\colon {}_R Ra_{k+1})$ is u-finitely generated by Proposition 7. By assumption, I_k is τ_u-finitely presented, and so L_k is u-finitely generated [3, Proposition 19.3]. Thus, L_{k+1} is u-finitely generated by Proposition 4. Hence, I_{k+1} is τ_u-finitely presented.

It is well known that Noetherian rings are coherent ones, and w-Noetherian rings are w-coherent ones [13, p. 393]. Recall that R is w-coherent if each finite type ideal of R is of finitely presented type [13, Definition 6.9.14]. By the same way of Theorem 5, we can get that if R is an SM ring, then every finitely generated regular ideal of R is of finitely presented type, where such a ring is called *regularw-coherent* [7, Definition 2.4]. Then, we can get the following corollary, which is corresponding to the classical result. □

Corollary 2. *SM rings are regular w-coherent.*

Next, our purpose is to complete the Cartan–Eilenberg–Bass theorem for SM rings. To do so, we need the following.

Lemma 9. *Let $\{M_i\}$ be a family of rGV-torsion-free R-modules. If R is an SM ring, then $\varinjlim \mathrm{Hom}_R(I, M_i) \cong \mathrm{Hom}_R(I, \varinjlim M_i)$ for any finitely generated regular ideal I of R.*

Proof. Let I be a finitely generated regular ideal of R. Since R is an SM ring, we can get that I is τ_u-finitely generated by Theorem 5. Then, there exists an exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow I \longrightarrow 0$, where $F$is finitely generated and free over R and K is u-finitely generated. Thus, we have the following commutative diagram (21) with exact rows:

$$\begin{array}{ccccccc} 0 \longrightarrow & \varinjlim \mathrm{Hom}_R(I, M_i) & \longrightarrow & \varinjlim \mathrm{Hom}_R(F, M_i) & \longrightarrow & \varinjlim \mathrm{Hom}_R(K, M_i) \\ & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 \\ 0 \longrightarrow & \mathrm{Hom}_R(I, \varinjlim M_i) & \longrightarrow & \mathrm{Hom}_R(F, \varinjlim M_i) & \longrightarrow & \mathrm{Hom}_R(K, \varinjlim M_i). \end{array} \quad (21)$$

By Corollary 1, f_3 is a monomorphism. By [10, Lemma 2.1], f_2 is an isomorphism. Then, f_1 is an epimorphism by the Five Lemma. Note that f_1 is a monomorphism again by Corollary 1. Thus, f_1 is an isomorphism. □

Lemma 10. *Let N be an rGV-torsion-free R-module and let Mbe au-module. Then, $Hom_R(N, M) \cong Hom_R(N_u, M)$.*

Proof. For the exact sequence $0 \longrightarrow N \longrightarrow N_u \longrightarrow N_u/N \longrightarrow 0$, we can get an exact sequence $0 \longrightarrow \mathrm{Hom}_R(N_u/N, M) \longrightarrow \mathrm{Hom}_R(N_u, M) \longrightarrow \mathrm{Hom}_R(N, M) \longrightarrow \mathrm{Ext}_R(N_u/N, M)$. Note that$N_u/N$ is rGV--torsion and M is a u-module. Then, $\mathrm{Hom}_R(N_u/N, M) = \mathrm{Ext}_R(N_u/N, M) = 0$ by Proposition 1 (1) and Proposition 2. Thus, $\mathrm{Hom}_R(N, M) \cong \mathrm{Hom}_R(N_u, M)$. □

Proposition 8. *Let $\{M_i\}$ be a family of u-modules. If R is an SM ring, then $\varinjlim Ext^1_R(R/I, M_i) \cong Ext^1_R(R/I, \varinjlim M_i)$ for any regular u-ideal$I$ of R.*

Proof. Let I be a regular u-ideal of R. From the exact sequence $0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$, we have the following commutative diagram (22) with exact rows:

$$\begin{array}{ccccccc} \varinjlim \mathrm{Hom}_R(R, M_i) & \longrightarrow & \varinjlim \mathrm{Hom}_R(I, M_i) & \longrightarrow & \varinjlim \mathrm{Ext}^1_R(R/I, M_i) & \longrightarrow 0 \\ \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & \\ \mathrm{Hom}_R(R, \varinjlim M_i) & \longrightarrow & \mathrm{Hom}_R(I, \varinjlim M_i) & \longrightarrow & \mathrm{Ext}^1_R(R/I, \varinjlim M_i) & \longrightarrow 0. \end{array} \quad (22)$$

By [10, Lemma 2.1], f_1 is an isomorphism. Since R is SM, we can get that I is u-finitely generated by Proposition 7. Then, there exists a finitely generated subideal I_0 of I such that $I = (I_0)_u$. It is easy to verify that I_0 is regular. Thus, $\varinjlim \mathrm{Hom}_R((I_0)_u, M_i) \cong \varinjlim \mathrm{Hom}_R(I_0, M_i)$ by Lemma 10. $\varinjlim \mathrm{Hom}_R(I_0, M_i) \cong \mathrm{Hom}_R(I_0, \varinjlim M_i)$ by Lemma 9. $\mathrm{Hom}_R(I_0, \varinjlim M_i) \cong \mathrm{Hom}_R((I_0)_u, \varinjlim M_i)$ by Theorem 3 and Lemma 10. Then, $\varinjlim \mathrm{Hom}_R(I, M_i) \cong \mathrm{Hom}_R(I, \varinjlim M_i)$.

Recall that an R-module E is *reg-injective* if $\mathrm{Ext}^1_R(R/I, E) = 0$ for any regular ideal I of R [12, Definition 5.2]. □

Proposition 9. *Let R be a commutative ring and let E be au-module over R. Then, the following statements are equivalent.*

(1) E is a reg-injective R-module.

(2) $\mathrm{Ext}^1_R(R/I, E) = 0$ for any regular u-ideal I of R.

(3) For any regular u-ideal I of R, every homomorphism $f\colon I \longrightarrow E$ can be extended to R.

Proof

(1)⇒(2) This is trivial.

(2)⇒(1) Let I be a regular ideal of R. Then, I_u is a regular ideal of R and (I_u/I) is rGV-torsion. Since E is a u-module, we can get that $\mathrm{Ext}^1_R(I_u/I, E) = 0$ by Proposition 2. From the exact sequence $0 \longrightarrow I_u/I \longrightarrow R/I \longrightarrow R/I_u \longrightarrow 0$, we can get an exact sequence

$$0 = \mathrm{Ext}^1_R(R/I_u, E) \longrightarrow \mathrm{Ext}^1_R(R/I, E) \longrightarrow \mathrm{Ext}^1_R(I_u/I, E) = 0. \quad (23)$$

Thus, $\mathrm{Ext}_R^1(R/I, E) = 0$. Therefore, E is injective.

(2)⇔(3) This is clear.

Recall that a reg-injective R-module E is $\sum$-*reg-injective* if any direct sum of its copies is reg-injective [12, Definition 6.4]. □

Theorem 6. *The following statements are equivalent for a ringR.*

(1) R is an SM ring.

(2) The direct sum of any number of rGV-torsion-free reg-injective R-modules is reg-injective.

(3) The direct sum of any number of GV-torsion-free reg-injective R-modules is reg-injective.

(4) The direct sum of any countable number of rGV-torsion-free reg-injective R-modules is reg-injective.

(5) The direct sum of any countable number of GV-torsion-free reg-injective R-modules is reg-injective.

(6) Every rGV-torsion-free reg-injective R-module is $\sum$-reg-injective.

(7) Every GV-torsion-free reg-injective R-module is $\sum$-reg-injective.

(8) The direct limit of rGV-torsion-free reg-injective R-modules is reg-injective.

(9) The direct limit of GV-torsion-free reg-injective R -modules is reg-injective.

Proof

(1)⇔(2)⇔(4)⇔(6) See [9, Theorem 4.1.10].

(1)⇔(3)⇔(5)⇔(7) See [12, Theorem 6.10].

(1)⇒(8) Let $\{E_i\}$ be a family of rGV-torsion-free reg-injective R-modules. Then, each E_i is a u-module by definition. Since R is an SM ring, we can get that $\mathrm{Ext}_R^1(R/I, \varinjlim E_i) \cong \varinjlim \mathrm{Ext}_R^1(R/I, E_i) = 0$ for any regular u-ideal I of R by Proposition 8. Note that $\varinjlim E_i$ is a u-module by Theorem 3. Then, $\varinjlim E_i$ is reg-injective by Proposition 9.

(8)⇒(9) This follows by the fact that GV-torsion-free R-modules are rGV-torsion-free.

(9)⇒(5) This follows by the fact that a direct sum is a direct limit of finite sums. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

Hwankoo Kim was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2021R1I1A3047469). De Chuan Zhou was supported by the National Natural Science Foundation of China (no. 12101515). Xiao Lei Zhang was supported by the National Natural Science Foundation of China (no. 12061001).

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