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Research Article

Approximating Fixed Points and the Solution of a Nonlinear Fractional Difference Equation via an Iterative Method

Faizan Ahmad Khan

Department of Mathematics, University of Tabuk, Tabuk 71491, Saudi Arabia

Correspondence should be addressed to Faizan Ahmad Khan; fkhan@ut.edu.sa

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The main intent of this article is to innovate a new iterative method to approximate fixed points of contraction and nonexpansive mappings. We prove that the new iterative method is stable for contraction and has a better rate of convergence than some distinctive iterative methods. Furthermore, some convergence results are proved for nonexpansive mappings. Finally, the solution of a nonlinear fractional difference equation is approximated via the proposed iterative method. Some numerical examples are constructed to support the analytical results and to illustrate the efficiency of the proposed iterative method.

1. Introduction

The fixed point theory is an imperative arm of mathematics. It has become not only a field with significant advancement but also a vital tool for solving different kinds of problems in several fields of mathematics. The fixed point theory is a powerful tool because it has a variety of applications in different fields like differential and integral equations, variational inequalities, approximation theory, economics, biological sciences, medical sciences, engineering, optimization theory, fractal theory, game theory, and control theory. Indeed, the strength of fixed point theory lies in its wide range of applications inside and beyond mathematics.

All over this paper, we presume that $\mathbb{Z}^+$ is the set of all non-negative integers, $\mathscr{B}$ a Banach space, $\varnothing \neq \mathscr{D} \subset \mathscr{B}$, and $F(\mathscr{G}) = \{k \in \mathscr{D}: \mathscr{G}: \mathscr{D} \longrightarrow \mathscr{D} \text{ and } \mathscr{G}k = k\}$. A mapping $\mathscr{G}: \mathscr{D} \longrightarrow \mathscr{D}$ is said to be a contraction mapping if $\exists \beta \in [0, 1)$, such that $\forall s, t \in \mathscr{D}$:

$$\|\mathscr{G}s - \mathscr{G}t\| \leq \beta \|s - t\|. \tag{1}$$

If $\beta = 1$, then $\mathscr{G}$ is a nonexpansive mapping on $\mathscr{D}$.

In 1965, an underlying existence result for fixed points of nonexpansive mappings was proved independently by Browder [1], Göhde [2], and Kirk [3]. After that, fixed point theory for nonexpansive and allied classes of mappings has been examined broadly and has provoked a parallel study in Banach space geometry.

On the other hand, it is well known that the Picard iterative method failed to estimate the fixed points of nonexpansive mappings. So, in 1953, Mann [4] introduced a one-step iterative method to estimate the fixed points of nonexpansive mappings. But, it can be effortlessly seen that Mann's iterative method was unsuccessful to estimate the fixed points of pseudo-contractive mappings. Therefore, in 1974, Ishikawa [5] defined a two-step Mann iterative method to estimate the fixed points of such type mappings. The speed of convergence of iterative sequence is important from the practical point of view. The faster iterative method saves time while approximating fixed points of nonlinear mappings. Recently, Ali et al. [6], and Garodia and Uddin [7] introduced new iterative methods to achieve a better rate of convergence. Quite recently, several Man-type and Ishikawa-type iterative methods have been studied by different researchers for the approximation of fixed points of single-valued nonlinear functions, e.g., see [8–10]. The following iterative methods are generated by an initial point $\tau_0 \in \mathscr{D}$, where $\mathscr{G}$ is a self-mapping on $\mathscr{D}$.

The S-iterative method (Agarwal et al. [11]) is as follows:

$$\begin{cases} \tau_{n+1} = (1-\mu_n)\mathcal{G}\tau_n + \mu_n\mathcal{G}\sigma_n, \\ \sigma_n = (1-\theta_n)\tau_n + \theta_n\mathcal{G}\tau_n, \ n \in \mathbb{Z}^+. \end{cases} \tag{2}$$

The Picard-S iterative method (Gursoy and Karakaya [12]) can be given as

$$\begin{cases} \tau_{n+1} = \mathcal{G}\sigma_n, \\ \sigma_n = (1-\mu_n)\mathcal{G}\tau_n + \mu_n\mathcal{G}\xi_n, \\ \xi_n = (1-\theta_n)\tau_n + \theta_n\mathcal{G}\tau_n, \ n \in \mathbb{Z}^+. \end{cases} \tag{3}$$

The Vatan iterative method (Karakaya et al. [13]) can be given as

$$\begin{cases} \tau_{n+1} = \mathcal{G}\left((1-\mu_n)\sigma_n + \mu_n\mathcal{G}\sigma_n\right), \\ \sigma_n = \mathcal{G}\left((1-\theta_n)\tau_n + \theta_n\mathcal{G}\tau_n\right), \ n \in \mathbb{Z}^+. \end{cases} \tag{4}$$

The Thakur-New iterative method (Thakur et al. [14]) can be given as

$$\begin{cases} \tau_{n+1} = \mathcal{G}\sigma_n, \\ \sigma_n = \mathcal{G}\left((1-\mu_n)\tau_n + \mu_n\xi_n\right), \\ \xi_n = (1-\theta_n)\tau_n + \theta_n\mathcal{G}\tau_n, n \in \mathbb{Z}^+. \end{cases} \tag{5}$$

The M^* iterative method (Ullah and Arshad [15]) can be given as

$$\begin{cases} \tau_{n+1} = \mathcal{G}\sigma_n, \\ \sigma_n = \mathcal{G}\left((1-\mu_n)\tau_n + \mu_n\mathcal{G}\xi_n\right), \\ \xi_n = (1-\theta_n)\tau_n + \theta_n\mathcal{G}\tau_n, \ n \in \mathbb{Z}^+. \end{cases} \tag{6}$$

The M iterative method (Ullah and Arshad [16]) can be given as

$$\begin{cases} \tau_{n+1} = \mathcal{G}\sigma_n, \\ \sigma_n = \mathcal{G}\xi_n, \\ \xi_n = (1-\mu_n)\tau_n + \mu_n\mathcal{G}\tau_n, \ n \in \mathbb{Z}^+. \end{cases} \tag{7}$$

Strongly inspired by the above equations, we define the following a novel iterative method:

$$\begin{cases} \tau_0 \in \mathcal{D}, \\ \tau_{n+1} = \mathcal{G}\left((1-\mu_n)\sigma_n + \mu_n\mathcal{G}\sigma_n\right), \\ \sigma_n = \mathcal{G}\left((1-\theta_n)\mathcal{G}\tau_n + \theta_n\xi_n\right), \\ \xi_n = \mathcal{G}\left(\mathcal{G}\tau_n\right), n \in \mathbb{Z}^+, \end{cases} \tag{8}$$

where $\{\mu_n\}$ and $\{\theta_n\}$ are control sequences in (0, 1).

Remark 1. This iterative method (8) is different from all the iterative methods existing in the literature.

The rest of the manuscript is unified as follows: Section 2 contains some definitions and lemmas that will be used in the main findings. In Section 3, the rate of convergence of the proposed iterative method is compared with some known and remarkable iterative methods by analytically and numerically. The stability of the new iterative method is also discussed with respect to contraction. In Section 4, weak and strong convergence theorems are proven for nonexpansive mappings via the newly defined method. In Section 5, the solution of a nonlinear fractional difference equation is estimated via the proposed iterative method. The conclusion of the paper is given in the last section.

2. Preliminaries

For the purpose of convenience, we recall the following concepts and results that will be used in the sequel.

Lemma 1 (see [17]). *Let $\{u_n\}$ and $\{\epsilon_n\}$ be sequences in $\mathbb{R}_+$ that satisfy the following inequality:*

$$u_{n+1} \leq (1-\nu_n)\, u_n + \epsilon_n, \tag{9}$$

where $\nu_n \in (0,1)$, $\forall n \in \mathbb{Z}^+$ with $\sum_{n=0}^{\infty} \nu_n = \infty$. If $\lim_{n\longrightarrow\infty} (\epsilon_n/\nu_n) = 0$, then $\lim_{n\longrightarrow\infty} u_n = 0$.

Definition 1 (see [18]). A Banach space $\mathcal{B}$ satisfies Opial's condition if for each $\tau_n \rightharpoonup p \in \mathcal{B}$, $\lim_{n\longrightarrow\infty} \inf\|\tau_n - p\| < \lim_{n\longrightarrow\infty} \inf\|\tau_n - q\|$ holds true, and $\forall q \in \mathcal{B}$ with $q \neq p$.

Definition 2 (see [19]). Let $\mathcal{B}$ be a Banach space, $\varnothing \neq \mathcal{D} \subset \mathcal{B}$ and $\mathcal{G}\colon \mathcal{D} \longrightarrow \mathcal{D}$ be a function. $\mathcal{G}$ satisfies condition (I) if $\exists$ is an increasing mapping $\xi\colon [0,\infty) \longrightarrow [0,\infty)$ such that $\xi(0) = 0$, $\xi(w) > 0$, $\forall w > 0$, and $d(p, \mathcal{G}p) \geq \xi(d(p, F(\mathcal{G})))$, $\forall p \in \mathcal{D}$.

Lemma 2 (see [20]). *Let $\{\tau_n\}$ and $\{\sigma_n\}$ be sequences in a uniformly convex Banach space $\mathcal{B}$ with $\lim_{n\longrightarrow\infty} \sup\|\tau_n\| \leq c$, $\lim_{n\longrightarrow\infty} \sup\|\sigma_n\| \leq c$,m and $\lim_{n\longrightarrow\infty} \|s_n\tau_n + (1-s_n)\sigma_n\| = c$, for some $c \geq 0$. Then, $\lim_{n\longrightarrow\infty} \|\tau_n - \sigma_n\| = 0$, where $0 < a \leq s_n \leq b < 1$ and $\forall n \geq 1$.*

Lemma 3 (see [21]). *If $\mathcal{G}\colon \mathcal{D} \longrightarrow \mathcal{B}$ is a nonexpansive function, where $\mathcal{B}$ is a uniformly convex Banach space and $\mathcal{D}$ is a convex closed subset of $\mathcal{B}$. Then, $I - \mathcal{G}$ is semiclosed on $\mathcal{B}$, where I stands for identity map on $\mathcal{D}$.*

Definition 3 (see [22]). Let $(\mathcal{B}, \|\cdot\|)$ be a normed space and $\mathcal{G}$ be a self-mapping on $\mathcal{B}$ and let two iterative methods $\{\tau_n\}$ and $\{\sigma_n\}$ converge identical to the point k. Further, assume that the following error estimates are available:

$$\begin{aligned} \|\tau_n - k\| &\leq d_n, \\ \|\sigma_n - k\| &\leq e_n, \end{aligned} \tag{10}$$

where $\{d_n\}$, $\{e_n\} \subset \mathbb{R}_+$ such that $d_n \longrightarrow 0$ and $e_n \longrightarrow 0$ as $n \longrightarrow \infty$.

Definition 4 (see [22]). Let $\{d_n\}$, $\{e_n\} \subset \mathbb{R}_+$ such that $d_n \longrightarrow d$ and $e_n \longrightarrow e$ as $n \longrightarrow \infty$. If

$$\lim_{n\longrightarrow\infty} \frac{|d_n - d|}{|e_n - e|} = 0, \tag{11}$$

then $\{d_n\}$ converges to d faster than $\{e_n\}$ to e.

And if

$$0 < \lim_{n \longrightarrow \infty} \frac{|d_n - d|}{|e_n - e|} < \infty, \tag{12}$$

then $\{d_n\}$ and $\{e_n\}$ have the equal rate of convergence.

Definition 5 (see [23]). Let $\{\psi_n\} \subset \mathscr{D}$ be an approximate sequence of $\{\tau_n\}$. Then, an iterative method is defined as

$$\begin{cases} \tau_0 \in \mathscr{D}, \\ \tau_{n+1} = \xi(\mathscr{G}, \tau_n), \ n \in \mathbb{Z}^+, \end{cases} \tag{13}$$

for some relation ξ, such that $\tau_n \longrightarrow k$ as $n \longrightarrow \infty$, is called $\mathscr{G}$-stable; if $\epsilon_n = \|\psi_{n+1} - \xi(\mathscr{G}, \psi_n)\|$, $n \in \mathbb{Z}^+$, we get $\lim_{n \longrightarrow \infty} \epsilon_n = 0 \Leftrightarrow \lim_{n \longrightarrow \infty} \psi_n = k$.

3. Rate of Convergence and the Stability of the Proposed Method

The motive of the current section is to demonstrate convergence, rate of convergence, and stability results for contraction in an arbitrary Banach space by the proposed iterative method.

Theorem 1. *Let $\mathscr{B}$ be a Banach space and $\mathscr{D}$ a nonempty, convex closed subset of $\mathscr{B}$ and $\mathscr{G}: \mathscr{D} \longrightarrow \mathscr{D}$ be a contraction mapping. Then, the iterative sequence $\{\tau_n\}$ defined in the method (8) converges to one and only one point of $F(\mathscr{G})$. Moreover, the sequence $\{\tau_n\}$ is stable with respect to contraction mapping.*

Proof. Since $\mathscr{G}$ is a contraction mapping, $\beta \in [0, 1)$ and $\forall s \in \mathscr{D}$, and $k \in F(\mathscr{G})$:

$$\|\mathscr{G}s - k\| = \|\mathscr{G}s - \mathscr{G}k\| \leq \beta\|s - \|k. \tag{14}$$

By iterative method (8), we get

$$\|\xi_n - k\| = \|\mathscr{G}(\mathscr{G}\tau_n) - k\| \leq \beta^2\|\tau_n - k\|. \tag{15}$$

Using equation (15), we get

$$\begin{aligned}
\|\sigma_n - k\| &= \|\mathscr{G}((1 - \theta_n)\mathscr{G}\tau_n + \theta_n\xi_n) - \|k \\
&\leq \beta\|(1 - \theta_n)\mathscr{G}\tau_n + \theta_n\xi_n - \|k \leq \beta\big((1 - \theta_n)\|\mathscr{G}\tau_n - k\| + \theta_n\|\xi_n - k\|\big) \\
&\leq \beta^2\big((1 - \theta_n)\|\tau_n - k\| + \theta_n\beta\|\tau_n - k\|\big) \\
&= \beta^2(1 - (1 - \beta)\theta_n)\|\tau_n - k\|.
\end{aligned} \tag{16}$$

Since $0 \leq \beta < 1$ and $\theta_n \in (0, 1)$, so with the fact $0 < (1 - (1 - \beta)\theta_n) < 1$, we get

$$\|\sigma_n - k\| \leq \beta^2\|\xi_n - k\|. \tag{17}$$

Using equation (17), we get

$$\begin{aligned}
\|\tau_{n+1} - k\| &= \|\mathscr{G}((1 - \mu_n)\sigma_n + \mu_n\mathscr{G}\sigma_n) - k\| \\
&\leq \beta\|(1 - \mu_n)\sigma_n + \mu_n\mathscr{G}\sigma_n - k\| \\
&\leq \beta\big((1 - \mu_n)\|\sigma_n - k\| + \mu_n\|\mathscr{G}\sigma_n - k\|\big) \\
&\leq \beta\big((1 - \mu_n)\|\sigma_n - k\| + \beta\mu_n\|\sigma_n - k\|\big) \\
&\leq \beta(1 - (1 - \beta)\mu_n)\|\sigma_n - k\| \\
&\leq \beta^3\|\tau_n - k\|.
\end{aligned} \tag{18}$$

Inductively, we get

$$\|\tau_{n+1} - k\| \leq \beta^{3(n+1)}\|\tau_0 - k\|. \tag{19}$$

Since $0 \leq \beta < 1$, it concludes that $\{\tau_n\}$ converges to k.

Here, we prove the stability of the method (8). Let $\{\psi_n\}$ be an estimate sequence of $\{\tau_n\}$ in $\mathscr{D}$, then the sequence defined by the iterative method (8) is $\tau_{n+1} = \xi(\mathscr{G}, \tau_n)$ and $\epsilon_n = \|\psi_{n+1} - \xi(\mathscr{G}, \psi_n)\|$, $n \in \mathbb{Z}^+$. Now, we show that $\lim_{n \longrightarrow \infty} \epsilon_n = 0 \Leftrightarrow \lim_{n \longrightarrow \infty} \psi_n = k$.

Let $\lim_{n \longrightarrow \infty} \epsilon_n = 0$, then by the method (8), we obtain

$$\begin{aligned}
\|\psi_{n+1} - k\| &\leq \|\psi_{n+1} - \xi(\mathscr{G}, \psi_n)\| + \|\xi(\mathscr{G}, \psi_n) - k\| = \epsilon_n + \|\xi(\mathscr{G}, \psi_n) - k\| \leq \epsilon_n \\
&\quad + \beta^3(1 - (1 - \beta)\mu_n)\|\psi_n - k\|.
\end{aligned} \tag{20}$$

Put $u_n = \|\psi_n - k\|$ and $v_n = (1-\beta)\mu_n \in (0,1)$, then

$$u_{n+1} \le \beta^3 (1 - v_n)u_n + \varepsilon_n. \tag{21}$$

Since $\lim_{n\longrightarrow\infty} \epsilon_n = 0$, $(\epsilon_n/v_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Thus, by Lemma 1, $\lim_{n\longrightarrow\infty} u_n = 0$, that is, $\lim_{n\longrightarrow\infty} \psi_n = k$.

Conversely, assume that $\lim_{n\longrightarrow\infty} \psi_n = k$, then we have

$$\begin{aligned}\epsilon_n &= \left\|\psi_{n+1} - \xi(\mathcal{G},\psi_n)\right\| \\ &\le \left\|\psi_{n+1} - k\right\| + \left\|\xi(\mathcal{G},\psi_n) - k\right\| \\ &\le \left\|\psi_{n+1} - k\right\| + \beta^3 (1 - (1-\beta)\mu_n)\left\|\psi_n - k\right\|.\end{aligned} \tag{22}$$

This shows that $\lim_{n\longrightarrow\infty} \epsilon_n = 0$. Thus, the iterative method (8) is stable with respect to contraction.

The following theorem shows that the iterative method (8) has the better speed of convergence than the methods (2)–(7) for contraction. □

Theorem 2. *Let $\mathcal{B}$ be a Banach space and $\mathcal{D}$ a nonempty, convex closed subset of $\mathcal{B}$, and $\mathcal{G}\colon \mathcal{D} \longrightarrow \mathcal{D}$ be a contraction. Assume that the sequence $\{\tau_{1,n}\}$ is defined by S (2), $\{\tau_{2,n}\}$ by Picard-S (3), $\{\tau_{3,n}\}$ by Vatan (4), $\{\tau_{4,n}\}$ by Thakur-New (5), $\{\tau_{5,n}\}$ by M^* (6), $\{\tau_{6,n}\}$ by M (7) and $\{\tau_n\}$ by (8) iterative method. Then, the iterative method (8) converges to a fixed point k of $\mathcal{G}$ faster than the M^*, S, Vatan, Picard-S, Thakur-New and M iterative methods.*

Proof. In view of equation (19), we obtain

$$\left\|\tau_{n+1} - k\right\| \le \beta^{3(n+1)}\left\|\tau_0 - k\right\| = \alpha_n,\ n \in \mathbb{Z}^+. \tag{23}$$

As proved by Sahu ([24], Theorem 3.6), we get

$$\left\|\tau_{1,n} - k\right\| \le \beta^{(n+1)} (1 - (1-\beta)\mu_n\theta_n)^{n+1}\left\|\tau_{1,0} - k\right\| = \alpha_{1,n},\ n \in \mathbb{Z}^+. \tag{24}$$

From Gursoy and Karakaya's study [12], we get

$$\left\|\tau_{2,n} - k\right\| \le \beta^{2(n+1)} (1 - (1-\beta)\mu_n\theta_n)^{n+1}\left\|\tau_{2,0} - k\right\| = \alpha_{2,n},\ n \in \mathbb{Z}^+. \tag{25}$$

As proved by Ullah and Arshad ([25], Theorem 4), we get

$$\left\|\tau_{3,n} - k\right\| \le \beta^{2(n+1)} (1 - (1-\beta)\mu_n)^{n+1}\left\|\tau_{3,0} - k\right\| = \alpha_{3,n},\ n \in \mathbb{Z}^+. \tag{26}$$

Now, by iterative method (5), we obtain

$$\begin{aligned}\left\|\xi_n - k\right\| &= \left\|(1-\theta_n)\tau_n + \theta_n\mathcal{G}\tau_n - k\right\| \\ &\le (1-\theta_n)\left\|\tau_n - k\right\| + \beta\theta_n\left\|\tau_n - k\right\| \\ &= (1 - \theta_n + \beta\theta_n)\left\|\tau_n - k\right\| \\ &= (1 - (1-\beta)\theta_n)\left\|\tau_n - k\right\|.\end{aligned} \tag{27}$$

Since $\beta \in (0,1]$ and $\theta_n \in (0,1)$, so with the fact $0 < 1 - (1-\beta)\theta_n < 1$, we have

$$\left\|\xi_n - k\right\| \le \left\|\tau_n - k\right\|. \tag{28}$$

By equation (28), we get

$$\begin{aligned}\left\|\sigma_n - k\right\| &= \left\|\mathcal{G}((1-\mu_n)\tau_n + \mu_n\xi_n) - k\right\| \\ &\le \beta\left[(1-\mu_n)\tau_n + \left\|\mu_n\xi_n - k\right\|\right] \\ &\le \beta\left[(1-\mu_n)\left\|\tau_n - k\right\| + \mu_n\left\|\xi_n - k\right\|\right] \\ &\le \beta\left[(1-\mu_n)\left\|\tau_n - k\right\| + \mu_n\left\|\tau_n - k\right\|\right] \\ &= \beta\left\|\tau_n - k\right\|.\end{aligned} \tag{29}$$

Thus, using the equation (29), we obtain that

$$\left\|\tau_{n+1} - k\right\| = \left\|\mathcal{G}\sigma_n - k\right\| \le \beta\left\|\sigma_n - k\right\| \le \beta^2\left\|\tau_n - k\right\|. \tag{30}$$

Inductively, we get

$$\left\|\tau_{n+1} - k\right\| \le \beta^{2(n+1)}\left\|\tau_0 - k\right\|. \tag{31}$$

Let

$$\left\|\tau_{4,n} - k\right\| \le \beta^{2(n+1)}\left\|\tau_{4,0} - k\right\| = \alpha_{4,n},\ n \in \mathbb{Z}^+. \tag{32}$$

By iterative method (6) and using equation (28), we get

$$\left\|\xi_n - k\right\| \le \left\|\tau_n - k\right\|. \tag{33}$$

Now, using equation (33), we get

$$\begin{aligned}\left\|\sigma_n - k\right\| &= \left\|\mathcal{G}((1-\mu_n)\tau_n + \mu_n\mathcal{G}\xi_n) - k\right\| \\ &\le \beta\left[(1-\mu_n)\tau_n + \left\|\mu_n\mathcal{G}\xi_n - k\right\|\right] \\ &\le \beta\left[(1-\mu_n)\left\|\tau_n - k\right\| + \beta\mu_n\left\|\xi_n - k\right\|\right] \\ &\le \beta\left[(1-\mu_n)\left\|\tau_n - k\right\| + \beta\mu_n\left\|\tau_n - k\right\|\right] \\ &= \beta[1 - (1-\beta)\mu_n]\left\|\tau_n - k\right\| \le \beta\left\|\tau_n - k\right\|.\end{aligned} \tag{34}$$

Thus, using equation (34), we obtain that

$$\left\|\tau_{n+1} - k\right\| = \left\|\mathcal{G}\sigma_n - k\right\| \le \beta\left\|\sigma_n - k\right\| \le \beta^2\left\|\tau_n - k\right\|. \tag{35}$$

Inductively, we get

$$\left\|\tau_{n+1} - k\right\| \le \beta^{2(n+1)}\left\|\tau_0 - k\right\|. \tag{36}$$

Let

$$\left\|\tau_{5,n} - k\right\| \le \beta^{2(n+1)}\left\|\tau_{5,0} - k\right\| = \alpha_{5,n},\ n \in \mathbb{Z}^+. \tag{37}$$

By iterative method (7) and using equation (28), we get

$$\left\|\xi_n - k\right\| \le \left\|\tau_n - k\right\|. \tag{38}$$

Now, using equation (38), we have

$$\begin{aligned}\left\|\sigma_n - k\right\| &= \left\|\mathcal{G}\xi_n - k\right\| \le \beta\left\|\xi_n - k\right\| \\ &\le \beta\left\|\tau_n - k\right\|.\end{aligned} \tag{39}$$

By equation (39), we obtain

$$\begin{aligned}\left\|\tau_{n+1} - k\right\| &= \left\|\mathcal{G}\sigma_n - k\right\| \le \beta\left\|\sigma_n - k\right\| \\ &\le \beta^2\left\|\tau_n - k\right\|.\end{aligned} \tag{40}$$

Inductively, we obtain

$$\left\|\tau_{n+1} - k\right\| \le \beta^{2(n+1)}\left\|\tau_0 - k\right\|. \tag{41}$$

Let

$$\|\tau_{6,n} - k\| \le \beta^{2(n+1)}\|\tau_{6,0} - k\| = \alpha_{6,n},\ n \in \mathbb{Z}^+. \tag{42}$$

Now,

$$\begin{aligned}\frac{\alpha_n}{\alpha_{1,n}} &= \frac{\beta^{3(n+1)}\|\tau_0 - k\|}{[\beta(1-(1-\beta)\mu_n\theta_n)]^{n+1}\|\tau_{1,0} - k\|} \\ &= \beta^{n+1}\left(\frac{\beta}{1-(1-\beta)\mu_n\theta_n}\right)^{n+1}\frac{\|\tau_0 - k\|}{\|\tau_{1,0} - k\|}.\end{aligned} \tag{43}$$

Since $\beta < 1$ and $(\beta/1-(1-\beta)\mu_n\theta_n) < 1$, we have $(\alpha_n/\alpha_{1,n}) \longrightarrow 0$ as $n \longrightarrow \infty$. Thus, $\{\tau_n\}$ has a better speed of convergence than $\{\tau_{1,n}\}$ and

$$\begin{aligned}\frac{\alpha_n}{\alpha_{2,n}} &= \frac{\beta^{3(n+1)}\|\tau_0 - k\|}{\beta^{2(n+1)}[(1-(1-\beta)\mu_n\theta_n)]^{n+1}\|\tau_{2,0} - k\|}, \\ &= \left(\frac{\beta}{1-(1-\beta)\mu_n\theta_n}\right)^{n+1}\frac{\|\tau_0 - k\|}{\|\tau_{2,0} - k\|}\end{aligned} \tag{44}$$

implies $(\alpha_n/\alpha_{2,n}) \longrightarrow 0$ as $n \longrightarrow \infty$. Thus, $\{\tau_n\}$ has better speed of convergence than $\{\tau_{2,n}\}$.

Next,

$$\begin{aligned}\frac{\alpha_n}{\alpha_{3,n}} &= \frac{\beta^{3(n+1)}\|\tau_0 - k\|}{\beta^{2(n+1)}([1-(1-\beta)\mu_n])^{n+1}\|\tau_{3,0} - k\|} \\ &= \left(\frac{\beta}{1-(1-\beta)\mu_n}\right)^{n+1}\frac{\|\tau_0 - k\|}{\|\tau_{3,0} - k\|}.\end{aligned} \tag{45}$$

Thus, $(\alpha_n/\alpha_{3,n}) \longrightarrow 0$ as $n \longrightarrow \infty$. Hence, $\{\tau_n\}$ has better speed of convergence than $\{\tau_{3,n}\}$.

Furthermore,

$$\begin{aligned}\frac{\alpha_n}{\alpha_{4,n}} &= \frac{\beta^{3(n+1)}\|\tau_0 - k\|}{\beta^{2(n+1)}\|\tau_{4,0} - k\|} \\ &= \beta^{n+1}\frac{\|\tau_0 - k\|}{\|\tau_{4,0} - k\|} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \\ \frac{\alpha_n}{\alpha_{5,n}} &= \frac{\beta^{3(n+1)}\|\tau_0 - k\|}{\beta^{2(n+1)}\|\tau_{5,0} - k\|} \\ &= \beta^{n+1}\frac{\|\tau_0 - k\|}{\|\tau_{5,0} - k\|} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{46}$$

And

$$\begin{aligned}\frac{\alpha_n}{\alpha_{6,n}} &= \frac{\beta^{3(n+1)}\|\tau_0 - k\|}{\beta^{2(n+1)}\|\tau_{6,0} - k\|} \\ &= \beta^{n+1}\frac{\|\tau_0 - t\|}{\|\tau_{6,0} - k\|} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{47}$$

Thus, the sequence $\{\tau_n\}$ has better speed of convergence than the sequences $\{\tau_{4,n}\}$, $\{\tau_{5,n}\}$, and $\{\tau_{6,n}\}$.

The following example supports the above result. □

Example 1. Take $\mathscr{B} = \mathbb{R}$ and $\mathscr{D} = [1, 100] \subset \mathbb{R}$. Let $\mathscr{G}: \mathscr{D} \longrightarrow \mathscr{D}$ be given by $\mathscr{G}s = \sqrt{s^2 - s + 1}$, for all $s \in \mathscr{D}$. Then, $\mathscr{G}$ is a contraction mapping and admits an unique fixed point $k = 1$. Here, we choose the control sequences $\mu_n = 0.25$ and $\theta_n = 0.35$, for all $n \in \mathbb{Z}^+$ with the initial guess $\tau_0 = 5$.

The proposed method (8) has better speed of convergence than S, Picard-S, Vatan, Thakur-New, M^*, and M iterative methods with control sequences $\mu_n = 0.25$, $\theta_n = 0.35$, $n \in \mathbb{Z}^+$, and the initial point $\tau_0 = 5$ (Tables 1 and 2 and Figure 1). With the same inputs, we compare the CPU time for distinct iterative methods (Table 2).

3.1. Observations. In the present article, we compare the speed of convergence of different iterative methods only in number of iterations. In Table 1, we observe that proposed method (8) converges to a fixed point $k = 1$ in 10 iterations and other methods, as shown in Figure 1, and converges more than 10 iterations. Thus, method (8) converges faster than methods (2)–(7).

We also compare the convergence behavior of the proposed iterative method for different initial points. We noticed that the rate of convergence also depends on the initial points, and one can easily see it in Table 3 and Figure 2.

4. Convergence Theorems

First, we demonstrate the following fruitful lemmas that help us to obtain the sequel.

Lemma 4. *Let $\mathscr{B}$ be a uniformly convex Banach space, $\mathscr{D}(\neq \emptyset)$ be a convex closed subset of $\mathscr{B}$, and let $\mathscr{G}: \mathscr{D} \longrightarrow \mathscr{D}$ be a nonexpansive mapping. If $\{\tau_n\}$ is defined by (8), then $\lim_{n\longrightarrow\infty} \|\tau_n - k\|$ exists when $\forall k \in F(\mathscr{G})$.*

Proof. As $\mathscr{G}$ is a nonexpansive mapping, so we get

$$\|\mathscr{G}\tau_n - k\| \le \|\tau_n - k\|. \tag{48}$$

Table 1: A comparison table for the speed of convergence of iterative methods.

Iter.	New	S	Picard-S	Vatan	Thakur-New	M^*	M
1	5.000000	5.000000	5.000000	5.000000	5.000000	5.000000	5.000000
2	3.410995	4.567208	4.158387	4.013700	4.158384	4.058541	4.073272
3	2.064573	4.143397	3.358521	3.090238	3.358514	3.172745	3.199868
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
10	1.000000	1.449346	1.000866	1.000098	1.000866	1.000196	1.000234
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
15	1.000000	1.046473	1.000003	1.000000	1.000003	1.000000	1.000001
16	1.000000	1.023580	1.000001	1.000000	1.000001	1.000000	1.000000
17	1.000000	1.011771	1.000000	1.000000	1.000000	1.000000	1.000000
18	1.000000	1.005826	1.000000	1.000000	1.000000	1.000000	1.000000
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
32	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000

Table 2: A comparison table for the CPU time for convergence of iterative methods.

Iterative methods	New (s)	S (s)	Picard-S (s)	Vatan (s)	Thakur-New (s)	M^* (s)	M (s)
CPU time (in seconds)	0.015	0.028	0.018	0.016	0.018	0.018	0.016

$\forall \tau_n \in \mathcal{D}$ and $\forall k \in F(\mathcal{G})$. By the iterative method (8), we obtain

$$\|\xi_n - k\| = \|\mathcal{G}(\mathcal{G}\tau_n) - k\| \le \|\tau_n - k\|. \tag{49}$$

Using equation (49), we have

$$\begin{aligned}\|\sigma_n - k\| &= \|\mathcal{G}((1-\theta_n)\mathcal{G}\tau_n + \theta_n\xi_n) - k\| \le \|(1-\theta_n)\mathcal{G}\tau_n + \theta_n\xi_n - k\| \\ &\le (1-\theta_n)\|\mathcal{G}\tau_n - k\| + \theta_n\|\xi_n - k\| \\ &\le (1-\theta_n)\|\tau_n - k\| + \theta_n\|\xi_n - k\| \\ &\le \|\tau_n - k\|.\end{aligned} \tag{50}$$

Using equation (50), we have

$$\begin{aligned}\|\tau_{n+1} - k\| &= \|\mathcal{G}((1-\mu_n)\sigma_n + \mu_n\mathcal{G}\sigma_n) - k\| \le \|(1-\mu_n)\sigma_n + \mu_n\mathcal{G}\sigma_n - k\| \\ &\le (1-\mu_n)\|\sigma_n - k\| + \mu_n\|\mathcal{G}\sigma_n - k\| \\ &\le (1-\mu_n)\|\sigma_n - k\| + \mu_n\|\sigma_n - k\| \\ &\le (1-\mu_n)\|\tau_n - k\| + \mu_n\|\tau_n - k\| \\ &= \|\tau_n - k\|.\end{aligned} \tag{51}$$

Hence, $\{\|\tau_n - k\|\}$ is nonincreasing $\forall k \in F(\mathcal{G})$. Hence, $\lim_{n\longrightarrow\infty}\|\tau_n - k\|$ exists. □

Lemma 5. *Let all the assumptions of Lemma 4 be true. If $\{\tau_n\}$ is defined in the method (8), then $F(\mathcal{G}) \neq \varnothing$ if and only if $\{\tau_n\}$ is bounded and $\lim_{n\longrightarrow\infty}\|\tau_n - \mathcal{G}\tau_n\| = 0$.*

Proof. Presume that $F(\mathcal{G}) \neq \varnothing$ and $k \in F(\mathcal{G})$. Then, $\lim_{n\longrightarrow\infty}\|\tau_n - k\|$ exists by Lemma 4 and $\{\tau_n\}$ is bounded. Presume that

$$\lim_{n\longrightarrow\infty}\|\tau_n - k\| = \alpha. \tag{52}$$

By equation (50) and (52), we obtain

$$\lim_{n\longrightarrow\infty}\sup\|\sigma_n - k\| \le \lim_{n\longrightarrow\infty}\sup\|\tau_n - k\| \le \alpha. \tag{53}$$

As $\mathcal{G}$ is a nonexpansive mapping, we get

$$\begin{aligned}\|\mathcal{G}\sigma_n - k\| &= \|\mathcal{G}\sigma_n - \mathcal{G}k\| \le \|\sigma_n - k\| \\ \lim_{n\longrightarrow\infty}\sup\|\mathcal{G}\sigma_n - k\| &\le \lim_{n\longrightarrow\infty}\sup\|\sigma_n - k\| \le \alpha.\end{aligned} \tag{54}$$

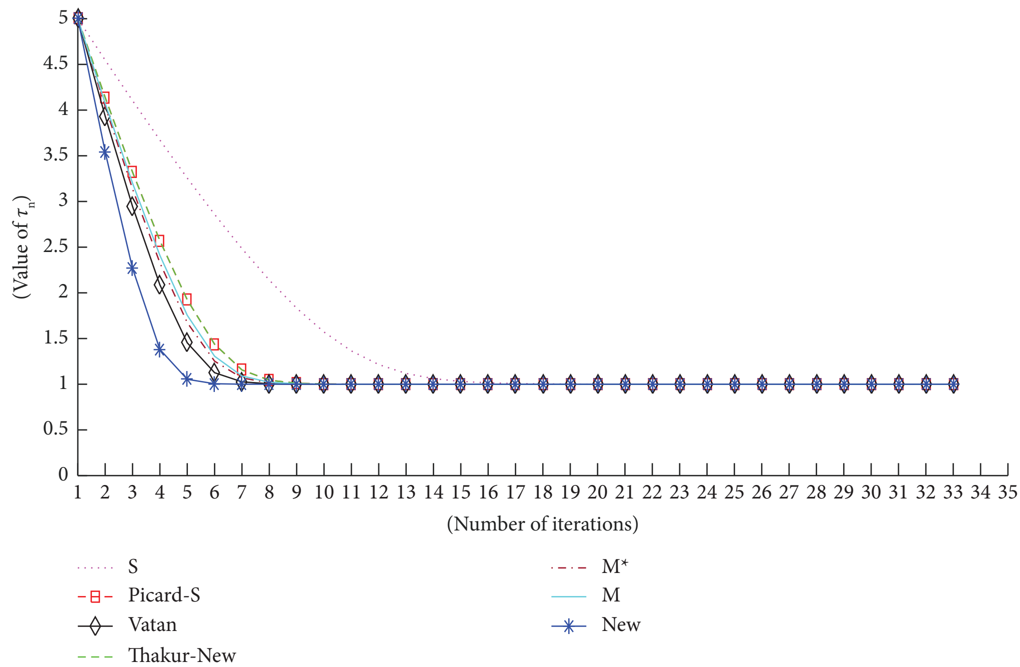

FIGURE 1: Comparison by graph.

TABLE 3: A comparison table for the speed of convergence of iterative method (8) for different initial points.

Iter.	$\tau_0 = 0.5$	$\tau_0 = 1.5$	$\tau_0 = 2.0$	$\tau_0 = 4.0$	$\tau_0 = 15$	$\tau_0 = 25$
1	0.500000	1.500000	2.000000	4.000000	15.000000	25.000000
2	0.981067	1.069972	1.217159	2.537170	13.132886	23.087463
3	0.998508	1.006144	1.022586	1.474128	11.282689	21.180400
⋮	⋮	⋮	⋮	⋮	⋮	⋮
7	1.000000	1.000000	1.000001	1.000037	4.254545	13.633143
8	1.000000	1.000000	1.000000	1.000003	2.752868	11.777916
9	1.000000	1.000000	1.000000	1.000000	1.603111	9.943726
⋮	⋮	⋮	⋮	⋮	⋮	⋮
15	1.000000	1.000000	1.000000	1.000000	1.000000	1.166649
⋮	⋮	⋮	⋮	⋮	⋮	⋮
20	1.000000	1.000000	1.000000	1.000000	1.000000	1.000001
21	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000

Now,

$$\begin{aligned}\alpha = \lim_{n\longrightarrow\infty}\|\tau_{n+1} - k\| &= \lim_{n\longrightarrow\infty}\|\mathcal{G}((1-\mu_n)\sigma_n + \mu_n\mathcal{G}\sigma_n) - k\| \\ &\leq \lim_{n\longrightarrow\infty}\|(1-\mu_n)\sigma_n + \mu_n\mathcal{G}\sigma_n - k\| \\ &= \lim_{n\longrightarrow\infty}\|(1-\mu_n)(\sigma_n - k) + \mu_n(\mathcal{G}\sigma_n - k)\| \\ &\leq \lim_{n\longrightarrow\infty}\left((1-\mu_n)\|\sigma_n - k\| + \mu_n\|\mathcal{G}\sigma_n - k\|\right) \\ &\leq \lim_{n\longrightarrow\infty}\left((1-\mu_n)\|\tau_n - k\| + \mu_n\|\tau_n - k\|\right) \leq \alpha.\end{aligned} \tag{55}$$

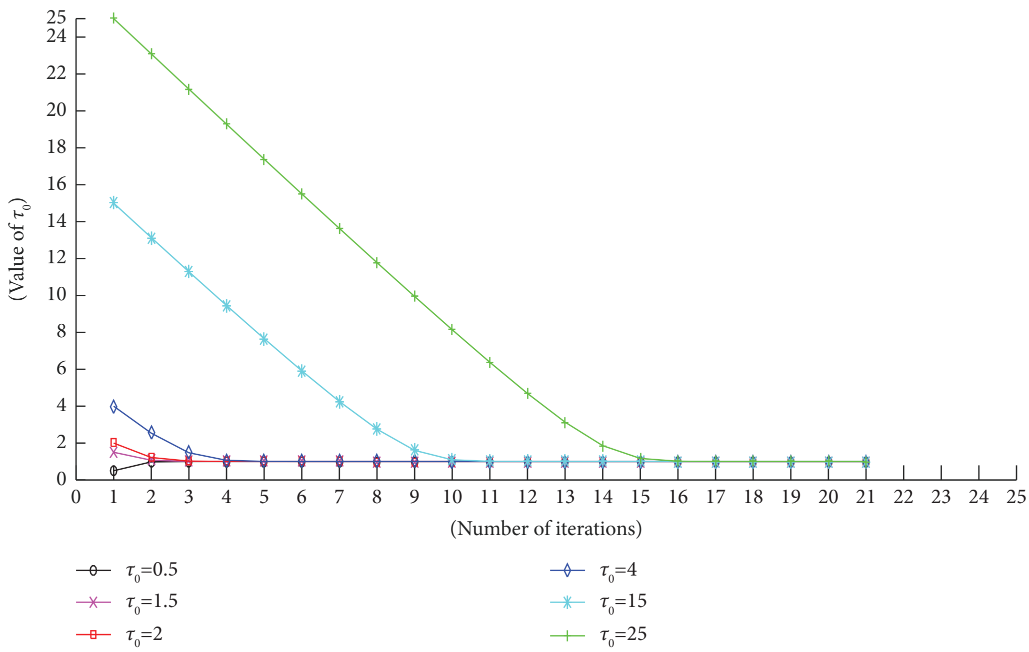

Figure 2: Comparison by the graph.

Thus,

$$\lim_{n\longrightarrow\infty}\left\|\left(1-\mu_n\right)\left(\sigma_n-k\right)+\mu_n\left(\mathcal{G}\sigma_n-k\right)\right\|=\alpha. \quad (56)$$

By (53)–(56) and applying Lemma 2, we get

$$\lim_{n\longrightarrow\infty}\left\|\sigma_n-\mathcal{G}\sigma_n\right\|=0. \quad (57)$$

Now, by using equation (57), we obtain

$$\begin{aligned}\left\|\tau_{n+1}-\mathcal{G}\tau_{n+1}\right\| &= \left\|\mathcal{G}\left((1-\mu_n)\sigma_n+\mu_n\mathcal{G}\sigma_n\right)-\mathcal{G}\tau_{n+1}\right\| \le \left\|(1-\mu_n)\sigma_n+\mu_n\mathcal{G}\sigma_n-\tau_{n+1}\right\| \\ &= \left\|(1-\mu_n)\sigma_n-(1-\mu_n)\mathcal{G}\sigma_n+\mathcal{G}\sigma_n-\tau_{n+1}\right\| \\ &\le (1-\mu_n)\left\|\sigma_n-\mathcal{G}\sigma_n\right\|+\left\|\mathcal{G}\sigma_n-\tau_{n+1}\right\| \\ &= (1-\mu_n)\left\|\sigma_n-\mathcal{G}\sigma_n\right\|+\left\|\mathcal{G}\sigma_n-\mathcal{G}\left((1-\mu_n)\sigma_n+\mu_n\mathcal{G}\sigma_n\right)\right\| \\ &\le (1-\mu_n)\left\|\sigma_n-\mathcal{G}\sigma_n\right\|+\left\|\sigma_n-(1-\mu_n)\sigma_n-\mu_n\mathcal{G}\sigma_n\right\| \\ &= (1-\mu_n)\left\|\sigma_n-\mathcal{G}\sigma_n\right\|+\mu_n\left\|\sigma_n-\mathcal{G}\sigma_n\right\| \\ &= \left\|\sigma_n-\mathcal{G}\sigma_n\right\|.\end{aligned} \quad (58)$$

Thus,

$$\lim_{n\longrightarrow\infty}\left\|\tau_n-\mathcal{G}\tau_n\right\|=0. \quad (59)$$

Conversely, presume that $\{\tau_n\}$ is bounded and $\lim_{n\longrightarrow\infty}\|\tau_n-\mathcal{G}\tau_n\|=0$. Let $k\in A(\mathcal{D},\{\tau_n\})$, then we have

$$\begin{aligned} r(\mathcal{G}k,\{\tau_n\}) &= \lim_{n\longrightarrow\infty}\sup\left\|\tau_n-\mathcal{G}k\right\| \\ &\le \lim_{n\longrightarrow\infty}\sup\left(\left\|\tau_n-\mathcal{G}\tau_n\right\|+\left\|\mathcal{G}\tau_n-\mathcal{G}k\right\|\right) \\ &\le \lim_{n\longrightarrow\infty}\sup\left\|\tau_n-k\right\| \\ &= r(k,\{\tau_n\}) \\ &= r(\mathcal{D},\{\tau_n\}).\end{aligned} \quad (60)$$

This implies that $\mathcal{G}k\in A(\mathcal{D},\{\tau_n\})$. Thus, we know that $A(\mathcal{D},\{\tau_n\})$ consists of only one element because $\mathcal{B}$ is uniformly convex; hence, $\mathcal{G}k=k$. □

Theorem 3. *Let all the assumptions of Lemma 4 be true. Presume that $\mathcal{B}$ contents Opial's property, then the sequence $\{\tau_n\}$ given in (8) converges to a point of $F(\mathcal{G})$ weakly.*

Proof. Assume $\{\tau_{n_j}\}$ and $\{\tau_{n_k}\}$ are two subsequences of $\{\tau_n\}$ such that $\tau_{n_j}\rightharpoonup k$ and $\tau_{n_j}\rightharpoonup q$ as $j\longrightarrow\infty$ ($\rightharpoonup$, stands for weakly convergent), where $k,q\in F(\mathcal{G})$. In view of Lemma 4, $\lim_{n\longrightarrow\infty}\|\tau_n-k\|$ exists and by Lemma 5, $\lim_{n\longrightarrow\infty}\|\tau_n-\mathcal{G}\tau_n\|=0$. In view of Lemma 3, $(I-\mathcal{G})k=0$, i.e., $k=\mathcal{G}k$, in the same way $q=\mathcal{G}q$.

Now, we prove that $k = q$. If $k \neq q$, then using Opial's condition, we get

$$\begin{aligned}\lim_{n\longrightarrow\infty}\|\tau_n - k\| &= \lim_{n_j\longrightarrow\infty}\|\tau_{n_j} - k\| \\ &< \lim_{n_j\longrightarrow\infty}\|\tau_{n_j} - q\| \\ &= \lim_{n\longrightarrow\infty}\|\tau_n - q\| \\ &= \lim_{n_k\longrightarrow\infty}\|\tau_{n_k} - q\| \\ &< \lim_{n_k\longrightarrow\infty}\|\tau_{n_k} - k\| \\ &= \lim_{n\longrightarrow\infty}\|\tau_n - k\|.\end{aligned} \tag{61}$$

A contradiction implies that $k = q$. Thus, $\tau_n \rightharpoonup k$ and $\forall k \in F(\mathscr{G})$. □

Theorem 4. *Let all the assumptions of Lemma 4 be true. Then, $\{\tau_n\}$ defined by (8) goes to an fixed element of $\mathscr{G}$ if and only if $\lim_{n\longrightarrow\infty}\inf d(\tau_n, F(\mathscr{G})) = 0$, where $d(\tau_n, F(\mathscr{G})) = \inf\{\|\tau_n - k\|: k \in F(\mathscr{G})\}$.*

Proof. One can prove the first part easily. For reverse, presume $\lim_{n\longrightarrow\infty}\inf d(\tau_n, F(\mathscr{G})) = 0$. In view of Result 4.1, $\lim_{n\longrightarrow\infty}\|\tau_n - k\|$ exists, $\forall k \in F(\mathscr{G})$ and $\lim_{n\longrightarrow\infty} d(\tau_n, F(\mathscr{G})) = 0$ is given.

We now prove that $\{\tau_n\}$ converges to an element of $\mathscr{D}$. Since $\lim_{n\longrightarrow\infty} d(\tau_n, F(\mathscr{G})) = 0$, for $\beta > 0$, $\exists P \in \mathbb{N}$ with $\forall n \geq P$.

$$\begin{aligned} d(\tau_n, F(\mathscr{G})) &< \frac{\beta}{2} \\ \Rightarrow \inf\{\|\tau_n - k\|: k \in F(\mathscr{G})\} &< \frac{\beta}{2}. \end{aligned} \tag{62}$$

Specifically, $\inf\{\|\tau_P - k\|: k \in F(\mathscr{G})\} < (\beta/2)$. So $\exists k \in F(\mathscr{G})$ with $\|\tau_P - k\| \langle (\beta/2)$. Now, for $m, n \geq P$, we have

$$\|\tau_{n+m} - \tau_n\| \leq \|\tau_{n+m} - k\| + \|\tau_n - k\| \leq \|\tau_P - k\| + \|\tau_P - k\| = 2\|\tau_P - k\| < \beta. \tag{63}$$

This shows that the sequence $\{\tau_n\}$ is Cauchy in $\mathscr{D}$, so that $\exists$an element $r \in \mathscr{D}$ and $\lim_{n\longrightarrow\infty}\tau_n = r$. Now, $\lim_{n\longrightarrow\infty} d(\tau_n, F(\mathscr{G})) = 0 \Rightarrow d(r, F(\mathscr{G})) = 0$, and thus, $r \in F(\mathscr{G})$. □

Theorem 5. *Let all the assumptions of Lemma 4 be true and $\mathscr{G}$ contents condition (I). Then, $\{\tau_n\}$ defined by (8) converges strongly to a point of $F(\mathscr{G})$.*

Proof. Using Lemma 5 and condition (I), we obtain

$$\begin{aligned} 0 \leq \lim_{n\longrightarrow\infty}\xi(d(\tau_n, F(\mathscr{G}))) &\leq \lim_{n\longrightarrow\infty}\|\tau_n - \mathscr{G}\tau_n\| = 0 \\ \Rightarrow \lim_{n\longrightarrow\infty}\xi(d(\tau_n, F(\mathscr{G}))) &= 0. \end{aligned} \tag{64}$$

This implies that

$$\lim_{n\longrightarrow\infty}(d(\tau_n, F(\mathscr{G}))) = 0. \tag{65}$$

Thus, the result is followed by Theorem 4. □

Example 2. Let $\mathscr{B} = \mathbb{R}^2$ with the norm $\|(s,t)\| = |s| + |t|$, $\forall (s,t) \in \mathscr{B}$, and $\mathscr{D} = \{(s,t): (s,t) \in [0,1] \times [0,1]\} \subset \mathscr{B}$. A function $\mathscr{G}: \mathscr{D} \longrightarrow \mathscr{D}$ is given as

$$\mathscr{G}(s,t) = (\cos(s), \cos(t)), \forall (s,t) \in \mathscr{D}. \tag{66}$$

Then, $\mathscr{G}$ is a nonexpansive mapping and has a fixed point $(k,q) = (0.739085, 0.739085)$, whether $\mathscr{G}$ is not a contraction.

The proposed method (8) goes to the fixed point $(k,q) = (0.739085, 0.739085)$ of the function $\mathscr{G}$ better than S, Picard-S, Vatan, Thakur-new, M^*, and M iterative methods with the control sequences $\mu_n = 0.85$, $\theta_n = 0.35$, $n \in \mathbb{Z}^+$, and initial point (0.5, 0.8) (Tables 4–6 and Figure 3). With the same inputs, we compare the CPU time for the conbergence of distinct iterative methods (Table 6).

5. Solution of a Nonlinear Fractional Difference Equation

The main intent of the present section is to approximate the solution of a fractional difference equation via an iterative method (8). Consider the following:

$$\begin{cases} D^\delta y(z) = F(z + \delta - 1, y(z + \delta - 1)), & z \in I_{1-\delta}, 0 < \delta \leq 1, \\ y(0) = y_0, \end{cases} \tag{67}$$

where D^δ indicates the Caputo-like discrete fractional difference of order δ, $F: [0,\infty) \times \mathscr{B} \longrightarrow \mathscr{B}$ is a continuous function, $I_{1-\delta} = \{1-\delta, 2-\delta, \ldots\}$ and $\mathscr{B} = C(I_{1-\delta})$ is a real Banach space with the norm

$$\|y\|_\infty = \max_{z\in I_{1-\delta}}|y(z)|, \forall y \in \mathscr{B}. \tag{68}$$

It is shown in [26] that $y(z)$ is a solution of the initial value problem (IVP) (67) if and only if $y(z)$ is a solution of the following relation:

Table 4: A comparison table for the rate of convergence of iterative methods for Example 2.

Iter.	New	S	Picard-S
1	(0.500000, 0.800000)	(0.500000, 0.800000)	(0.500000, 0.800000)
2	(0.728448, 0.742483)	(0.817381, 0.718359)	(0.684134, 0.752887)
3	(0.738520, 0.739267)	(0.712404, 0.746080)	(0.726516, 0.742226)
⋮	⋮	⋮	⋮
6	(0.739085, 0.739085)	(0.740115,0.738814)	(0.738936, 0.739122)
⋮	⋮	⋮	⋮
10	(0.739085, 0.739085)	(0.739099, 0.739082)	(0.739085, 0.739085)
11	(0.739085, 0.739085)	(0.739081, 0.739086)	(0.739085, 0.739085)
12	(0.739085, 0.739085)	(0.739087, 0.739085)	(0.739085, 0.739085)
13	(0.739085, 0.739085)	(0.739085, 0.739085)	(0.739085, 0.739085)

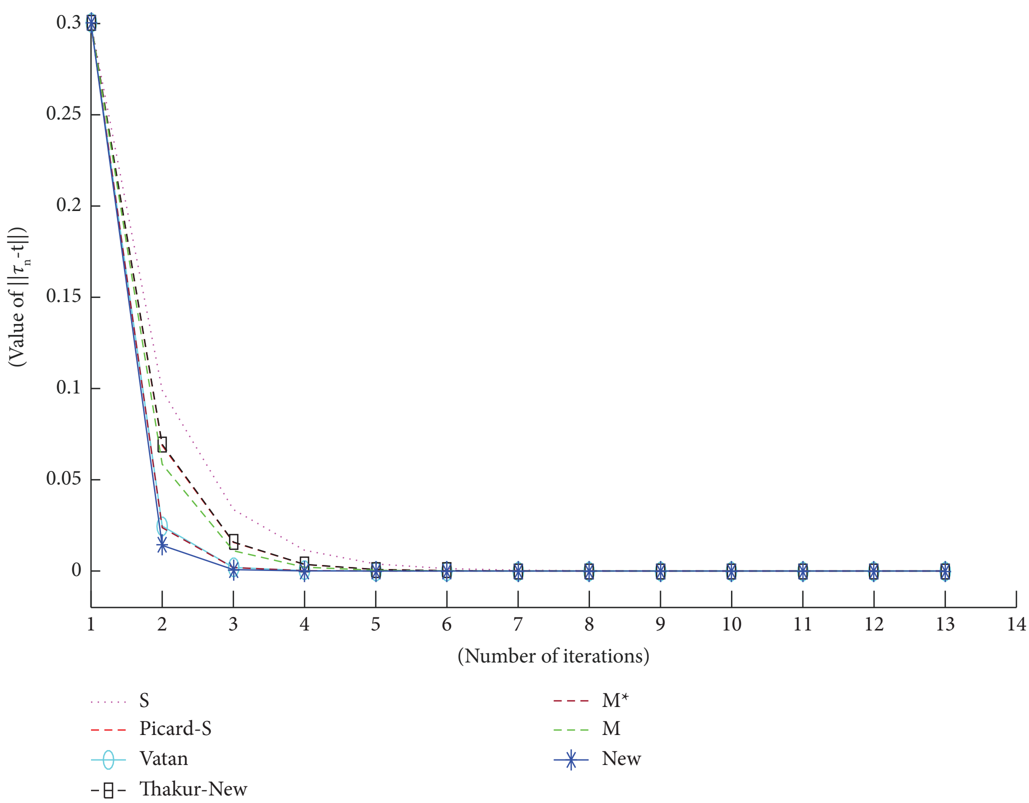

Figure 3: Comparison by the graph.

Table 5: A comparison table for the rate of convergence of iterative methods for Example 2.

Iter.	Vatan	Thakur-New	M^*	M
1	(0.500000, 0.800000)	(0.500000, 0.800000)	(0.500000, 0.800000)	(0.500000, 0.800000)
2	(0.758961, 0.734316)	(0.683455, 0.752848)	(0.758026, 0.734260)	(0.691844, 0.750366)
3	(0.737514, 0.739464)	(0.726326, 0.742215)	(0.737582, 0.739468)	(0.730107, 0.741200)
4	(0.739210, 0.739055)	(0.736173, 0.739798)	(0.739205, 0.739055)	(0.737394, 0.739482)
5	(0.739075, 0.739088)	(0.738422, 0.739248)	(0.739076, 0.739088)	(0.738767, 0.739160)
6	(0.739086, 0.739085)	(0.738934, 0.739122)	(0.739086, 0.739085)	(0.739025, 0.739099)
7	(0.739085, 0.739085)	(0.739051, 0.739094)	(0.739085, 0.739085)	(0.739074, 0.739088)
8	(0.739085, 0.739085)	(0.739077, 0.739087)	(0.739085, 0.739085)	(0.739083, 0.739086)
9	(0.739085, 0.739085)	(0.739083, 0.739086)	(0.739085, 0.739085)	(0.739085, 0.739085)
10	(0.739085, 0.739085)	(0.739085, 0.739085)	(0.739085, 0.739085)	(0.739085, 0.739085)

Table 6: A comparison table for the CPU time for convergence of iterative methods.

Iterative methods	New (s)	S (s)	Picard-S (s)	Vatan (s)	Thakur-New (s)	M^* (s)	M (s)
CPU time (in seconds)	0.017	0.017	0.023	0.018	0.022	0.022	0.018

$$\begin{cases} y(z) = y_0 + \dfrac{1}{\Gamma(\delta)} \displaystyle\sum_{b=1-\delta}^{z-\delta} (z-b-1)^{(\delta-1)} F(b+\delta-1, y(b+\delta-1)), \ 0<\delta\leq 1, \\ y(0) = y_0. \end{cases} \tag{69}$$

The following lemma plays a key part to demonstrate the main finding of this segment.

Lemma 6 (see [26]). *We have*

$$\sum_{b=1-\delta}^{z-\delta} (z-b-1)^{(\delta-1)} = \frac{\Gamma(z+\delta)}{\delta\Gamma(z)}. \tag{70}$$

Now, set $I_k = \{1,2,3,\ldots,k\}$ *and* $\mathscr{B} = C(I_k)$ *is a real Banach space, where* $k \in \mathbb{N}$*. Presume that the following assumptions are true.*

(C_1)*: Assume* F: $[0,k]\times\mathscr{B} \longrightarrow \mathscr{B}$ *is a locally Lipschitz continuous function with constant* L*, i.e.,*

$$\|F(z,y) - F(z,w)\|_\infty \leq L\|y-w\|_\infty. \tag{71}$$

(C_2)*:*

$$\frac{L\Gamma(k+\delta)}{\Gamma(\delta+1)\Gamma(k)} < 1. \tag{72}$$

The existence and uniqueness of a solution of the IVP (67) can be found in [26].

Presently, we are going to demonstrate the main result of the present section.

Theorem 6. *Presume that the conditions* (C_1) *and* (C_2) *are satisfied. Let an operator* $\mathscr{G}$: $\mathscr{B} \longrightarrow \mathscr{B}$ *be given by*

$$\mathscr{G}y(z) = y_0 + \frac{1}{\Gamma(\delta)} \sum_{b=1-\delta}^{z-\delta} (z-b-1)^{(\delta-1)} F(b+\delta-1, y(b+\delta-1)), \tag{73}$$

for $z \in I_k$*. Then, the sequence* $\{\tau_n\}$ *developed by iterative method (8) converges to a unique solution of IVP (67).*

Proof. It is enough to show the operator $\mathscr{G}$ is a contraction mapping on $\mathscr{B}$. For any $y, w \in \mathscr{B}$, we obtain that

$$\begin{aligned} \|\mathscr{G}y(z) - \mathscr{G}w(z)\|_\infty &\leq \frac{1}{\Gamma(\delta)} \sum_{b=1-\delta}^{z-\delta} (z-b-1)^{(\delta-1)} \|F(b+\delta-1, y(b+\delta-1)) - F(b+\delta-1, w(b+\delta-1))\|_\infty \\ &\leq \frac{1}{\Gamma(\delta)} \sum_{b=1-\delta}^{z-\delta} (z-b-1)^{(\delta-1)} \|y-w\|_\infty \\ &\leq \frac{L\Gamma(z+\delta)}{\delta\Gamma(\delta)\Gamma(z)} \|y-w\|_\infty \\ &\leq \frac{L\Gamma(k+\delta)}{\Gamma(\delta+1)\Gamma(k)} \|y-w\|_\infty. \end{aligned} \tag{74}$$

Thus, $\mathscr{G}$ is a contraction on $\mathscr{B}$. Hence, by applying Theorem 1, the sequence $\{\tau_n\}$ converges to a unique solution of IVP (67). □

6. Conclusion

The main intent of this article was to propose a novel and effective iterative method for the estimation of fixed points of contraction and nonexpansive mappings in the frame work of Banach space. It is also shown that the new iterative method is stable with respect to contraction mapping. The rate of convergence of the distinct iterative methods has been discussed. Therefore, the newly introduced iterative method is more efficient and effective than the previously defined iterative methods. The researchers may apply the new iterative method to approximate the solution of nonlinear problems to achieve a better rate of convergence. Besides, the results of the present paper generalize and amplify the relevant results in the literature. Thus, it can be concluded that the work done in the paper is new and useful in the area of nonlinear analysis. The fixed point iterative methods are very useful to estimate the solution of nonlinear differential equations, nonlinear fractional differential equations, nonlinear integrodifferential equations, etc. The interested researchers may apply the proposed iterative method to estimate the solution of above discussed problems.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares no conflicts of interest.

References

[1] F. E. Browder, "Non-expansive nonlinear operators in a Banach space," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 54, pp. 1041–1044, 1965.

[2] D. Göhde, "Zum Prinzip der kontraktiven Abbildung, (German)," *Mathematische Nachrichten*, vol. 30, pp. 251–258, 1965.

[3] W. A. Kirk, "A fixed point theorem for mappings which do not increase distances," *The American Mathematical Monthly*, vol. 72, no. 9, pp. 1004–1006, 1965.

[4] W. R. Mann, "Mean value methods in iteration," *Proceedings of the American Mathematical Society*, vol. 4, no. 3, pp. 506–510, 1953.

[5] S. Ishikawa, "Fixed points by a new iteration method," *Proceedings of the American Mathematical Society*, vol. 44, no. 1, pp. 147–150, 1974.

[6] F. Ali, J. Ali, and J. J. Nieto, "Some observations on generalized non-expansive mappings with an application," *Computational and Applied Mathematics*, vol. 39, no. 2, p. 74, 2020.

[7] C. Garodia and I. Uddin, "A new iterative method for solving split feasibility problem," *Journal of Applied Analysis & Computation*, vol. 10, no. 3, pp. 986–1004, 2020.

[8] M. G. Alshehri, F. A. Khan, and F. Ali, "An iterative algorithm to approximate fixed points of non-linear operators with an application," *Mathematics*, vol. 10, no. 7, p. 1132, 2022.

[9] W. Sintunavarat and A. Pitea, "On a new iteration scheme for numerical reckoning fixed points of Berinde mappings with convergence analysis," *The Journal of Nonlinear Science and Applications*, vol. 9, no. 05, pp. 2553–2562, 2016.

[10] S. Hassan, M. De la Sen, P. Agarwal, Q. Ali, and A. Hussain, "A new faster iterative scheme for numerical fixed points estimation of Suzuki's generalized nonexpansive mappings," *Mathematical Problems in Engineering*, vol. 2020, Article ID 3863819, 9 pages, 2020.

[11] R. P. Agrawal, D. O'Regan, and D. R. Sahu, "Iterative construction of fixed points of nearly asymptotically non-expansive mappings," *Journal of Nonlinear and Convex Analysis*, vol. 8, no. 1, pp. 61–79, 2007.

[12] F. Gursoy and V. Karakaya, "A Picard-S hybrid type iteration method for solving a differential equation with retarded argument," 2014, https://arxiv.org/abs/1403.2546.

[13] V. Karakaya, N. E. H. Bouzara, K. Dogan, and Y. Atalan, "On different results for a new two-step iteration method under weak contraction mapping in Banach spaces," 2015, https://arxiv.org/abs/1507.00200.

[14] B. S. Thakur, D. Thakur, and M. Postolache, "A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized nonexpansive mappings," *Applied Mathematics and Computation*, vol. 275, pp. 147–155, 2016.

[15] K. Ullah and M. Arshad, "New iteration process and numerical reckoning fixed points in Banach space," *UPB Scientific Bulletin, Series A: Applied Mathematics and Physics*, vol. 79, no. 4, pp. 113–122, 2017.

[16] K. Ullah and M. Arshad, "Numerical reckoning fixed points for Suzuki's generalized nonexpansive mappings via new iteration process," *Filomat*, vol. 32, no. 1, pp. 187–196, 2018.

[17] X. Weng, "Fixed point iteration for local strictly pseudocontractive mapping," *Proceedings of the American Mathematical Society*, vol. 113, no. 3, pp. 727–731, 1991.

[18] Z. Opial, "Weak convergence of the sequence of successive approximations for non-expansive mappings," *Bulletin of the American Mathematical Society*, vol. 73, no. 4, pp. 591–597, 1967.

[19] H. F. Senter and W. G. Dotson, "Approximating fixed points of non-expansive mappings," *Proceedings of the American Mathematical Society*, vol. 44, no. 2, pp. 375–380, 1974.

[20] J. Schu, "Weak and strong convergence to fixed points of asymptotically non-expansive mappings," *Bulletin of the Australian Mathematical Society*, vol. 43, no. 1, pp. 153–159, 1991.

[21] F. E. Browder, "Semicontractive and semiaccretive nonlinear mappings in Banach spaces," *Bulletin of the American Mathematical Society*, vol. 74, no. 4, pp. 660–665, 1968.

[22] V. Berinde, "Picard iteration converges faster than Mann iteration for a class of quasi-contractive operators," *Fixed Point Theory and Applications*, vol. 2004, pp. 97–105, 2004.

[23] A. M. Ostrowski, "The round-off stability of iterations," *Zeitschrift für Angewandte Mathematik und Mechanik*, vol. 47, no. 2, pp. 77–81, 1967.

[24] D. R. Sahu, "Applications of the S-iteration process to constrained minimization problems and split feasibility problems," *Fixed Point Theory*, vol. 12, pp. 187–204, 2011.

[25] K. Ullah and M. Arshad, "On different results for new three step iteration process in Banach spaces," *Springer Plus*, vol. 5, no. 1, p. 1616, 2016.

[26] A. Chen, Y. Chen, and Y. Zhou, "Existence of solutions to anti-periodic boundary value problem for nonlinear fractional differential equations with impulses," *Advances in Difference Equations*, vol. 2011, Article ID 915689, 2017 pages, 2011.