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Research Article

Extension of *m*-Symmetric Hilbert Space Operators

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We introduce a new class of operators, which we will call the class of P-quasi-m-symmetric operators that includes m-symmetric operators and k-quasi m-symmetric operators. Some basic structural properties of this class of operators are established based on the operator matrix representation associated with such operators.

1. Introduction

Throughout this paper, $\mathscr{H}$ stands for a complex Hilbert space of infinite-dimension with inner product $\langle\cdot|\cdot\rangle$. By $\mathscr{B}(\mathscr{H})$, we denote the Banach algebra of all bounded linear operators on $\mathscr{H}$. For every $A \in \mathscr{B}(\mathscr{H})$, we denote by $\ker(A)$ and $\mathscr{R}(A)$ the null space and the range of A, respectively. Also, $\sigma_p(A), \sigma_{ap}(A), \sigma(T)$, and $\sigma_s(A)$ denote the point spectrum, the approximate spectrum, the spectrum, and the surjective spectrum of A.

The authors in[1] introduced the class of Helton operators as followsan operator $A \in \mathscr{B}(\mathscr{H})$ is said to be in the m th Helton class of B if

$$\sum_{0\leq j\leq m} (-1)^{m-j}\binom{m}{j}B^j A^{m-j} = 0. \tag{1}$$

We refer the interested reader to [1–6] for complete details.

The class of m-symmetric operators or m-self-adjoint operators on a Hilbert $\mathscr{H}$ space has attracted much attention and has been the subject of intensive studies by several authors. m-symmetric operators were introduced in [3, 4, 7] as followsan Hilbert space operator $A \in \mathscr{B}(\mathscr{H})$ is said to be an m-symmetric if A satisfies the following identity:

$$\sum_{0\leq j\leq m} (-1)^{j}\binom{m}{j}A^{*j} A^{m-j} = 0, \tag{2}$$

for some positive integer m, where A^* is the adjoint operator of A. For $m = 1$, equation (2) is reduced to $A^* - A = 0$ (A is symmetric or self-adjoint). If $m = 2$, equation (2) is reduced to $A^{*2} - 2A^*A + A^2 = 0$ (A is 2-symmetric). It has been proven that a power of m-symmetric transformation is again m-symmetric, and the product of two m-symmetric transformations is also m-symmetric under suitable conditions (see [8]).

Another extension of the relation in (2), we recall that if $T, A \in \mathscr{B}\mathscr{H})$, for which A is positive, T is called an (A, m) -symmetric ([9]) when

$$\sum_{0\leq j\leq m} (-1)^{j}\binom{m}{j}T^{*j} A T^{m-j} = 0. \tag{3}$$

The authors Chō and Sid Ahmed in [10] generalized the concept of those operators on a Hilbert space. They introduced the (A, m)-symmetric commuting tuple of operators.

Using the identity (2), the authors in [11] have introduced the concept of k-quasi-m-symmetric operators as followsan operator $A \in \mathscr{B}(\mathscr{H})$ is said to be k-quasi-m -symmetric operator if A satisfies the following identity:

$$A^{*k}\left(\sum_{0\leq j\leq m} (-1)^{j}\binom{m}{j}A^{*j} A^{m-j}\right)A^k = 0, \tag{4}$$

for some positive integers k and m. Obviously, every m -symmetric operator is k-quasi-m-symmetric operator.

Many algebraic and spectral properties for k-quasi-m-symmetric operators has been studied by the authors in [11] parallel to those obtained for the classes of n-quasi-m-isometric operators and its variants studied intensively by many authors in the papers [12–16]. Recall from [17] that an operator $A \in \mathscr{B}(\mathscr{H})$ is said to have the single-valued extension property (or SVEP) if for every open subset W of $\mathbb{C}$ and any $\mathscr{H}$-valued analytic function f on W such that $(A-\lambda)f(\lambda) \equiv 0$ on W, we have $f(\lambda) \equiv 0$ on W. $A \in \mathscr{B}(\mathscr{H})$ admits Bishop's property (β) if, for every open subset $\mathbb{D}$ of $\mathbb{C}$ and every sequence $g_n\colon \mathbb{D} \longrightarrow \mathscr{H}$ of analytic functions with $(T-z)g_n(z)$ converges uniformly to 0 in norm on compact subsets of $\mathbb{D}$, $g_n(z)$ converges uniformly to 0 in norm on compact subsets of $\mathbb{D}$.

For $A \in \mathscr{B}(\mathscr{H})$, we set $\Theta_k(A) := \sum_{0\le j\le k}(-1)^j \binom{k}{j} A^{*j}A^{k-j}$. Note that

$$\Theta_{k+1}(A) = A^*\Theta_k(A) - \Theta_k(A)A. \tag{5}$$

Hence, if A is m-symmetric operator, then A is p-symmetric operator for $p \ge m$.

The aim of this paper is to introduce the class of P-quasi-m-symmetric operators, where P is a nonconstant complex polynomial and m is a positive constant. This class of operators seems a natural generalization of k-quasi-m-symmetric operators. We show that many results for m-symmetric and k-quasi m-symmetric operators remain true for our new class.

2. Main Results

In this section, we study the concept of polynomial-quasi m-symmetric operators.

Definition 1. An operator $A \in \mathscr{B}(\mathscr{H})$ is said to be P-quasi-m-symmetric for some nonconstant polynomial P, if there exists a nonconstant complex polynomial P such that

$$P(A)^*\left(\sum_{0\le k\le m}(-1)^k\binom{m}{k}A^{*k}A^{m-k}\right)P(A) = 0, \tag{6}$$

for some positive integer m.

Remark 1

(1) If $P(z) = z^n$ for some positive integer n, then A is said to be n-quasi-m-symmetric (see [11])

(2) If $P(z) = z$, then A is said to be quasisymmetric

Example 1. Consider $A = \begin{pmatrix} I_{\mathscr{H}} & I_{\mathscr{H}} \\ 0 & 0 \end{pmatrix} \in \mathscr{B}(\mathscr{H}\oplus\mathscr{H})$. It is obvious that $T = I_{\mathscr{H}}$ is P-quasi-m-symmetric operator and $P(S) = 0$ with $S = 0$ for all complex polynomial $P = \sum_{0\le k\le m} a_k z^k$. By observing that $A^k = A$ for all $k \ge 1$, it follows from direct calculation that

$$\begin{aligned}
P(A) &= \sum_{0\le k\le m} a_k A^k \\
&= a_0 I_{\mathscr{H}\oplus\mathscr{H}} + \sum_{1\le k\le m} a_k A \\
&= \begin{pmatrix} \sum_{0\le k\le m} a_k I_{\mathscr{H}} & \sum_{1\le k\le m} a_k I_{\mathscr{H}} \\ 0 & a_0 I_{\mathscr{H}} \end{pmatrix}, \\
\sum_{0\le k\le m}(-1)^k\binom{m}{k}A^{*k}A^{m-k} &= A^m + \sum_{1\le k\le m}(-1)^k\binom{m}{k}A^*A \\
&= A - A^*A \\
&= \begin{pmatrix} I_{\mathscr{H}} & I_{\mathscr{H}} \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} I_{\mathscr{H}} & I_{\mathscr{H}} \\ I_{\mathscr{H}} & I_{\mathscr{H}} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ -I_{\mathscr{H}} & -I_{\mathscr{H}} \end{pmatrix}.
\end{aligned} \tag{7}$$

Moreover,

$$\begin{aligned}
&P(A)^*\left(\sum_{0\le k\le m}(-1)^k\binom{m}{k}A^{*k}A^{m-k}\right)P(A) \\
&= \begin{pmatrix} \sum_{0\le k\le m} \overline{a_k} I_{\mathscr{H}} & 0 \\ \sum_{1\le k\le m} \overline{a_k} I_{\mathscr{H}} & \overline{a_0} I_{\mathscr{H}} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ -I_{\mathscr{H}} & -I_{\mathscr{H}} \end{pmatrix}\begin{pmatrix} \sum_{0\le k\le m} a_k I_{\mathscr{H}} & \sum_{1\le k\le m} a_k I_{\mathscr{H}} \\ 0 & a_0 I_{\mathscr{H}} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ -\overline{a_0} I_{\mathscr{H}} & -\overline{a_0} I_{\mathscr{H}} \end{pmatrix}\begin{pmatrix} \sum_{0\le k\le m} a_k I_{\mathscr{H}} & \sum_{1\le k\le m} a_k I_{\mathscr{H}} \\ 0 & a_0 I_{\mathscr{H}} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ -\overline{a}_0 \sum_{0\le k\le m} a_k I_{\mathscr{H}} & -\overline{a_0} \sum_{1\le k\le m} a_k I_{\mathscr{H}} - |a_0|^2 I_{\mathscr{H}} \end{pmatrix}.
\end{aligned} \tag{8}$$

We deduce that A is a P-quasi-m-symmetric operator for all polynomial P satisfying $P(0) = 0$.

Theorem 1. *Let P be a nonconstant complex polynomial and let $A \in \mathscr{B}(\mathscr{H})$. Assume that $\overline{P(A)(\mathscr{H})}$ is not dense. Then the following statements are equivalent:*

(1) A is P-quasi-m-symmetric operator

(2) $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$ on $\mathscr{H} = \overline{P(A)(\mathscr{H})} \oplus \ker(P(A)^*)$, where $A_1 = A|_{\overline{P(A)(\mathscr{H})}}$ is m-symmetric and $P(A_3) = 0$ (i.e.; A_3 is algebraic operator). Furthermore, $\sigma(A) = \sigma(A_1) \cup \sigma(A_3)$.

Proof

(1)⇒(2). Consider the matrix representation of A with respect to the decomposition $\mathscr{H} = \overline{P(A)(\mathscr{H})} \oplus \ker(P(A)^*) A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$. Let Q be the orthogonal projection onto $\overline{P(A)(\mathscr{H})}$, then we have $\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} = AQ = QAQ$. Since A is P-quasi-m-symmetric operator, we have

$$Q\left(\sum_{0\le k\le m} (-1)^k \begin{pmatrix} m \\ k \end{pmatrix} A^{*k} A^{m-k}\right) Q = 0. \tag{9}$$

That is,

$$\sum_{0\le k\le m} (-1)^k \begin{pmatrix} m \\ k \end{pmatrix} A_1^{*k} A_1^{m-k} = 0. \tag{10}$$

Therefore, A_1 is an m-symmetric operator.

Let $w = w_1 + w_2 \in \overline{P(A)(\mathscr{H})} \oplus \ker(P(A)^*)$, we have

$$\begin{aligned} \langle P(A_3)w_2 | w_2\rangle &= \langle P(A)(I-Q)w | (I-Q)w\rangle \\ &= \langle (I-Q)w | P(A)^*(I-Q)w\rangle = 0. \end{aligned} \tag{11}$$

So that $P(A_3) = 0$.

The proof of the statement $\sigma(A) = \sigma(A_1) \cup \sigma(A_3)$ is similar to the one given in [18], Lemma 3.9, so we omit it.

(2)⇒(1) Assume that $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$ on $\mathscr{H} = \overline{\mathscr{R}(P(A))} \oplus \ker(P(A)^*)$ where $A_1 = A|_{\overline{\mathscr{R}(P(A))}}$ is m-symmetry and $P(A_3) = 0$. A simple computation shows that

$$P(A) = \begin{pmatrix} P(A_1) & Z \\ 0 & 0 \end{pmatrix} \text{and}\, P(A)P(A)^* = \begin{pmatrix} P(A_1)P(A_1)^* + ZZ^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} F & 0 \\ 0 & 0 \end{pmatrix}, \tag{12}$$

where $F = F^* = P(A_1)P(A_1)^* + ZZ^*$.

Moreover,

$$\begin{aligned} &P(A)P(A)^*\left(\sum_{0\le k\le m} (-1)^k \begin{pmatrix} m \\ k \end{pmatrix} A^{*k} A^{m-k}\right) P(A)P(A)^* \\ &= \begin{pmatrix} F\left(\sum_{0\le k\le m} (-1)^k \begin{pmatrix} m \\ k \end{pmatrix} A_1^{*k} A_1^{m-k}\right) F & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ &= 0. \end{aligned} \tag{13}$$

Hence, $P(A)^*\left(\sum_{0\le k\le m} (-1)^k \begin{pmatrix} m \\ k \end{pmatrix} A^{*k} A^{m-k}\right) P(A) = 0$, and therefore A is P-quasi-m-symmetric. □

In the following corollary, we show the relationship between P-quasi-m-symmetric and P-quasi-$(m+1)$-symmetric operators.

Corollary 1. *Let P be a complex polynomial and let $A \in \mathscr{B}(\mathscr{H})$. If A is P-quasi-m-symmetric operator then A is P-quasi-q-symmetric operator for all positive integer $q \ge m$.*

Proof. Since A is P-quasi-m-symmetric operator, two cases can be distinguished.

(1) If $\overline{\mathscr{R}(P)} = \mathscr{H}$, then A is m-symmetric operator, and hence, A is q-symmetric operator for all $q \ge m$ by [8].

(2) If $\overline{\mathscr{R}(P)} \ne \mathscr{H}$, taking into account Theorem 1, we can write $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$ on $\mathscr{H} = \overline{P(A)(\mathscr{H})} \oplus \ker(P(A)^*)$, where $A_1 = A|_{\overline{P(A)(\mathscr{H})}}$ is an m-symmetric and $P(A_3) = 0$ (i.e., A_3 is algebraic operator). From [8], A_1 is a q-symmetric operator for all $q \ge m$. Consequently, the desired results follow from the statement (2) of Theorem 1. □

Corollary 2. *Let P be a complex polynomial, and let $A \in \mathscr{B}(\mathscr{H})$ be a P-quasi-m-symmetric operator such that $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$ on $\mathscr{H} = \overline{\mathscr{R}(P(A))} \oplus \mathscr{N}(P(A)^*)$. If $\sigma(A_1) \cap \sigma(A_3) = \varnothing$, then A is similar to a direct sum of a m-symmetric operator and an algebraic operator.*

Proof. By Theorem 1, we write the matrix representation of A on $\mathscr{H} = \overline{\mathscr{R}(P(A))} \oplus \mathscr{N}(P(A)^*)$ as follows $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$, where $A_1 = A_{|\overline{\mathscr{R}(P(A))}}$ is a m-symmetric operator and

$P(A_3)=0$. Since $\sigma(A_1)\cap\sigma(A_3)=\varnothing$, then there exists an operator C such that $A_1C-CA_3=A_2$ by [19]. Hence,

$$A=\begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} = \begin{pmatrix} I & C \\ 0 & I \end{pmatrix}^{-1}\begin{pmatrix} A_1 & 0 \\ 0 & A_3 \end{pmatrix}\begin{pmatrix} I & C \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & C \\ 0 & I \end{pmatrix}^{-1}(A_1\oplus A_2)\begin{pmatrix} I & C \\ 0 & I \end{pmatrix}. \tag{14}$$

Consequently, the desired result follows from Theorem 1. □

Theorem 2. *Let P be a nonconstant complex polynomial and A be a P-quasi-m-symmetric operator, and* **M** *is an closed invariant subspace for A . Then, the restriction $A_{|M}$ is also a P -quasi-m-symmetric operator.*

Proof. Let us consider the following matrix representation of A:

$$A=\begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} \text{on } \mathscr{H}=\mathbf{M}\oplus\mathbf{M}^{\perp}. \tag{15}$$

Since A is P-quasi m-symmetric, we have

$$\begin{aligned} 0 &= P\begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}^{*n}\left(\sum_{0\le k\le m}\binom{m}{k}\begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}^{*k}\begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}^{m-k}\right)P\begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix} \\ &= \begin{pmatrix} P(A_1)^* & 0 \\ X^* & P(A_3)^* \end{pmatrix}\begin{pmatrix} \sum_{0\le k\le m}(-1)^k\binom{m}{k}A_1^{*k}A_1^{m-k} & Z \\ Y & W \end{pmatrix}\begin{pmatrix} P(A_1) & X \\ 0 & P(A_3) \end{pmatrix} \\ &= \begin{pmatrix} P(A_1)^*\sum_{0\le k\le m}(-1)^k\binom{m}{k}A_1^{*k}A_1^{m-k}P(A_1) & ** \\ ** & ** \end{pmatrix}. \end{aligned} \tag{16}$$

Therefore,

$$P(A_1)^*\left(\sum_{0\le k\le m}\binom{m}{k}A_1^{*k}A_1^{m-k}\right)P(A_1)=0. \tag{17}$$

Thus, A_1 is an P-quasi m-symmetric operator. □

Proposition 1. *Let P be a nonconstant complex polynomial and let $A\in\mathscr{B}(\mathscr{H})$. Suppose that A is a P-quasi-m-symmetric operator. If $\mathscr{R}(P(A))$ is dense. Then, A is an m-symmetric operator.*

Proof. We have $P(A)^*\left(\sum_{0\le k\le m}(-1)^k\binom{m}{k}A^{*k}A^{m-k}\right)P(A)=0$, and therefore,

$$\sum_{0\le k\le m}(-1)^k\binom{m}{k}A^{*k}A^{m-k}=0 \text{ on } \overline{\mathscr{R}(P(A))}. \tag{18}$$

□

Proposition 2. *Let P be a nonconstant complex polynomial and $A\in\mathscr{B}(\mathscr{H})$. If A is a P-quasi-m-symmetric operator, then A^n is also a P-quasi-m-symmetric operator for any positive integer n.*

Proof. If $P(A)$ has a dense range, then A is an m-symmetric operator and so is A^n for any positive integer n (see [20], Theorem 2.4).

If $\overline{\mathscr{R}(P(A))}\neq\mathscr{H}$, taking into account Theorem 1, we can use the matrix representation as $A=\begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$on $\mathscr{H}=\overline{\mathscr{R}(P(A))}\oplus\mathscr{N}(P(A)^*)$, where A_1 is an m-symmetric operator and A_3 is algebraic operator. As

$$A^n=\begin{pmatrix} A_1^n & \sum_{0\le j\le n-1}A_1^jA_2A_3^{n-1-j} \\ 0 & A_3^n \end{pmatrix}. \tag{19}$$

Since A_1^n is m-symmetric and A_3^n is algebraic, it follows that A^n is P-quasi-m-symmetric operator. □

Theorem 3. *Let P be a complex polynomial and consider $A=\begin{pmatrix} T & R \\ 0 & S \end{pmatrix}\in\mathscr{B}(\mathscr{H}\oplus\mathscr{H})$, such that T is P-quasi-m-symmetric operator and $P(S)=0$. If $\sigma_s(T)\cap\sigma_a(S)=\varnothing$, then A is similar to a P-quasi- m-symmetric operator.*

Proof. Since $\sigma_s(T)\cap\sigma_a(S)=\varnothing$, it follows from [17] (Theorem 3.5.1) that there exist some operator $B\in\mathscr{B}(\mathscr{H})$ for which $TB-BS=R$. Since

$$\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}\begin{pmatrix} T & R \\ 0 & S \end{pmatrix}=\begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix}\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}. \tag{20}$$

Therefore, A is similar to $D = \begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix}$.

Since T is P-quasi-m-symmetric operator and $P(S) = 0$, it follows

$$\begin{aligned} &P(D)^*\left(\sum_{0\le k\le m}(-1)^k\binom{m}{k}D^{*k}D^{m-k}\right)P(D) \\ &= \begin{pmatrix} P(T)^* & 0 \\ 0 & P(S)^* \end{pmatrix}\left\{\begin{pmatrix} \sum_{0\le k\le m}(-1)^k\binom{m}{k}T^{*k}T^{m-k} & 0 \\ 0 & \sum_{0\le k\le m}(-1)^k\binom{m}{k}S^{*k}S^{m-k} \end{pmatrix}\right. \begin{pmatrix} P(T) & 0 \\ 0 & P(S) \end{pmatrix} \\ &= \begin{pmatrix} P(T)^*\left(\sum_{0\le k\le m}(-1)^k\binom{m}{k}T^{*k}T^{m-k}\right)P(T) & 0 \\ 0 & 0 \end{pmatrix} \\ &= 0. \end{aligned} \tag{21}$$

Consequently, A is similar to an P-quasi-m-symmetric operator. □

Question 1. Let $T, R, S \in \mathcal{B}(\mathcal{H})$. If T is P-quasi m-symmetric operator and $P(S) = 0$ for some complex polynomial, then the operator matrix $A = \begin{pmatrix} T & R \\ 0 & S \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ is P-quasi-m-symmetric operator.

Theorem 4 (see [21], Theorem 2.5). *Let $\mathcal{H}$ and $\mathcal{K}$ be infinite complex Hilbert spaces and let $A \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ the operator matrix of the form $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$. Assume that A_3 has Bishop's property (β). Then, the following assertions are equivalent:*

(i) A has Bishop's property (β)

(ii) A_1 has Bishop's property (β)

Theorem 5. *Let P be a nonconstant complex polynomial and $A \in \mathcal{B}(\mathcal{H})$. If A is P-quasi-3-symmetric operator, then A has Bishop's property (β).*

Proof. If $\mathcal{R}(P(A)$ is dense, then A is 3-symmetric operator, and therefore, A has Bishop's property (β) by [11]. If $\mathcal{R}(P(A)$ is not dense, we have by Theorem 1 the matrix representation $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$, where $A_1 = A_{|\mathcal{R}(P(A))}$ is a 3-symmetric operator and A_3 is algebraic operator.

Since every algebraic operator has Bishop's property (β) and A_1 has Bishop´s property (β) from [11], the desired result follows from Theorem 4. □

Corollary 3. *Let P be a nonconstant complex polynomial and $A \in \mathcal{B}(\mathcal{H})$. If A is P-quasi-3-symmetric operator, then A has SVEP.*

Definition 2. Let P be a nonconstant complex polynomial and $A \in \mathcal{B}(\mathcal{H}]$. We say that A is a strict P-quasi m-symmetric operator if A is P-quasi m-symmetric but A is not P-quasi $(m-1)$-symmetric operator.

The idea of the proof of the following theorem is inspired from [22].

Theorem 6. *Let P be a complex polynomial and $A \in \mathcal{B}(\mathcal{H})$. Assume that A is a strict P-quasi-m-symmetric operator, then the family of operators*

$$\{P(A)^*\Theta_k(A)P(A), k = 0, 1, \cdots, m-1, m \ge 2.\}, \tag{22}$$

is linearly independent.

Proof. Since A is P-quasi m-symmetric operator, we have

$$P(A)^*\Theta_m(A)P(A) = 0. \tag{23}$$

Note that from (5), we have

$$\Theta_k(A) = A^*\Theta_{k-1}(A) - \Theta_{k-1}(A)A \text{ for all } k \ge 1. \tag{24}$$

Now assume that for some complex numbers α_k,

$$\sum_{0\le k\le m-1}\alpha_k P(A)^*\Theta_k(A)P(A) = 0. \tag{25}$$

Multiplying the (25) on the left by A^*, we get

$$\sum_{0\le k\le m-1}\alpha_k P(A)^*A^*\Theta_k(A))P(A) = 0. \tag{26}$$

Similarly, multiplying (25) on the right by A, we get

$$\sum_{0\le k\le m-1}\alpha_k P(A)^*\Theta_k(A))AP(A) = 0. \tag{27}$$

Subtracting (26) and (27), we obtain

$$\sum_{0\le k\le m-1}\alpha_k P(A)^*\left(A^*\Theta_k(A) - \Theta_k(A)A\right)P(A) = 0. \tag{28}$$

If we use (5), we see that

$$\sum_{0\leq k\leq m-1} \alpha_k P(A)^* \theta_{k+1}(A)P(A) = 0. \quad (29)$$

The same procedure applied to (29) gives

$$\sum_{0\leq k\leq m-1} \alpha_k P(A)\Theta_{k+2}(A)P(A) = 0. \quad (30)$$

Applying the above process we obtain,

$$\sum_{0\leq k\leq m-1} \alpha_k P(A)\Theta_{k+r}(A)P(A) = 0 \text{ for all } r \in \mathbb{N}. \quad (31)$$

From Corollary 1, it is well known that if A is P-quasi m-symmetric operator, then A is P-quasi-q-symmetric operator for all $q \geq m$. This implies the following implications:

For $r = m-1$,

$$\sum_{0\leq k\leq m-1} \alpha_k P(A)\Theta_{k+m-1}(A)P(A) = 0 \Rightarrow \alpha_0 P(A)\Theta_{m-1}(A)P(A) = 0, \quad (32)$$

so $\alpha_0 = 0$ from the fact that A is a strict P-quasi m-symmetric operator.

For $r = m-2$,

$$\sum_{0\leq k\leq m-1} \alpha_k P(A)\Theta_{k+m-2}(A)P(A) = 0 \Rightarrow \alpha_1 P(A)\Theta_{m-1}(A)P(A) = 0, \quad (33)$$

so $\alpha_1 = 0$ for the same reason.

Doing this iteratively, we can find for $r = m-3, \ldots, r = 1$ and $r = 0$ that $\alpha_k = 0$ for $k = 2, \cdots, m-1$.

Then, the following implication is true:

$$\sum_{0\leq k\leq m-1} \alpha_k P(A)\Theta_k(A)P(A) = 0 \Rightarrow \alpha_0 \Rightarrow \alpha_1 \Rightarrow \cdots \Rightarrow \alpha_{m-1} \Rightarrow 0. \quad (34)$$

□

The set of P-quasi m-symmetric operators is closed in norm as shown in the following theorem:

Theorem 7. *Let P be a nonconstant complex polynomial and $A \in \mathscr{B}(\mathscr{H})$, then the set of P-quasi-m-symmetric operators is closed in norm in $\mathscr{B}(\mathscr{H})$.*

Proof. Suppose that $(A_q)_q$ is a sequence of P-quasi-m-symmetric operators such that

$$\lim_{q\longrightarrow\infty} \left\| A_m - A \right\| = 0. \quad (35)$$

Since for every positive integer q, A_q is P-quasi-m-symmetric operator, we get

$$P\left(A_q\right)^* \Theta\left(A_q\right) P\left(A_q\right) = 0. \quad (36)$$

However,

$$\begin{aligned}
\left\|P\left(A_q\right)^* \Theta_q\left(A_q\right) P\left(A_q\right)\right\| &= \left\|P\left(A_q\right)^* \Theta_m\left(A_q\right) P\left(A_q\right) - P(A)^* \Theta_m(A) P(A)\right\| \\
&= \left\|P\left(A_q\right)^* \left(\sum_{0\leq j\leq m} (-1)^j \binom{m}{j} A^{*j} A_q^{m-j}\right) p\left(A_q\right) - P(A)^* \left(\sum_{0\leq j\leq m} (-1)^j \binom{m}{j} A^{*j} A^{m-j}\right) P(A)\right\| \\
&\leq \left\|P\left(A_q\right)^* \left(\sum_{0\leq j\leq m} (-1)^j \binom{m}{j} A^{*j} A_q^{m-j}\right) P\left(A_q\right) - P\left(A_q\right)^* \left(\sum_{0\leq j\leq m} (-1)^j \binom{m}{j} A^{*j} A^{m-j}\right) P(A)\right\| \\
&\quad + \left\|P\left(A_q\right) \left(\sum_{0\leq j\leq m} (-1)^j \binom{m}{j} A^{*j} A^{m-j}\right) P(A) - P(A)^* \left(\sum_{0\leq j\leq m} (-1)^j \binom{m}{j} A^{j} A^{m-j}\right) P(A)\right\| \\
&\leq \left\|P\left(A_q\right)\right\| \left\| \sum_{0\leq j\leq m} (-1)^j \binom{m}{j} A^{*j} \left(A_q^{m-j} P\left(A_q\right) - A^{m-j} P(A)\right)\right\| \\
&\quad + \left\| \sum_{0\leq j\leq m} (-1)^j \binom{m}{j} \left(P\left(A_q\right)^* A^{*j} A^{m-j} - P(A)^* A^{*j} A^{m-j} P\left(A_q\right) - A^{m-j} P(A)\right)\right\| \\
&\quad + \|P(A)\| \left\| \sum_{0\leq j\leq m} (-1)^j \binom{m}{j} A^{*j} \left(A_q^{m-j} P\left(A_q\right) - A^{m-j} P(A)\right)\right\| \\
&\quad + \left\| \sum_{0\leq j\leq m} (-1)^j \binom{m}{j} \left(P\left(A_q\right)^* A^{*j} A^{m-j} - P(A) * A^{*j} A^{m-j}\right) P(A)\right\|.
\end{aligned} \quad (37)$$

Since $P(A_q)^*\Theta_m(A_q)P(A_q)=0$ we get by taking $q\longrightarrow\infty$, that $P(A)^*\Theta_q(A)P(A)=0$, and therefore A belongs to the of P-quasi-m-symmetric operators. □

Lemma 1. *Let P be a nonconstant complex polynomial and let $A\in\mathscr{B}(\mathscr{H})$, the following statements are true:*

(1) A is a P-quasi m-symmetric operator if and only if $A\otimes I$ is P-quasi m-symmetric operator

(2) A is a P-quasi m-symmetric operator if and only if $I\otimes A$ is P-quasi m-symmetric operator

Proof. Set $P=\sum_{0\le k\le m}a_kz^k$, we have

$$\begin{aligned}P(A\otimes I)&=\sum_{0\le k\le m}a_k(A\otimes I)^k=\sum_{0\le k\le m}a_kA^k\otimes\\&=P(A)\otimes I,\\&\quad P(A\otimes I)^*\Theta_m(A\otimes I)P(A\otimes I)\\&=P(A\otimes I)^*\left(\sum_{0\le j\le m}(-1)^j\binom{m}{j}(A\otimes I)^{*j}\text{big}(A\otimes I)^{m-j}\right)P(A\otimes I)\\&=P(A)^*\left(\sum_{0\le j\le m}(-1)^j\binom{m}{j}A^{*j}A^{m-j}\right)P(A)\otimes I\\&=P(A)^*\Theta_m(A)P(A)\otimes I.\end{aligned}\tag{38}$$

□

Proposition 3. *Let P be a nonconstant complex polynomial and $A,V\in\mathscr{B}(\mathscr{H})$. If A is P-quasi-m-symmetric for some positive integer m and V is an isometry, then VAV^* is an P-quasi-m-symmetric operator.*

Proof. Let $P(z)=\sum_{0\le k\le n}a_kz^k$ such that A is P-quasi-m-symmetric operator. Since V is an isometry, then $V^*V=I$. A direct calculation shows that

$$\begin{aligned}(VAV^*)^l&=\underbrace{(VAV^*)(VAAV^*)\cdots(VAV^*)}_{l\text{–times}}\\&=\left(VA^2V^*\cdots VA\right)V^*\\&=VA^lV^*,\end{aligned}\tag{39}$$

and moreover

$$\begin{aligned}P(VAV^*)&=\sum_{0\le k\le n}a_k(VAV^*)^k\\&=\sum_{0\le k\le n}a_k\left(VA^kV^*\right)\\&=V\sum_{0\le k\le n}a_k\left(A^k\right)V^*\\&=VP(A)V^*.\end{aligned}\tag{40}$$

We have

$$\begin{aligned}&P(VAV^*)^*\left(\sum_{0\le k\le m}(-1)^k\binom{m}{k}(VAV^*)^{*k}(VAV^*)^{m-k}\right)P((VAV^*)\\&=(VP(A)V^*)^*V\left(\sum_{0\le k\le m}(-1)^k\binom{m}{k}(A)^{*k}(A)^{m-k}\right)V^*(VP(A)V^*))\\&=V\underbrace{\left(P(A)^*\left(\sum_{0\le k\le m}(-1)^k\binom{m}{k}(A)^{*k}(A)^{m-k}\right)p(A)\right)V^*}_{=0}\\&=0.\end{aligned}\tag{41}$$

Therefore, VAV^* is a P-quasi-m-symmetric operator [23]. □

Data Availability

Data sharing not applicable to this paper as no data sets were generated or analyzed during the current study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

Authors' Contributions

The author wrote and reviewed the manuscript.

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