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Research Article

Two Logarithmically Improved Regularity Criteria for the 3D Nematic Liquid Crystal Flows

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In this paper, we study the regularity criterion for the local smooth solution of the 3D nematic liquid crystal flows. More precisely, it is proved the smooth solution (u,d) can be extended beyond T provided that $\int_0^T (\|\nabla_h u_h\|_{\dot{B}^0_{\infty,\infty}} + \|\nabla d\|^2_{\dot{B}^0_{\infty,\infty}})/\sqrt{1+\log(1+\|\nabla u\|_{\dot{B}^0_{\infty,\infty}} + \|\nabla d\|_{\dot{B}^0_{\infty,\infty}})}\,dt < \infty$ or $\int_0^T (\|\nabla_h u_h\|^{4/3-2r}_{\dot{B}^{-r}_{\infty,\infty}} + \|\nabla d\|^2_{\dot{B}^0_{\infty,\infty}})/\sqrt{1+\log(1+\|\nabla u\|_{\dot{B}^0_{\infty,\infty}} + \|\nabla d\|_{\dot{B}^0_{\infty,\infty}})}\,dt < \infty, 0 \le r \le 1$.

1. Introduction

Liquid crystals is a state of the matter which has both properties of the liquid and the solid crystal. And as a kind of liquid crystals, the nematic liquid crystal can flow like fluids and has very nice properties. Ericksen et al. during 1960s (see [1, 2]) established the hydrodynamic theory for describing the nematic liquid crystal flows. Owing to the complexity of original Ericksen-Leslie equations and for further research, Lin [3] simplify the original Ericksen-Leslie equations, which still retains most of the essential features of original equations. In this paper, we investigate the following simplified version for nematic liquid crystal flows in 3-dimensions

$$\begin{cases} \partial_t u + u\cdot\nabla u - \mu\Delta u + \nabla p = -\lambda\nabla\cdot(\nabla d\odot\nabla d),\\ \partial_t d + u\cdot\nabla d = \gamma\left(\Delta\ d + |\nabla d|^2 d\right),\\ \nabla\cdot u = 0, |d| = 1,\\ u(x,0) = u_0(x), d(x,0) = d_0(x), \end{cases} \tag{1}$$

here $u = u(x,t)\in\mathbb{R}^3$ denotes the velocity field, $d = d(x,t)\in\mathbb{S}^2$ (the unit sphere in $\mathbb{R}^3$) the macroscopic average of molecular orientation field and $p = p(x,t)$ represents the scalar pressure, $\nabla\cdot u = 0$ is the incompressible condition. And μ, λ, γ are positive constants, which shall be assumed to be all equal to 1 in consideration of their concrete values playing no role in our arguments. The notation $\nabla d\odot\nabla d$ represents the 3×3 matrix whose the (i,j)th component is given by

$$\sum_{k=1}^{3}\partial_i d_k \partial_j d_k\,(i,j\le 3). \tag{2}$$

It is well-known that the system (1) has a unique local smooth solution (see [4]). More precisely, if initial data $u_0\in H^s(\mathbb{R}^n,\mathbb{R}^n)$ with $\nabla\cdot u_0 = 0$ and $d_0\in H^{s+1}(\mathbb{R}^n,\mathbb{S}^2)$ for $s\ge n$, then

$$\begin{aligned} &u\in C([0,T];H^s(\mathbb{R}^n))\cap C^1\left([0,T];H^{s-1}(\mathbb{R}^n)\right),\\ &d\in C\left([0,T];H^{s+1}\left(\mathbb{R}^n,\mathbb{S}^2\right)\right)\cap C^1\left([0,T];H^s\left(\mathbb{R}^n,\mathbb{S}^2\right)\right). \end{aligned} \tag{3}$$

However, the global existence of solutions is an difficult problem. Hence much efforts have been paid to study the regularity criteria to extend local solutions. For the regularity criteria readers may refer to [5–13] and references therein.

On one hand, the above system (1) reduces to the incompressible Navier-Stokes equations when the orientation field d equals a constant. It is well-known that Navier-Stokes equations has an unique smooth solution (see [14]) provided that the solution u satisfies

$$\int_0^T \|\omega\|_{L^\infty} dt < \infty, \tag{4}$$

where $\omega = \nabla \times u$. Later, Kozono and Taniuchi [15], Kozono et al. [16] generalized the criterion (4) to

$$\int_0^T \|\omega\|_{\mathrm{BMO}} dt < \infty, \int_0^T \|\omega\|_{\dot{B}^0_{\infty,\infty}} dt < \infty, \tag{5}$$

respectively, where BMO is the space of Bounded Mean Oscillation and $\dot{B}^0_{\infty,\infty}$ represents the homogeneous Besov space. And based on (5), Fan et al. [17], Guo and Gala [18] respectively improve (5) by the following conditions

$$\int_0^T \frac{\|\nabla u\|_{\dot{B}^0_{\infty,\infty}}}{\sqrt{1 + \ln\left(e + \|\nabla u\|_{\dot{B}^0_{\infty,\infty}}\right)}} dt < \infty, \tag{6}$$

$$\int_0^T \frac{\|\nabla u\|^2_{\dot{B}^{-1}_{\infty,\infty}}}{\sqrt{1 + \ln\left(e + \|\nabla u\|_{\dot{B}^0_{\infty,\infty}}\right)}} dt < \infty. \tag{7}$$

It is obvious that the logarithmic improvement is here, in time only, and that can be seen as a natural Gronwall type extension of the Prodi-Serrin conditions. On the other hand, when the velocity field $u = 0$, the system (1) becomes to the heat flow of harmonic maps onto a sphere. And Wang [19] established a blow up criterion, which implies the unique smooth solution $d \in C^\infty(\mathbb{R}^n; (0,T])$ is global if

$$\int_0^T \|\nabla d\|_{L^n} dt < \infty. \tag{8}$$

Inspired by the conditions (4) and (8), Huang and Wang [4] established a BKM type blow-up criterion for the system (1). That is, if T is the maximal time, $0 < T < \infty$, then

$$\int_0^T \left(\|\omega\|_{L^\infty} + \|\nabla d\|^2_{L^\infty}\right) dt = \infty. \tag{9}$$

Naturally, similar to (6) and (7), Liu and Zhao [20] extend (9) to the Logarithmically improved regularity criterion. Namely, the local smooth solution (u, d) can continuously past any time $T > 0$ if the following holds

$$\int_0^T \frac{\|\omega\|_{\dot{B}^0_{\infty,\infty}} + \|\nabla d\|^2_{\dot{B}^0_{\infty,\infty}}}{\sqrt{1 + \ln\left(e + \|\nabla u\|_{\dot{B}^0_{\infty,\infty}} + \|\nabla d\|_{\dot{B}^0_{\infty,\infty}}\right)}} dt < \infty, \tag{10}$$

or

$$\int_0^T \frac{\|\nabla u\|^2_{\dot{B}^{-1}_{\infty,\infty}} + \|\nabla d\|^2_{\dot{B}^0_{\infty,\infty}}}{\sqrt{\ln\left(e + \|\nabla u\|_{\dot{B}^0_{\infty,\infty}} + \|\nabla d\|_{\dot{B}^0_{\infty,\infty}}\right)}} dt < \infty. \tag{11}$$

In view of it is difficult to reduce the condition on d, we are mainly concerned with reducing the condition on u. Inspired by the references above, we will use the components of ∇u to replace the condition (11). Our main results are stated as follows:

Theorem 1. *Assume (u, d) is a local smooth solution to the system (1) on the time interval $[0, T)$ for some $0 < T < \infty$. And let initial datum $u_0 \in H^3(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$, $d_0 \in H^4(\mathbb{R}^3, \mathbb{S}^2)$. If (u, d) satisfies*

$$\int_0^T \left(\frac{\|\nabla_h u_h\|_{\dot{B}^0_{\infty,\infty}} + \|\nabla d\|^2_{\dot{B}^0_{\infty,\infty}}}{\sqrt{1 + \log\left(1 + \|\nabla u\|_{\dot{B}^0_{\infty,\infty}} + \|\nabla d\|_{\dot{B}^0_{\infty,\infty}}\right)}} \right) dt < \infty, \tag{12}$$

then (u, d) can be extended beyond T smoothly, where $\nabla_h = (\partial_1, \partial_2, 0)$, $u_h = (u_1, u_2, 0)$. That is to say, if the solution blows up at T, then

$$\int_0^T \left(\frac{\|\nabla_h u_h\|_{\dot{B}^0_{\infty,\infty}} + \|\nabla d\|^2_{\dot{B}^0_{\infty,\infty}}}{\sqrt{1 + \log\left(1 + \|\nabla u\|_{\dot{B}^0_{\infty,\infty}} + \|\nabla d\|_{\dot{B}^0_{\infty,\infty}}\right)}} dt \right) = \infty. \tag{13}$$

Remark 1

(1) *In view of the fact that the norms $\|\omega\|_{\dot{B}^0_{\infty,\infty}}$ and $\|\nabla u\|_{\dot{B}^0_{\infty,\infty}}$ are approximate, it is obvious that the condition*(10) *is weaker than the condition* (8) *and* (9) *in some sense. And it can be seen that if the condition* (10) *reduces to*

$$\int_0^T \left(\|\nabla_h u_h\|_{\dot{B}^0_{\infty,\infty}} + \|\nabla d\|^2_{\dot{B}^0_{\infty,\infty}}\right) dt < \infty, \tag{14}$$

then the conclusion of Theorem (1) still remains valid, which is also an improved result compared to the regularity criterion (9).

(2) Noting that the norm $\|\nabla u_h\|_{\dot{B}^{-1}_{\infty,\infty}}$ is equivalent to $\|u_h\|_{\dot{B}^0_{\infty,\infty}}$, combining (10)–(12), the condition (12) can be replaced by the following condition:

$$\int_0^T \left(\frac{\|u_h\|^2_{\dot{B}^0_{\infty,\infty}} + \|\nabla d\|^2_{\dot{B}^0_{\infty,\infty}}}{1 + \log\left(1 + \|\nabla u\|_{\dot{B}^0_{\infty,\infty}} + \|\nabla d\|_{\dot{B}^0_{\infty,\infty}}\right)} dt \right) < \infty. \tag{15}$$

Remark 2. It is well-known that $L^\infty(\mathbb{R}^3) \subset \mathrm{BMO}(\mathbb{R}^3) \subset \dot{B}^0_{\infty,\infty}(\mathbb{R}^3)$. Thus the conclusion of Theorem 1*still remains true if the condition* (12) *is substituted by*

$$\int_0^T \left(\frac{\|\nabla_h u_h\|_{\mathrm{BMO}} + \|\nabla d\|^2_{\mathrm{BMO}}}{\sqrt{1 + \log\left(1 + \|\nabla u\|_{\mathrm{BMO}} + \|\nabla d\|_{\mathrm{BMO}}\right)}} dt \right) < \infty. \tag{16}$$

Theorem 2. *Assume (u, d) is a local smooth solution to the system (1) on the time interval $[0, T)$ for some $0 < T < \infty$. And let initial datum $u_0 \in H^3(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$, $d_0 \in H^4(\mathbb{R}^3, \mathbb{S}^2)$. If (u, d) satisfies*

$$\int_0^T \left(\frac{\|\nabla_h u_h\|^{4/3-2r}_{\dot{B}^{-r}_{\infty,\infty}} + \|\nabla d\|^2_{\dot{B}^0_{\infty,\infty}}}{\sqrt{1 + \log\left(1 + \|\nabla u\|_{\dot{B}^0_{\infty,\infty}} + \|\nabla d\|_{\dot{B}^0_{\infty,\infty}}\right)}} dt \right) < \infty, \tag{17}$$

here $0 \le r \le 1$, *then* (u, d) *can be extended beyond* T *smoothly. That is to say, if the solution blows up at* T, *then*

$$\int_0^T \left(\frac{\|\nabla_h u_h\|_{\dot{B}^{-r}_{\infty,\infty}}^{4/3-2r} + \|\nabla d\|_{\dot{B}^0_{\infty,\infty}}^2}{\sqrt{1 + \log\left(1 + \|\nabla u\|_{\dot{B}^0_{\infty,\infty}} + \|\nabla d\|_{\dot{B}^0_{\infty,\infty}}\right)}} \mathrm{d}t \right) < \infty. \tag{18}$$

Remark 3. Owing to $\dot{B}^{-r}_{\infty,\infty}$ *containing the case* $\dot{B}^0_{\infty,\infty}$ *as* $r = 0$, *the condition* (17) *is an improvement in some sense compared to the condition* (12). *However,* $4/3 - 2r = 4/3$ *when* $r = 0$, *hence the condition* (12) *is better than the condition* (17) *in the end point.*

2. Preliminaries

In this section, we collect some useful analytic tools which play an important part in our proof.

Lemma 1 (Page 82 in **[21]**). *Let* $1 < q < p < \infty$ *and* α *be a positive real number. Then there exists a constant* C *such that*

$$\|f\|_{L^p} \le C\|f\|_{\dot{B}^{-\alpha}_{\infty,\infty}}^{1-\theta} \|f\|_{\dot{B}^{\beta}_{q,q}}^{\theta}, \text{with } \beta = \alpha\left(\frac{p}{q} - 1\right), \theta = \frac{q}{p}. \tag{19}$$

In particular, when $\beta = 1, q = 2$ and $p = 4$, we have $\alpha = 1$ and

$$\|f\|_{L^4} \le C\|f\|_{\dot{B}^{-1}_{\infty,\infty}}^{1/2} \|\nabla f\|_{L^2}^{1/2}. \tag{20}$$

Lemma 2. (Product and Commutator estimate[22, 23]). Let$s > 0, 1 < p < \infty$, *and* $1/p = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$ *with* $p_2, p_3 \in (1, +\infty)$ *and* $p_1, p_4 \in [1, +\infty]$. *Then,*

$$\left\|\Lambda^s (fg)\right\|_{L^p} \le C\left(\|g\|_{L^{p_1}}\left\|\Lambda^s f\right\|_{L^{p_2}} + \left\|\Lambda^s g\right\|_{L^{p_3}}\|f\|_{L^{p_4}}\right), \tag{21}$$

$$\left\|[\Lambda^s, f\cdot\nabla]g\right\|_{L^p} \le C\left(\|\nabla f\|_{L^{p_1}}\left\|\Lambda^s g\right\|_{L^{p_2}} + \left\|\Lambda^s f\right\|_{L^{p_3}}\|\nabla g\|_{L^{p_4}}\right), \tag{22}$$

where $[\Lambda^s, f]g = \Lambda^s(fg) - f\Lambda^s g$.

Lemma 3 (see [24], Lemma 2). *Let* $\nabla f \in \dot{B}^{-1}_{\infty,\infty}, f \in H^s$ *for all* $s > 3/2$. *Then there exists a positive constant* C *such that*

$$\|f\|_{L^\infty} \le C\left(1 + \|\nabla f\|_{\dot{B}^{-1}_{\infty,\infty}}\left[\log^+(1 + \|f\|_{H^s})\right]^{1/2}\right), \tag{23}$$

where H^s *denotes the standard Sobolev space and*

$$\log^+ x = \begin{cases} \log x, & x > e, \\ 1, & 0 \le x \le e. \end{cases} \tag{24}$$

3. Proof of Main Results

Proof of Theorem 1. In this section, we shall first show the proof of Theorem 1. Since the existence of local smooth solutions is obvious owing to the initial value condition for (u, d) in Theorem 1, we only need to show the priori estimate for the local smooth solution. And by the condition (12), we will give the following priori estimate

$$\lim_{t\longrightarrow T^-} \sup\left(\|\nabla\Delta u\|_{L^2}^2 + \left\|\Delta^2 d\right\|_{L^2}^2\right) < \infty, \tag{25}$$

which is enough to guarantee the smooth solution (u, d) pasts time T smoothly.

Firstly, we will show the L^1 estimate of u and ∇d together because the terms $\nabla\cdot(\nabla d \odot \nabla d)$ and $u\cdot\nabla d$ can be cancelled when integrating. Applying u to the equation $(1)_1$ and integrating over $\mathbb{R}^3$ yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 = -\int_{\mathbb{R}^3} \nabla d\cdot\Delta\, d\cdot u\, dx, \tag{26}$$

where the following equalities have been used

$$\int_{\mathbb{R}^3} (u\cdot\nabla u)\cdot u\mathrm{d}x = 0, \int_{\mathbb{R}^3} \nabla p\cdot u\mathrm{d}x = 0. \tag{27}$$

Then, multiplying equation $(1)_2$ by $-\Delta\, d$ and integrating over $\mathbb{R}^3$ one has

$$\frac{1}{2}\frac{d}{dt}\|\nabla d\|_{L^2}^2 + \|\Delta\, d\|_{L^2}^2 = \int_{\mathbb{R}^3} u\cdot\nabla d\cdot\Delta\, d - |\nabla d|^2 d\Delta\, d\mathrm{d}x. \tag{28}$$

By adding the above equalities and using the facts $|d| = 1$, $\Delta(|d|^2) = 0 \Rightarrow |\nabla d|^2 = -d\cdot\Delta\, d$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2\right) + \|\nabla u\|_{L^2}^2 + \|\Delta\, d\|_{L^2}^2 \le \|\Delta\, d\|_{L^2}^2. \tag{29}$$

Integrating above inequality (29) in time yields

$$\sup_{0<t<T}\left(\|u(t)\|_{L^2}^2 + \|\nabla d(t)\|_{L^2}^2\right) + \int_0^T \|\nabla u(t)\|_{L^2}^2\mathrm{d}t \le C\left(\left\|u_0\right\|_{L^2}^2 + \left\|\nabla d_0\right\|_{L^2}^2\right). \tag{30}$$

Besides, it is sufficient to show the boundness for $\|d\|_{L^\infty}$. Multiplying equation $(1)_2$ by $p|d|^{p-2}d$ for $p > 2$ and integrating both sides on $\mathbb{R}^3$, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\|d\|_{L^p}^p + 2\left\|\nabla|d|^{p/2}\right\|_{L^2}^2 + p\|d\|_{L^{p+2}}^{p+2} = p\|d\|_{L^p}^p, \tag{31}$$

where we use the equality

$$\int_{\mathbb{R}^3} (u\cdot\nabla d)\cdot p|d|^{p-2} d\mathrm{d}x = -\int_{\mathbb{R}^3} \nabla\cdot u|d|^p\mathrm{d}x = 0. \tag{32}$$

Hence the (31) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|d\|_{L^p} \le \|d\|_{L^p}. \tag{33}$$

Applying the Gronwall inequality and letting $p \longrightarrow \infty$, it can be deduced from above that

$$\|d\|_{L^\infty} \le \mathrm{e}^T \left\|d_0\right\|_{L^\infty} < \infty. \tag{34}$$

Now we shall show the H^1 estimate of u and ∇d. Similarly going on the above process, multiplying the equation $(1)_1$ by $-\Delta u$ and integrating over $\mathbb{R}^3$, then taking Δ on the equation $(1)_2$, multiplying with $\Delta\, d$ and integrating over $\mathbb{R}^3$, and combining that two equations, one obtains that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u\|_{L^2}^2 + \|\Delta\, d\|_{L^2}^2\right) + \|\Delta u\|_{L^2}^2 + \|\nabla\Delta\, d\|_{L^2}^2 \\ &= \int_{\mathbb{R}^3} (u\cdot\nabla)u\cdot\Delta u \mathrm{d}x + \int_{\mathbb{R}^3} \nabla\cdot(\nabla d\odot\nabla d)\cdot\Delta u \mathrm{d}x \\ &\quad - \int_{\mathbb{R}^3} \Delta(u\cdot\nabla d)\cdot\Delta\, d\mathrm{d}x + \int_{\mathbb{R}^3} \Delta\left(|\nabla d|^2 d\right)\cdot\Delta\, d\mathrm{d}x \\ &\coloneqq I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{35}$$

In the following, we will estimate the terms $I_i\,(i = 1,2,3,4)$. For I_1, making use of the incompressibility condition and integration by parts several times, one can be concluded that

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^3} (u\cdot\nabla)u\cdot\Delta u \mathrm{d}x = \int_{\mathbb{R}^3} \sum_{i,j,k=1}^{3} u_i\partial_i u_j \partial_k\partial_k u_j \mathrm{d}x, \\ &= -\int_{\mathbb{R}^3}\left(\sum_{i,j,k=1}^{3}\partial_k u_i\partial_i u_j\partial_k u_j + \sum_{i,j,k=1}^{3}\frac{1}{2}u_i\partial_i\left(\partial_k u_j\right)^2\right)\mathrm{d}x \\ &= -\int_{\mathbb{R}^3}\left(\sum_{i,k=1}^{2}\sum_{j=1}^{3}\partial_k u_i\partial_i u_j\partial_k u_j + \sum_{i=1}^{2}\sum_{j=1}^{3}\partial_3 u_i\partial_i u_j\partial_3 u_j, + \sum_{k=1}^{2}\sum_{j=1}^{3}\partial_k u_3\partial_3 u_j\partial_k u_j + \sum_{j=1}^{3}\partial_3 u_3\partial_3 u_j\partial_3 u_j\right)\mathrm{d}x \\ &= -\int_{\mathbb{R}^3}\left(\sum_{i,k=1}^{2}\sum_{j=1}^{3}\partial_k u_i\partial_i u_j\partial_k u_j + \sum_{i,j=1}^{2}\partial_3 u_i\partial_i u_j\partial_3 u_j + \sum_{i=1}^{2}\partial_3 u_i\partial_i u_3\partial_3 u_3 + \sum_{j,k=1}^{2}\partial_k u_3\partial_3 u_j\partial_k u_j\right. \\ &\quad \left. + \sum_{k=1}^{2}\partial_k u_3\partial_3 u_3\partial_k u_3 + \sum_{j=1}^{3}\partial_3 u_3\partial_3 u_j\partial_3 u_j\right)\mathrm{d}x. \end{aligned} \tag{36}$$

Noting that $\partial_3 u_3 = -(\partial_1 u_1 + \partial_2 u_2)$, and using the Lemma 3, I_1 can be estimated as follows

$$
\begin{aligned}
I_1 &\leq C\int_{\mathbb{R}^3} \left|\nabla_h u_h \nabla u\right|^2 \mathrm{d}x \\
&\leq C\left\|\nabla_h u_h\right\|_{L^\infty} \|\nabla u\|_{L^2}^2 \\
&\leq C\left[1 + \left\|\nabla\nabla_h u_h\right\|_{\dot{B}^{-1}_{\infty,\infty}} \log^{1/2}\left(1 + \|\nabla\Delta u\|_{L^2}\right)\right] \|\nabla u\|_{L^2}^2 \\
&\leq C\left[1 + \left\|\nabla_h u_h\right\|_{\dot{B}^{0}_{\infty,\infty}} \log^{1/2}\left(1 + \|\nabla\Delta u\|_{L^2}\right)\right] \|\nabla u\|_{L^2}^2 \\
&\leq C \frac{\left\|\nabla_h u_h\right\|_{\dot{B}^{0}_{\infty,\infty}}}{\sqrt{1 + \log\left(1 + \|\nabla u\|_{\dot{B}^{0}_{\infty,\infty}} + \|\nabla d\|_{\dot{B}^{0}_{\infty,\infty}}\right)}} \log^{1/2}\left(1 + \|\nabla\Delta u\|_{L^2}\right) \\
&\quad \times \sqrt{1 + \log\left(1 + \|\nabla u\|_{\dot{B}^{0}_{\infty,\infty}} + \|\nabla d\|_{\dot{B}^{0}_{\infty,\infty}}\right)} \|\nabla u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 \\
&\leq C \frac{\left\|\nabla_h u_h\right\|_{\dot{B}^{0}_{\infty,\infty}}}{\sqrt{1 + \log\left(1 + \|\nabla u\|_{\dot{B}^{0}_{\infty,\infty}} + \|\nabla d\|_{\dot{B}^{0}_{\infty,\infty}}\right)}} \log^{1/2}\left(1 + \|\nabla\Delta u\|_{L^2} + \left\|\Delta^2 d\right\|_{L^2}\right) \\
&\quad \times \sqrt{1 + \log\left(1 + \|\nabla u\|_{\dot{B}^{0}_{\infty,\infty}} + \|\nabla d\|_{\dot{B}^{0}_{\infty,\infty}}\right)} \|\nabla u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 \\
&\leq C\left(1 + \frac{\left\|\nabla_h u_h\right\|_{\dot{B}^{0}_{\infty,\infty}}}{\sqrt{1 + \log\left(1 + \|\nabla u\|_{\dot{B}^{0}_{\infty,\infty}} + \|\nabla d\|_{\dot{B}^{0}_{\infty,\infty}}\right)}}\right) \log\left(1 + \|\nabla\Delta u\|_{L^2} + \left\|\Delta^2 d\right\|_{L^2}\right) \|\nabla u\|_{L^2}^2,
\end{aligned} \tag{37}
$$

where we have used the following inequality

$$
\begin{aligned}
\sqrt{1 + \log\left(1 + \|\nabla u\|_{\dot{B}^{0}_{\infty,\infty}} + \|\nabla d\|_{\dot{B}^{0}_{\infty,\infty}}\right)} &\leq C\sqrt{\log\left(1 + \|\nabla u\|_{L^\infty} + \|\nabla d\|_{L^\infty}\right.} \\
&\leq C\sqrt{\log\left(1 + \|\nabla u\|_{L^2}^{1/6} \|\nabla\Delta u\|_{L^2}^{5/6} + \|\nabla d\|_{L^2}^{1/2} \|\Delta^2 d\|_{L^2}^{1/2}\right.} \\
&\leq C\sqrt{\log\left(1 + \|\nabla\Delta u\|_{L^2}^2 + \left\|\Delta^2 d\right\|_{L^2}^2\right.}.
\end{aligned} \tag{38}
$$

In view of I_2 and I_3 containing terms that could be cancelled, adding I_2 and I_3 together and by the incompressibility condition $\nabla \cdot u = 0$, it follows that

$$\begin{aligned} I_2 + I_3 &= \int_{\mathbb{R}^3} \sum_{i,j,k=1}^{3} \left[\left(\partial_i \partial_j d_k \partial_j d_k + \partial_i d_k \partial_j \partial_j d_k\right)\Delta u_i - \left(\Delta u_i \partial_i d_k \Delta d_k + 2\nabla u_i \partial_i \nabla d_k \Delta d_k + u_i \partial_i \Delta d_k \Delta d_k\right)\right] \mathrm{d}x \\ &= \int_{\mathbb{R}^3} \sum_{i,j,k=1}^{3} -2\nabla u_i \partial_i \nabla d_k \Delta d_k \mathrm{d}x \\ &\leq C \int_{\mathbb{R}^3} |\nabla u\|\nabla\nabla d\|\Delta\ d| \mathrm{d}x. \end{aligned} \tag{39}$$

Hence, it can be deduced from $H\ddot{o}$ lder inequality, Young inequality and the inequality (20) that

$$\begin{aligned} I_2 + I_3 &\leq C\|\nabla u\|_{L^2}\|\Delta\ d\|_{L^4}^2 \leq C\|\nabla u\|_{L^2}\|\nabla d\|_{\dot{B}^0_{\infty,\infty}}\|\nabla\Delta\ d\||_{L^2} \\ &\leq C\|\nabla d\|_{\dot{B}^0_{\infty,\infty}}^2\|\nabla u\|_{L^2}^2 + \frac{1}{4}\|\nabla\Delta\ d\|_{L^2}^2 \\ &\leq C\frac{\|\nabla d\|_{\dot{B}^0_{\infty,\infty}}^2}{\sqrt{1+\log\left(1+\|\nabla u\|_{\dot{B}^0_{\infty,\infty}}+\|\nabla d\|_{\dot{B}^0_{\infty,\infty}}\right)}}\sqrt{1+\log\left(1+\|\nabla u\|_{\dot{B}^0_{\infty,\infty}}+\|\nabla d\|_{\dot{B}^0_{\infty,\infty}}\right)} \\ &\quad\cdot \|\nabla u\|_{L^2}^2 + \frac{1}{4}\|\nabla\Delta\ d\|_{L^2}^2 \\ &\leq C\frac{\|\nabla d\|_{\dot{B}^0_{\infty,\infty}}^2}{\sqrt{1+\log\left(1+\|\nabla u\|_{\dot{B}^0_{\infty,\infty}}+\|\nabla d\|_{\dot{B}^0_{\infty,\infty}}\right)}}\log\left(1+\|\nabla\Delta u\|_{L^2}+\left\|\Delta^2 d\right\|_{L^2}\right)\|\nabla u\|_{L^2}^2 \\ &\quad+\frac{1}{4}\|\nabla\Delta\ d\|_{L^2}^2, \end{aligned} \tag{40}$$

where the fact that the norms $\|\Delta\ d\|_{\dot{B}^{-1}_{\infty,\infty}}$ and $\|\nabla d\|_{\dot{B}^0_{\infty,\infty}}$ are equivalent has been used. For I_4, by the product estimate (21) and inequality (20), we obtain

$$
\begin{aligned}
I_4 &= \int_{\mathbb{R}^3} \Delta\left(|\nabla d|^2 d\right) \cdot \Delta\, d \mathrm{d}x \le \left\| \Delta\left(|\nabla d|^2 d\right)\right\|_{L^{\frac{4}{3}}} \|\Delta\, d\|_{L^4} \\
&\le C\left(\|\nabla\Delta\, d\|_{L^2}\|\nabla d\|_{L^4}\|d\|_{L^\infty} + \|\Delta\, d\|_{L^4}\|\nabla d\|_{L^4}^2\right)\|\Delta\, d\|_{L^4} \\
&\le C\|\Delta\, d\|_{L^4}^2\|\nabla d\|_{L^4}^2 + \frac{1}{8}\|\nabla\Delta\, d\|_{L^2}^2 \\
&\le C\|\nabla d\|_{\dot{B}^0_{\infty,\infty}}\|\nabla\Delta\, d\|_{L^2}\|d\|_{L^\infty}\|\Delta\, d\|_{L^2} + \frac{1}{8}\|\nabla\Delta\, d\|_{L^2}^2 \\
&\le C\|\nabla d\|_{\dot{B}^0_{\infty,\infty}}^2\|\Delta\, d\|_{L^2}^2 + \frac{1}{4}\|\nabla\Delta\, d\|_{L^2}^2 \\
&\le C\frac{\|\nabla d\|^2_{\dot{B}^0_{\infty,\infty}}}{\sqrt{1+\log\left(1+\|\nabla u\|_{\dot{B}^0_{\infty,\infty}} + \|\nabla d\|_{\dot{B}^0_{\infty,\infty}}\right)}}\log\left(1+\|\nabla\Delta u\|_{L^2} + \left\|\Delta^2 d\right\|_{L^2}\right)\|\Delta\, d\|_{L^2}^2 \\
&\quad + \frac{1}{4}\|\nabla\Delta\, d\|_{L^2}^2,
\end{aligned}
\tag{41}
$$

here the following Gagliardo-Nirenberg inequality has been used

$$
\|\nabla d\|_{L^4} \le C\|d\|_{L^\infty}^{1/2}\|\Delta\, d\|_{L^2}^{1/2}. \tag{42}
$$

Combining (35), (37), (39), and (41) one has

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u\|_{L^2}^2 + \|\Delta\, d\|_{L^2}^2\right) + \|\Delta u\|_{L^2}^2 + \|\nabla\Delta\, d\|_{L^2}^2 \\
&\quad \le C\left(1 + \frac{\left\|\nabla_h u_h\right\|_{\dot{B}^0_{\infty,\infty}} + \|\nabla d\|^2_{\dot{B}^0_{\infty,\infty}}}{\sqrt{1+\log\left(1+\|\nabla u\|_{\dot{B}^0_{\infty,\infty}} + \|\nabla d\|_{\dot{B}^0_{\infty,\infty}}\right)}}\right)\log\left(1+\|\nabla\Delta u\|_{L^2} + \left\|\Delta^2 d\right\|_{L^2}\right) \\
&\qquad \times\left(\|\nabla u\|_{L^2}^2 + \|\Delta\, d\|_{L^2}^2\right).
\end{aligned}
\tag{43}
$$

Noting (12), one may conclude that for any small constant $\epsilon > 0$, there exists $T_0 < T$ such that

$$
\int_{T_0}^{T} \frac{\left\|\nabla_h u_h\right\|_{\dot{B}^0_{\infty,\infty}} + \|\nabla d\|^2_{\dot{B}^0_{\infty,\infty}}}{\sqrt{1+\log\left(1+\|\nabla u\|_{\dot{B}^0_{\infty,\infty}} + \|\nabla d\|_{\dot{B}^0_{\infty,\infty}}\right)}}\mathrm{d}t < \epsilon. \tag{44}
$$

For any $T_0 \le t < T$, we set

$$
M(t) = \sup_{T_0 \le s \le t}\left(\|\nabla\Delta u(s)\|_{L^2}^2 + \left\|\Delta^2 d(s)\right\|_{L^2}^2\right). \tag{45}
$$

Using (44) and (45), and applying Gronwall inequality to (43) in the interval $[T_0, t]$ gives

$$\begin{aligned}
&\|\nabla u(t)\|_{L^2}^2+\|\Delta\, d(t)\|_{L^2}^2+\int_{T_0}^t\left(\|\Delta u(s)\|_{L^2}^2+\|\nabla\Delta\, d(s)\|_{L^2}^2\right)\mathrm{d}s\\
&\quad\le\left(\left\|\nabla u(T_0)\right\|_{L^2}^2+\left\|\Delta\, d(T_0)\right\|_{L^2}^2\right)\\
&\qquad\times\exp\left\{C\int_{T_0}^t\left(1+\frac{\left\|\nabla_h u_h\right\|_{\dot B^0_{\infty,\infty}}+\|\nabla d\|_{\dot B^0_{\infty,\infty}}^2}{\sqrt{1+\log\left(1+\|\nabla u\|_{\dot B^0_{\infty,\infty}}+\|\nabla d\|_{\dot B^0_{\infty,\infty}}\right)}}\log\left(1+\|\nabla\Delta u\|_{L^2}^2+\left\|\Delta^2 d\right\|_{L^2}^2\right)\right)\mathrm{d}s\right\}\\
&\quad\le\left(\left\|\nabla u(T_0)\right\|_{L^2}^2+\left\|\Delta\, d(T_0)\right\|_{L^2}^2\right)\exp\{C(T-T_0)+C\epsilon\log(1+M(t))\}\\
&\quad\le C_0C(T)\exp\{C\epsilon\log(1+M(t))\}\\
&\quad\le C_0C(T)(1+M(t))^{C\epsilon},
\end{aligned}\tag{46}$$

where the letter C_0 means a constant depending on $(\|\nabla u(T_0)\|_{L^2}^2+\|\Delta\, d(T_0)\|_{L^2}^2)$, $C(T)$ depends on the maximum value of time T, and C is a generic constant which may be different from line to line.

At last, the boundedness of the norm $\|\nabla\Delta u\|_{L^2}$ and $\|\Delta^2 d\|_{L^2}$ are needed so as to guarantee the validness of inequality (25) and (46). Employing $\nabla\Delta$ and Δ^2 to the equations $(1)_1$ and $(1)_2$ respectively, and taking the L^2 inner product with $(\nabla\Delta u,\Delta^2 d)$, we see that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla\Delta u\|_{L^2}^2+\left\|\Delta^2 d\right\|_{L^2}^2\right)+\left\|\Delta^2 u\right\|_{L^2}^2+\left\|\nabla\Delta^2 d\right\|_{L^2}^2\\
&\quad=-\int_{\mathbb{R}^3}\nabla\Delta(u\cdot\nabla u)\cdot\nabla\Delta u\mathrm{d}x-\int_{\mathbb{R}^3}\nabla\Delta(\nabla d\cdot\Delta\, d)\cdot\nabla\Delta u\mathrm{d}x\\
&\qquad-\int_{\mathbb{R}^3}\Delta^2(u\cdot\nabla d)\cdot\Delta^2 d\mathrm{d}x-\int_{\mathbb{R}^3}\Delta^2\left(|\nabla d|^2 d\right)\cdot\Delta^2\mathrm{d}x\\
&\quad\coloneqq J_1+J_2+J_3+J_4.
\end{aligned}\tag{47}$$

For J_1, applying $\nabla\cdot u=0$, $H\ddot{o}$ lder inequality, the commutator estimate (22) and Young inequality, we have

$$\begin{aligned}
J_1&=-\int_{\mathbb{R}^3}[\nabla\Delta,u\cdot\nabla]u\cdot\nabla\Delta u\mathrm{d}x\\
&\le\|[\nabla\Delta,u\cdot\nabla]u\|_{L^{\frac{4}{3}}}\|\nabla\Delta u\|_{L^4}\\
&\le C\left(\|\nabla u\|_{L^2}\|\nabla\Delta u\|_{L^4}+\|\nabla\Delta u\|_{L^4}\|\nabla u\|_{L^2}\right)\|\nabla\Delta u\|_{L^4}\\
&\le C\|\nabla u\|_{L^2}\|\nabla\Delta u\|_{L^4}^2\\
&\le C\|\nabla u\|_{L^2}\|\nabla u\|_{L^2}^{1/6}\|\Delta^2 u\|_{L^2}^{11/6}\\
&\le C\|\nabla u\|_{L^2}^{14}+\frac{1}{6}\left\|\Delta^2 u\right\|_{L^2}^2,
\end{aligned}\tag{48}$$

here we have used the following Gagliardo-Nirenberg inequality:

$$\|\nabla\Delta u\|_{L^4} \le C\|\nabla u\|_{L^2}^{1/6}\|\Delta^2 u\|_{L^2}^{5/6}. \tag{49}$$

For J_2, by the above inequalities used for J_1 and product estimate (21), we have

$$\begin{aligned}
J_2 &= \int_{\mathbb{R}^3} \Delta(\nabla d\cdot\Delta\, d)\cdot\Delta^2 u\,dx \le \|\Delta(\nabla d\Delta\, d)\|_{L^2}\|\Delta^2 u\|_{L^2}\\
&\le \left(\|\nabla d\|_{L^4}\|\Delta^2 d\|_{L^4} + \|\nabla\Delta\, d\|_{L^4}\|\Delta\, d\|_{L^4}\right)\|\Delta^2 u\|_{L^2}\\
&\le C\|\nabla d\|_{L^4}^2\|\Delta^2 d\|_{L^4}^2 + C\|\nabla\Delta\, d\|_{L^4}^2\|\Delta\, d\|_{L^4}^2 + \frac{1}{6}\|\nabla\Delta u\|_{L^2}^2\\
&\le C\|d\|_{L^\infty}\|\Delta\, d\|_{L^2}\|\Delta\, d\|_{L^2}^{1/6}\|\nabla\Delta^2 d\|_{L^2}^{11/6} + C\|\Delta\, d\|_{L^2}^{5/6}\|\nabla\Delta^2 d\|_{L^2}^{7/6}\|\Delta\, d\|_{L^2}^{3/2}\|\nabla\Delta^2 d\|_{L^2}^{1/2}\\
&\quad + \frac{1}{6}\|\nabla\Delta u\|_{L^2}^2\\
&\le C\|\Delta\, d\|_{L^2}^{7/6}\|\nabla\Delta^2 d\|_{L^2}^{11/6} + C\|\Delta\, d\|_{L^2}^{7/3}\|\Delta^2 d\|_{L^2}^{5/3} + \frac{1}{6}\|\Delta^2 u\|_{L^2}^2\\
&\le C\|\Delta\, d\|_{L^2}^{14} + \frac{1}{6}\|\nabla\Delta^2 d\|_{L^2}^2 + \frac{1}{6}\|\nabla\Delta u\|_{L^2}^2,
\end{aligned} \tag{50}$$

here we have used the following Gagliardo-Nirenberg inequalities:

$$\begin{aligned}
&\|\nabla d\|_{L^4} \le C\|d\|_{L^\infty}^{1/2}\|\Delta\, d\|_{L^2}^{1/2},\ \|\Delta\, d\|_{L^4} \le C\|\Delta\, d\|_{L^2}^{3/4}\|\nabla\Delta^2 d\|_{L^2}^{1/4},\\
&\|\nabla\Delta\, d\|_{L^4} \le C\|\Delta\, d\|_{L^2}^{5/12}\|\nabla\Delta^2 d\|_{L^2}^{7/12},\ \|\Delta^2 d\|_{L^4} \le C\|\Delta\, d\|_{L^2}^{1/12}\|\nabla\Delta^2 d\|_{L^2}^{11/12}.
\end{aligned} \tag{51}$$

For J_3, similar as (48) and by the above Gagliardo-Nirenberg inequalities, one may conclude

$$\begin{aligned}
J_3 &= -\int_{\mathbb{R}^3}\left[\Delta^2, u\cdot\nabla\right]d\cdot\Delta^2 d\,dx \le \left\|\left[\Delta^2, u\cdot\nabla\right]d\right\|_{L^{\frac{4}{3}}}\|\Delta^2 d\|_{L^4}\\
&\le \left(\|\nabla u\|_{L^2}\|\Delta^2 d\|_{L^4} + \|\nabla d\|_{L^4}\|\Delta^2 u\|_{L^2}\right)\|\Delta^2 d\|_{L^4}\\
&\le C\|\nabla u\|_{L^2}\|\Delta^2 d\|_{L^4}^2 + C\|\nabla d\|_{L^4}^2\|\Delta^2 d\|_{L^4}^2 + \frac{1}{6}\|\Delta^2 u\|_{L^2}^2\\
&\le C\|\nabla u\|_{L^2}\|\Delta^2 d\|_{L^2}^{1/6}\|\nabla\Delta^2 d\|_{L^2}^{11/6} + C\|d\|_{L^\infty}\|\Delta\, d\|_{L^2}\|\Delta\, d\|_{L^2}^{1/6}\|\nabla\Delta^2 d\|_{L^2}^{11/6} + \frac{1}{6}\|\Delta^2 u\|_{L^2}^2\\
&\le C\|\nabla u\|_{L^2}^{12}\|\Delta\, d\|_{L^2}^2 + C\|\Delta\, d\|_{L^2}^{14} + \frac{1}{6}\|\nabla\Delta^2 d\|_{L^2}^2 + \frac{1}{6}\|\Delta^2 u\|_{L^2}^2.
\end{aligned} \tag{52}$$

For J_4, by the product estimate (21) and the fact $|\nabla d|^2 = -d \cdot \Delta d$, we infer that

$$\begin{aligned} J_4 &= -\int_{\mathbb{R}^3} \Delta^2\left(|\nabla d|^2 d\right) \cdot \Delta^2 d \, d\, x = \int_{\mathbb{R}^3} \nabla\Delta\left(|\nabla d|^2 d\right) \cdot \left(\nabla\Delta^2 d\right) \mathrm{d}x \\ &\leq \left\|\nabla\Delta\left(|\nabla d|^2 d\right)\right\| L^2 \left\|\nabla\Delta^2 d\right\|_{L^2} \\ &\leq C\left(\left\|\nabla\Delta\left(|\nabla d|^2\right) d\right\|_{L^2} + \left\||\nabla d|^2 \nabla\Delta\, d\right\|_{L^2}\right)\left\|\nabla\Delta^2 d\right\|_{L^2} \\ &\leq C\left(\|\nabla d\|_{L^4}\left\|\Delta^2 d\right\|_{L^4}\|d\|_{L^\infty} + \|\Delta\, d\|_{L^4}\|\nabla\Delta\, d\|_{L^4}\|d\|_{L^\infty}\right)\left\|\nabla\Delta^2 d\right\|_{L^2} \\ &\leq C\left(\|\Delta\, d\|_{L^2}^{1/2}\|\Delta\, d\|_{L^2}^{1/12}\left\|\nabla\Delta^2 d\right\|_{L^2}^{11/12} + \|\Delta\, d\|_{L^2}^{3/4}\left\|\nabla\Delta^2 d\right\|_{L^2}^{1/4}\|\Delta\, d\|_{L^2}^{5/12}\left\|\nabla\Delta^2 d\right\|_{L^2}^{7/12}\right)\left\|\nabla\Delta^2 d\right\|_{L^2} \\ &\leq C\left(\|\Delta\, d\|_{L^2}^{7/12}\left\|\nabla\Delta^2 d\right\|_{L^2}^{11/12} + \|\Delta\, d\|_{L^2}^{7/6}\left\|\nabla\Delta^2 d\right\|_{L^2}^{5/6}\right)\left\|\nabla\Delta^2 d\right\|_{L^2} \\ &\leq C\|\Delta\, d\|_{L^2}^{14} + \frac{1}{6}\left\|\nabla\Delta^2 d\right\|_{L^2}^2. \end{aligned} \tag{53}$$

Inserting the above estimates (48), (50), (52), (53) to (47), and combining (46) yields

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(1 + \|\nabla\Delta u\|_{L^2}^2 + \left\|\Delta^2 d\right\|_{L^2}^2\right) + \left\|\Delta^2 u\right\|_{L^2}^2 + \left\|\nabla\Delta^2 d\right\|_{L^2}^2 \\ &\leq C\left(\|\nabla u\|_{L^2}^{14} + \|\Delta\, d\|_{L^2}^{14}\right) \leq CC_0 C(T)(1 + M(t))^{7C\epsilon}. \end{aligned} \tag{54}$$

Integrating the above inequality with respect to time from T_0 to t, $T_0 \leq t < T$, it follows that

$$\begin{aligned} &\left(1 + \|\nabla\Delta u(t)\|_{L^2}^2 + \left\|\Delta^2 d(t)\right\|_{L^2}^2\right) + \int_{T_0}^t \left(\left\|\Delta^2 u\right\|_{L^2}^2 + \left\|\nabla\Delta^2 d\right\|_{L^2}^2\right)\mathrm{d}\tau \\ &\leq 1 + \left\|\nabla\Delta u\left(T_0\right)\right\|_{L^2}^2 + \left\|\Delta^2 d\left(T_0\right)\right\|_{L^2}^2 + \int_{T_0}^t CC_0 C(T)(1 + M(\tau))^{7C\epsilon}\mathrm{d}\tau \\ &= 1 + \left\|\nabla\Delta u\left(T_0\right)\right\|_{L^2}^2 + \left\|\Delta^2 d\left(T_0\right)\right\|_{L^2}^2 + \int_{T_0}^t CC_0 C(T)(1 + M(\tau))\mathrm{d}\tau, \end{aligned} \tag{55}$$

here we choose $\epsilon = 1/7C$. The above inequality and equality (45) imply that

$$\begin{aligned} &(1 + M(t)) + \int_{T_0}^t \left(\|\nabla\Delta u\|_{L^2}^2\right) + \left\|\Delta^2 d\right\|_{L^2}^2 \mathrm{d}\tau \\ &\leq 1 + \left\|\nabla\Delta u\left(T_0\right)\right\|_{L^2}^2 + \left\|\Delta^2 d\left(T_0\right)\right\|_{L^2}^2 + \int_{T_0}^t CC_0 C(T)(1 + M(\tau))\mathrm{d}\tau. \end{aligned} \tag{56}$$

Therefore, employing Gronwall's inequality leads to

$$\begin{aligned}
&(1+M(t))+\int_{T_0}^{t}\left(\|\nabla\Delta u\|_{L^2}^2+\left\|\Delta^2 d\right\|_{L^2}^2\right)\mathrm{d}\tau\\
&\quad\leq\left(1+\left\|\nabla\Delta u\left(T_0\right)\right\|_{L^2}^2+\left\|\Delta^2 d\left(T_0\right)\right\|_{L^2}^2\right)\exp\left\{CC_0C(T)\left(T-T_0\right)\right\},
\end{aligned}\tag{57}$$

which indicates the truth of equality (25). Thus the Proof of Theorem 1 is completed. □

Proof of Theorem 2. For the proof of Theorem 2, we only need reestimate I_1 again. By the Lemma 1, we have

$$\begin{aligned}
I_1&\leq C\int_{\mathbb{R}^3}\left|\nabla_h u_h\right||\nabla u|^2\mathrm{d}x\leq C\|\nabla_h u_h\|_{L^4}\|\nabla u\|_{L^{8/3}}^2\\
&\leq C\left\|\nabla_h u_h\right\|_{\dot{B}_{\infty,\infty}^{-r}}^{1/2}\left\|\nabla_h u_h\right\|_{\dot{H}^r}^{1/2}\|\nabla u\|_{L^2}^{5/4}\|\Delta u\|_{L^2}^{3/4}\\
&\leq C\left\|\nabla_h u_h\right\|_{\dot{B}_{\infty,\infty}^{-r}}^{1/2}\left\|\nabla_h u_h\right\|_{L^2}^{1-r/2}\|\Delta u\|_{L^2}^{r/2}\|\nabla u\|_{L^2}^{5/4}\|\Delta u\|_{L^2}^{3/4}\\
&\leq C\left\|\nabla_h u_h\right\|_{\dot{B}_{\infty,\infty}^{-r}}^{1/2}\|\nabla u\|_{L^2}^{7-2r/4}\|\Delta u\|_{L^2}^{3+2r/4}\leq C\left\|\nabla_h u_h\right\|_{\dot{B}_{\infty,\infty}^{-r}}^{4/5-2r}\|\nabla u\|_{L^2}^{4/5-2r}\|\nabla u\|_{L^2}^2+\frac{1}{4}\|\Delta u\|_{L^2}^2\\
&\leq C\left(\left\|\nabla_h u_h\right\|_{\dot{B}_{\infty,\infty}^{-r}}^{4/3-2r}+\|\nabla u\|_{L^2}^2\right)\|\nabla u\|_{L^2}^2+\frac{1}{4}\|\Delta u\|_{L^2}^2\\
&\leq C\left(\|\nabla u\|_{L^2}^2+\frac{\left\|\nabla_h u_h\right\|_{\dot{B}_{\infty,\infty}^{-r}}^{4/3-2r}}{\sqrt{1+\log\left(1+\|\nabla u\|_{\dot{B}_{\infty,\infty}^{0}}+\|\nabla d\|_{\dot{B}_{\infty,\infty}^{0}}\right)}}\right)\log\left(1+\|\nabla\Delta u\|_{L^2}+\left\|\Delta^2 d\right\|_{L^2}\right)\\
&\quad\times\|\nabla u\|_{L^2}^2+\frac{1}{4}\|\Delta u\|_{L^2}^2.
\end{aligned}\tag{58}$$

Going the same process to (39)–(57), the desired result will be obtained. Thus the Proof of Theorem 2 is completed. □

Data Availability

There is no underlying data.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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