Hindawi
Journal of Mathematics
Volume 2022, Article ID 3482254, 7 pages
https://doi.org/10.1155/2022/3482254

Research Article

Orthogonally C^*-Ternary Jordan Homomorphisms and Jordan Derivations: Solution and Stability

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Received 28 February 2022; Revised 23 April 2022; Accepted 12 December 2022; Published 26 December 2022

Academic Editor: Li Guo

In this work, by using some orthogonally fixed point theorem, we prove the stability and hyperstability of orthogonally C^*-ternary Jordan homomorphisms between C^*-ternary Banach algebras and orthogonally C^*-ternary Jordan derivations of some functional equation on C^*-ternary Banach algebras.

1. Introduction and Preliminaries

A classical question in the sense of a functional equation says that "when is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?" Ulam [1] raised the question of stability of functional equations and Hyers [2] was the first to give an affirmative answer to the question of Ulam for additive mapping between Banach spaces. In 1987, Rassias [3] proved a generalized version of the Hyers' theorem for approximately additive maps. The study of stability problem of functional equations have been done by several authors on different spaces such as Banach, C^*-Banach algebras and modular spaces (for example see [4–13]). One of the stimulating aspects is to examine the stability of those functional equations whose general solutions exist and are useful in characterizing entropies [14].

Recently, Eshaghi Gordji et al. [15] introduced the notion of the orthogonal set, which contains the notion of orthogonality in normed space. The study on orthogonal sets has been done by several authors (for example, see [16–18])

Definition 1 (see [15]). Let $X \neq \varnothing$ and $\perp \subseteq X \times X$ be a binary relation. If there exists $x_0 \in X$ such that for all $y \in X$,

$$y \perp x_0 \text{ or } x_0 \perp y. \tag{1}$$

Then $\perp$ is called an orthogonally set (briefly O-set). We denote this O-set by $(X, \perp)$.

Let $(X, \perp)$ be an O-set and (X, d) be a generalized metric space, then $(X, \perp, d)$ is called orthogonally generalized metric space.

Let $(X, \perp, d)$ be an orthogonally metric space.

(i) A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called orthogonally sequence (briefly O-sequence) if for any $n \in \mathbb{N}$,

$$x_n \perp x_{n+1} \text{ or } x_{n+1} \perp x_n. \tag{2}$$

(ii) Mapping $f\colon X \longrightarrow X$ is called $\perp-$ continuous in $x \in X$ if for each O-sequence $\{x_n\}_{n\in\mathbb{N}}$ in X with $x_n \longrightarrow x$, then $f(x_n) \longrightarrow f(x)$. Clearly, every continuous map is $\perp-$ continuous at any $x \in X$.

(iii) $(X, \perp, d)$ is called orthogonally complete (briefly O-complete) if every Cauchy O-sequence is convergent to a point in X.

(iv) Mapping $f: X \longrightarrow X$ is called $\perp$-preserving if for all $x, y \in X$ with $x \perp y$, then $f(x) \perp f(y)$.

(v) A mapping $f: X \longrightarrow X$ is said to be orthogonally contraction (or $\perp - \lambda$-contraction) with Lipschitz constant $0 < \lambda < 1$ if

$$d(f(x), f(y)) \leq \lambda\, d(x, y) if\, x \perp y. \tag{3}$$

By using the concept of orthogonally sets, Bahraini et al. [19], proved the generalization of the Diaz and Margolis [20] fixed point theorem on these sets.

Theorem 1 (see [19]). *Let $(X, d, \perp)$ be an O-complete generalized metric space. Let $T: X \longrightarrow X$ be a $\perp$-preserving, $\perp$-continuous, and $\perp$-λ-contraction. Let $x_0 \in X$ be such that for all $y \in X$, $x_0 \perp y$ or for all $y \in X$, $y \perp x_0$, and consider the "O-sequence of successive approximations with initial element x_0": x_0, $T(x_0)$, $T^2(x_0)$, $\ldots$, $T^n(x_0)$, $\ldots$. Then, either $d(T^n(x_0), T^{n+1}(x_0)) = \infty$ for all $n \geq 0$, or there exists a positive integer n_0 such that $d(T^n(x_0), T^{n+1}(x_0)) < \infty$ for all $n > n_0$. If the second alternative holds, then*

(i) the O-sequence of $\{T^n(x_0)\}$ is convergent to a fixed point x^* of T.

(ii) x^* is the unique fixed point of T in $X^* = \{y \in X: d(T^n(x_0), y) < \infty\}$.

(iii) If $y \in X$, then

$$d(y, x^*) \leq \frac{1}{1-\lambda} d(y, T(y)). \tag{4}$$

A C^*-ternary Banach algebra $\mathfrak{A}$, endowed with a ternary product $(x, y, z) \longrightarrow [x, y, z]$ of $\mathfrak{A}^3$ into $\mathfrak{A}$, is a complex Banach space in which the product is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $[[\backslash[x, y, z], u, w] = [x, [y, z, u]w] = [x, y, [z, u, w]]$, for all x, y, z, u, w in $\mathfrak{A}$ and satisfies $\|[x, y, z]\| \leq \|x\|.\|y\|.\|z\|$, $\|[x, x, x]\| = \|x\|^3$ (see [21]). If $(\mathfrak{A}, \cdot)$ is a usual C^*-algebra, then an induced ternary multiplication can defined by $[u, v, w]: = u \cdot v^* \cdot w$. If a C^*-ternary Banach algebra $\mathfrak{A}$ has a unital "e" such that $u = [u, e, e] = [e, e, u]$ for all $u \in \mathfrak{A}$, then $\mathfrak{A}$ with binary product $u \cdot v: = [u, e, v]$ and $u^*: = [e, u, e]$, is a unital C^*-algebra (see [22]).

Definition 2. A $\mathbb{C}$-linear mapping between C^*-ternary Banach algebras $\mathfrak{A}, \mathfrak{B}$; i.e. $H: \mathfrak{A} \longrightarrow \mathfrak{B}$, is called

(1) C^*-ternary homomorphism if

$$H([x, y, z]) = [H(x), H(y), H(z)],\ H(x^*) = H(x)^*. \tag{5}$$

(2) C^*-ternary Jordan homomorphism if

$$H([x, x, x]) = [H(x), H(x), H(x)],\ H(x^*) = H(x)^*. \tag{6}$$

For all $x, y, z \in \mathfrak{A}$.

Definition 3. A $\mathbb{C}$-linear mapping $D: \mathfrak{A} \longrightarrow \mathfrak{A}$ is called

(1) C^*-ternary derivation if

$$D([x, y, z]) = [D(x), y, z] + [x, D(y), z] + [x, y, D(z)],\ D(x^*) = D(x)^*. \tag{7}$$

(2) C^*-ternary Jordan derivation if

$$D([x, x, x]) = [D(x), x, x] + [x, D(x), x] + [x, x, D(x)],\ D(x^*) = D(x)^*. \tag{8}$$

For all $x, y, z \in \mathfrak{A}$.

To prove main results we use the following equivalent assertions.

Lemma 1 (see [23]). *Let $f: \mathfrak{A} \longrightarrow \mathfrak{B}$ be a mapping such that*

$$\left\| f\left(\frac{y-x}{3}\right) + f\left(\frac{x-3z}{3}\right) + f\left(\frac{3x+3z-y}{3}\right) \right\| \leq \|f(x)\|, \tag{9}$$

for all $x, y, z \in \mathfrak{A}$. Then f is additive.

Lemma 2 (see [24]). *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two ternary Banach algebras. Let $f: \mathfrak{A} \longrightarrow \mathfrak{B}$ be an additive mapping. Then the following assertions are equivalent:*

(a) $f[a, a, a]) = [f(a), f(a), f(a)]$, *for all $a \in \mathfrak{A}$.*

(b) $f([a, b, c] + [b, c, a] + [c, a, b]) = [f(a), f(b), f(c)] + [f(b), f(c), f(a)] + [f(c), f(a), f(b)], \quad \forall a, b, c \in \mathfrak{A}$.

Lemma 3 (see[25]). *Let $\mathfrak{A}$ be a ternary Banach algebras. Let f be an additive mapping from $\mathfrak{A}$ into $\mathfrak{A}$. Then the following assertions are equivalent:*

(a) $f([a,a,a]) = [f(a),a,a] + [a,f(a),a] + [a,a,f(a)]$, *for all* $a \in \mathfrak{A}$

(b) $f([a,b,c] + [b,c,a] + [c,a,b]) = [f(a),b,c] + [a,f(b),c] + [a,b,f(c)] + [f(b),c,a] + [b,f(c),a] + [b,c,f(a)] + [f(c),a,b] + [c,f(a),b] + [c,a,f(b)]$, *for all* $a,b,c \in \mathfrak{A}$.

In this paper, motivated by the works of [15, 23, 26], we prove the stability of orthogonally C^*-ternary Jordan homomorphism and orthogonally C^*-ternary Jordan derivation of the functional equation

$$f\left(\frac{ty-x}{3}\right) + f\left(\frac{x-3tz}{3}\right) + tf\left(\frac{3x+3z-y}{3}\right) = tf(x). \tag{10}$$

On orthogonally C^*-ternary Banach algebras, where t belongs to the set of all complex numbers $e^{i\theta}$ with $0 \leq \theta \leq (2\pi/n_0)$ for some fixed positive integer number n_0.

2. Main Results

Throughout the paper, let $\mathbb{T}^1_{1/n_0}$ be the set of all complex numbers $e^{i\theta}$, where $0 \leq \theta \leq (2\pi/n_0)$ and n_0 is a fixed positive integer number and let $\mathfrak{A}, \mathfrak{B}$ be two C^*-ternary Banach algebras.

For simplicity, denote

$$\Phi_f(x,y,z,t) = f\left(\frac{ty-x}{3}\right) + f\left(\frac{x-3tz}{3}\right) + tf\left(\frac{3x+3z-y}{3}\right) - tf(x), \tag{11}$$

$$\begin{aligned}\Psi_f(x,y,z) = {} & f([x,y,z] + [y,z,x] + [z,x,y]) - [f(x),f(y),f(z)] - [f(y),f(z),f(x)] \\ & -[f(z),f(x),f(y)],\end{aligned} \tag{12}$$

where $x,y,z \in \mathfrak{A}$ and $t \in \mathbb{T}^1_{(1/n_0)}$.

Suppose that φ and ψ are two mappings from $\mathfrak{A}^3$ into $[0,\infty)$ such that for all $x,y,z \in \mathfrak{A}$ with $x \perp y$, $x \perp z$, $y \perp z$

$$\varphi(x,y,z) \leq \frac{L}{3}\varphi(3x,3y,3z), \tag{13}$$

$$\psi(x,y,z) \leq \frac{L}{3^3}\psi(3x,3y,3z), \tag{14}$$

for some constant $0 < L < 1$.

Now, we are ready to prove the stability of orthogonally C^*-ternary Jordan homomorphism in C^*-ternary Banach algebras.

Theorem 2. *Let* $f\colon \mathfrak{A} \longrightarrow \mathfrak{B}$ *be a mapping for which*

$$\left\|\Phi_f(x,y,z,t)\right\| \leq \varphi(x,y,z), \tag{15}$$

and

$$\left\|\Psi_f(x,y,z)\right\| \leq \psi(x,y,z), \tag{16}$$

and

$$\left\|f(x^*) - f(x)^*\right\| \leq \varphi(x,x,x). \tag{17}$$

For all $t \in \mathbb{T}^1_{(1/n_0)}$, and $x,y,z \in \mathfrak{A}$ with $x \perp y$, $x \perp z$, $y \perp z$, whose φ and ψ are defined as (13) and (14). Then there exists a unique orthogonally C^*-ternary Jordan homomorphism $H\colon \mathfrak{A} \longrightarrow \mathfrak{B}$ such that

$$\|H(x) - f(x)\| \leq \frac{L}{1-L}\varphi(x,2x,0), \tag{18}$$

for all $x \in \mathfrak{A}$.

Proof. Let Δ be the set of all mappings $g\colon \mathfrak{A} \longrightarrow \mathfrak{B}$ such that $g(x) \perp 3g((1/3)x)$ or $3g((1/3)x) \perp g(x)$, for all $x \in \mathfrak{A}$. Define d_φ on Δ by

$$d_\varphi(g,h) = \inf\{\alpha \in (0,\infty)\colon \|g(x) - h(x)\| \leq \alpha\varphi(x,2x,0) \quad \forall x \in \mathfrak{A}\}, \tag{19}$$

and suppose that, for all $g,h \in \Delta$, $h \perp g$ if and only if

$$h(x) \perp g(x) \text{ or } g(x) \perp h(x). \tag{20}$$

For all $x \in \mathfrak{A}$.

Clearly $(X, \Delta, \perp)$ is an O-complete generalized metric space. Define $\Lambda\colon \Delta \longrightarrow \Delta$ by $\Lambda g(x) = 3g((1/3)x)$, $x \in \mathfrak{A}$. Then

$$\|\Lambda g(x) - \Lambda h(x)\| = \left\|3g\left(\frac{1}{3}x\right) - 3h\left(\frac{1}{3}x\right)\right\| \leq 3\alpha\varphi\left(\frac{x}{3},\frac{2x}{3},0\right) \leq L\alpha\varphi(x,2x,0). \tag{21}$$

So, by definition of d_φ on Δ, for every $g, h \in \Delta$ with $g \perp h$ or $h \perp g$ and $d_\varphi(g,h) < \alpha$ we have $\|\Lambda g - \Lambda h\| \le \alpha L$. This shows that $d_\varphi(\Lambda g, \Lambda h) \le L d_\varphi(g,h)$, i.e. Λ is $\perp - \lambda$-contraction. The function Λ is $\perp$-continuous. In fact, if $\{g_n\}$ is an O-sequence in Δ which converges to $g \in \Delta$, then for given $\varepsilon > 0$, there exists $\alpha > 0$ with $\alpha < \varepsilon$ and $n \in \mathbb{N}$ such that

$$\|g_n(x) - g(x)\| \le \alpha\varphi(x, 2x, 0). \tag{22}$$

For all $x \in \mathfrak{A}$ and $n \in \mathbb{N}$. Therefore by the similar argument, for all $x \in \mathfrak{A}$ and $n \ge N$, we have

$$d_\varphi(\Lambda(g_n), \Lambda(g)) \le L\alpha < L\varepsilon. \tag{23}$$

Clearly, Λ is $\perp$-preserving.

We show that for any $f \in \Delta$, we have

$$d_\varphi\left(\Lambda^{n+1} f, \Lambda^n f\right) \le \infty. \tag{24}$$

In (11), put $t = 1$, $y = (2x/3^{n-1})$, $z = 0$ and $x: = (x/3^{n-1})$. By induction we have

$$\left\|3f\left(\frac{x}{3^n}\right) - f\left(\frac{x}{3^n}\right)\right\| \le \frac{L^{n-1}}{3^{n-1}}\varphi(x, 2x, 0). \tag{25}$$

Then for $L \in (0,1)$,

$$\|\Lambda^{n+1} f - \Lambda^n f\| = 3^n\left\|3f\left(\frac{x}{3^{n+1}}\right) - f\left(\frac{x}{3^n}\right)\right\| \le L^n\varphi(x, 2x, 0) \longrightarrow 0 \text{ as } n \longrightarrow +\infty, \tag{26}$$

and then, all conditions of Theorem 1 hold.

So, the O-sequence $\{\Lambda^n f\}$ converges to the unique fixed point H in the set of $\left\{g \in \Delta: d_\varphi(\Lambda^n f, g) < \infty\right\}$, i.e.,

$$H(x) = \lim_{n\longrightarrow\infty} \Lambda^n f = \lim_{n\longrightarrow\infty} 3^n f\left(\frac{x}{3^n}\right). \tag{27}$$

Also, for $f \in \Delta$,

$$d_\varphi(f, H) \le \frac{1}{1-L} d_\varphi(f, \Lambda f), \tag{28}$$

and by (26), $d(f, H) \le L\varphi(x, 2x, 0)$. Therefore, H satisfies in (18), i.e.,

$$\|H(x) - f(x)\| \le \frac{L}{1-L}\varphi(x, 2x, 0). \tag{29}$$

We claim that H is the unique desired orthogonally C^*-ternary Jordan homomorphism which satisfies in (18).

First of all, H is a additive. In fact, for all $t \in \mathbb{T}^1_{(1/n_0)}$, $x, y, z \in \mathfrak{A}$ with $x \perp y$, $x \perp z$, $y \perp z$ and using (13), we have

$$\begin{aligned}\|\Phi_H(x,y,z,t)\| &= \left\|H\left(\frac{ty - x}{3}\right) + H\left(\frac{x - 3tz}{3}\right) + tH\left(\frac{3x + 3z - y}{3}\right) - tH(x)\right\| = \lim_{n\longrightarrow\infty} 3^n \\ &\left\|f\left(\frac{ty - x}{3^{n+1}}\right) + f\left(\frac{x - 3tz}{3^{n+1}}\right) + tf\left(\frac{3x + 3z - y}{3^{n+1}}\right) - tf\left(\frac{x}{3^n}\right)\right\| \le \lim_{n\longrightarrow\infty} 3^n\varphi\left(\frac{x}{3^n}, \frac{y}{3^n}, \frac{z}{3^n}\right) = 0.\end{aligned} \tag{30}$$

So by Lemma 1, H is additive. By the same proof of Theorem 3 of [27], the mapping H is $\mathbb{C}$-linear. We show that H is unique. Let H' be another additive mapping satisfying (18). Then, we have

$$\begin{aligned}\|H(x) - H'(x)\| &= 3^n\left\|H\left(\frac{x}{3^n}\right) - H'\left(\frac{x}{3^n}\right)\right\| \le 3^n\left\|H\left(\frac{x}{3^n}\right) - f\left(\frac{x}{3^n}\right)\right\| + 3^n\left\|H'\left(\frac{x}{3^n}\right) - f\left(\frac{x}{3^n}\right)\right\| \\ &\le 2\cdot 3^n\varphi\left(\frac{x}{3^n}, \frac{2x}{3^n}, 0\right) \le 2Ł^n\varphi(x, 2x, 0).\end{aligned} \tag{31}$$

For all $x \in \mathfrak{A}$. Letting $n \longrightarrow \infty$ shows that H is unique. Now, by using (16)

$$\begin{aligned}\left\|\Psi_H(x,y,z)\right\| &= \|H([x,y,z]+[y,z,x]+[z,x,y]) - [H(x),H(y),H(z)] - [H(y),H(z),H(x)] - [H(z),H(x),H(y)]\| \\ &= \lim_{n\longrightarrow\infty} 3^{3n}\left\|\left(f\left[\frac{x}{3^n},\frac{y}{3^n},\frac{z}{3^n}\right]+\left[\frac{y}{3^n},\frac{z}{3^n},\frac{x}{3^n}\right]+\left[\frac{z}{3^n},\frac{x}{3^n},\frac{y}{3^n}\right]\right)-\left[f\left(\frac{x}{3^n}\right),f\left(\frac{y}{3^n}\right),f\left(\frac{z}{3^n}\right)\right]-\left[f\left(\frac{y}{3^n}\right),f\left(\frac{z}{3^n}\right),f\left(\frac{x}{3^n}\right)\right]-\left[f\left(\frac{z}{3^n}\right),f\left(\frac{x}{3^n}\right),f\left(\frac{y}{3^n}\right)\right]\right\| \\ &\le \lim_{n\longrightarrow\infty} 3^{3n}\psi\left(\frac{x}{3^n},\frac{y}{3^n},\frac{z}{3^n}\right) = 0,\end{aligned} \tag{32}$$

and then (14) implies that $\Psi_H(x,y,z) = 0$ for all $x,y,z \in \mathfrak{A}$ with $x\perp y,\ x\perp z,\ y\perp z$. On the other hand, by (13) and (17) we have

$$\begin{aligned}\left\|H(x^*) - H(x)^*\right\| &= \lim_{n\longrightarrow\infty} 3^n\left\|f\left(\frac{x^*}{3^n}\right) - f\left(\frac{x}{3^n}\right)^*\right\| \\ &\le \lim_{n\longrightarrow\infty} 3^n\varphi\left(\frac{x}{3^n},\frac{x}{3^n},\frac{x}{3^n}\right) = 0.\end{aligned} \tag{33}$$

For all $x \in \mathfrak{A}$. Therefore H is an orthogonally C^*-ternary Jordan homomorphism satisfying (18). □

In the next theorem, we prove that the self-mapping f with the same appropriate conditions which satisfied in the functional (11), can be approximated by an orthogonally C^*-ternary Jordan derivation.

Denote

$$\begin{aligned}W_f(x,y,z) = f([x,y,z]+[y,z,x]+[z,x,y]) - [f(x),y,z] - [x,f(y),z] - [x,y,f(z)] - [f(y),z,x] \\ -[y,f(z),x] - [y,z,f(x)] - [f(z),x,y] - [z,f(x),y] - [z,x,f(y)].\end{aligned} \tag{34}$$

Theorem 3. *Let $f\colon \mathfrak{A} \longrightarrow \mathfrak{A}$ be a mapping satisfying (17) such that*

$$\left\|\Phi_f(x,y,z,t)\right\| \le \varphi(x,y,z), \tag{35}$$

and

$$\left\|W_f(x,y,z)\right\| \le \psi(x,y,z). \tag{36}$$

For all $t \in \mathbb{T}^1_{(1/n_0)}$ and $x,y,z \in \mathfrak{A}$ with $x\perp y,\ x\perp z,\ y\perp z$ where mappings φ and ψ are satisfied in (13) and (14). Then, there exists a unique orthogonally C^*-ternary Jordan derivation $D\colon \mathfrak{A} \longrightarrow \mathfrak{A}$ such that

$$\|D(x) - f(x)\| \le \frac{L}{1-L}\varphi(x,2x,0), \tag{37}$$

for all $x \in \mathfrak{A}$.

Proof. Similar to proof of Theorem 2, there exists a self-mapping D on $\mathfrak{A}$ defined by $D(x):=\lim_{n\longrightarrow\infty}3^n f(x/3^n)$ satisfies (37). By using Lemma 3 and definition of $D(x)$ we have

$$\begin{aligned}\left\|W_D(x,y,z)\right\| &= \|D([x,y,z]+[y,z,x]+[z,x,y]) - [D(x),y,z] - [x,D(y),z] - [x,y,D(z)] \\ &\quad - [D(y),z,x] - [y,D(z),x] - [y,z,D(x)] - [D(z),x,y] - [z,D(x),y] - [z,x,D(y)]\| \\ &= \lim_{n\longrightarrow\infty} 3^{3n}\left\|f\left(\left[\frac{x}{3^n},\frac{y}{3^n},\frac{z}{3^n}\right]+\left[\frac{y}{3^n},\frac{z}{3^n},\frac{x}{3^n}\right]+\left[\frac{z}{3^n},\frac{x}{3^n},\frac{y}{3^n}\right]\right)-\left[f\left(\frac{x}{3^n}\right),\frac{y}{3^n},\frac{z}{3^n}\right]\right. \\ &\quad -\left[\frac{x}{3^n},f\left(\frac{y}{3^n}\right),\frac{z}{3^n}\right]-\left[\frac{x}{3^n},\frac{y}{3^n},f\left(\frac{z}{3^n}\right)\right] \\ &\quad \left.-\left[f\left(\frac{y}{3^n}\right),\frac{z}{3^n},\frac{x}{3^n}\right]-\left[\frac{y}{3^n},f\left(\frac{z}{3^n}\right),\frac{x}{3^n}\right]-\left[\frac{y}{3^n},\frac{z}{3^n},f\left(\frac{x}{3^n}\right)\right]-\left[f\left(\frac{z}{3^n}\right),\frac{x}{3^n},\frac{y}{3^n}\right]-\left[\frac{z}{3^n},f\left(\frac{x}{3^n}\right),\frac{y}{3^n}\right]-\left[\frac{z}{3^n},\frac{x}{3^n},f\left(\frac{y}{3^n}\right)\right]\right\| \\ &\le \lim_{n\longrightarrow\infty} 3^{3n}\psi\left(\frac{x}{3^n},\frac{y}{3^n},\frac{z}{3^n}\right) = 0,\end{aligned} \tag{38}$$

for all $x, y, z \in \mathfrak{A}$ with $x \perp y,\ x \perp z,\ y \perp z$. So $W_D(x, y, z) = 0$ for all $x, y, z \in \mathfrak{A}$ with $x \perp y,\ x \perp z,\ y \perp z$. Thus, the mapping $D\colon \mathfrak{A} \longrightarrow \mathfrak{A}$ is a unique orthogonally C^*-ternary Jordan derivation satisfies (37). Also, by the same argument in the proof of Theorem 2,

$$D(x^*) = D(x)^*. \tag{39}$$

□

Theorems 1 and 2 generalized the result of Rassias [3], whenever we define

$$\begin{aligned} \varphi(x, y, z) &= \theta(\|x\|^p + \|y\|^p + \|z\|^p),\\ \psi(x, y, z) &= \theta\left(\|x\|^{3p} + \|y\|^{3p} + \|z\|^{3p}\right). \end{aligned} \tag{40}$$

For all $\theta \in \mathbb{R}^+$ and $p \neq 1$, in the sense of orthogonal sets.

As a consequence of Theorem 1, we have hyperstability of orthogonally C^*-ternary Jordan homomorphism between C^*-ternary Banach algebras.

Theorem 4. *Let $p \neq 1$ and θ be nonnegative real numbers, and let $f\colon \mathfrak{A} \longrightarrow \mathfrak{B}$ be a mapping such that*

$$\left\| f\left(\frac{ty - x}{3}\right) + f\left(\frac{x - 3tz}{3}\right) + tf\left(\frac{3x + 3z - y}{3}\right) \right\| \leq \|tf(x)\|, \tag{41}$$

$$\left\| \Psi_f(x, y, z) \right\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p). \tag{42}$$

For all $t \in \mathbb{T}^1_{1/n_0}$ and all $x, y, z \in \mathfrak{A}$ with $x \perp y,\ x \perp z,\ y \perp z$. Then, the mapping $f\colon \mathfrak{A} \longrightarrow \mathfrak{B}$ is a orthogonally C^*-ternary Jordan homomorphism.

From Theorem 3, we obtain hyperstability of orthogonally C^*-ternary Jordan derivation.

Theorem 5. *Let $p \neq 1$ and θ be nonnegative real numbers. Let $f\colon \mathfrak{A} \longrightarrow \mathfrak{A}$ be a mapping satisfies (41) such that*

$$\left\| W_f(x, y, z) \right\| \leq \theta(\|x\|^p + \|y\|^p + \|z\|^p). \tag{43}$$

For all $x, y, z \in \mathfrak{A}$ with $x \perp y,\ x \perp z,\ y \perp z$. Then the mapping $f\colon \mathfrak{A} \longrightarrow \mathfrak{A}$ is an orthogonally C^*-ternary Jordan derivation.

3. Conclusions

In this paper, we introduced orthogonally C^*-ternary Jordan homomorphism and C^*-ternary Jordan derivation. Using an orthogonally fixed point theorem, we proved that orthogonally C^*-ternary Jordan homomorphism and orthogonally C^*-ternary Jordan derivation of the functional (11) can be stable and hyperstable in the orthogonally C^*-ternary Banach algebras. The Hyers–Ulam stability theory has many attractions and applications in the field of fractional calculus. For farther research in this field we suggest to see the paper [28, 29].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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