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Research Article

Pluriharmonic Mappings with the Convex Holomorphic Part

Ma Lihua [1] **and See Keong Lee** [2]

[1] *Key Laboratory of High Performance Computing and Stochastic Information Processing (HPCSIP) (Ministry of Education of China), School of Mathematics and Statistics, Hunan Normal University, Changsha, Hunan 410081, China*
[2] *School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM, Penang, Malaysia*

Correspondence should be addressed to Ma Lihua; maryjiahua@scau.edu.cn

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In 2018, Partyka et al. established several equivalent conditions for a sense-preserving locally injective harmonic mapping $f = h + \overline{g}$ in the unit disk $\mathbb{D}$ with convex holomorphic part h to be quasiconformal in terms of the relationships of two-point distortion of h, g, and f. In this study, we first generalize the above result to the case of pluriharmonic mappings $f_A = h + A\overline{g}$, where h is a convex mapping in the unit ball $\mathbb{B}^n$ and $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ with $\|A\| = 1$. Then, we establish a relationship of two-point distortion property between f and f_A.

1. Introduction and Main Results

For $n \geq 1$, let $\mathbb{C}^n$ denote the n-dimensional complex Euclidean space. Also, we identify each point $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ with a column vector. For two column vectors $z, w \in \mathbb{C}^n$, set $\langle z, w\rangle = \sum_{j=1}^{n} z_j \overline{w}_j$ and $|z| = \langle z, z\rangle^{1/2}$. We use $\mathbb{B}^n(z_0, r)$ and $\mathbb{S}^{n-1}(z_0, r)$ to denote the open ball $\{z \in \mathbb{C}^n\colon |z - z_0| < r\}$ and its boundary $\{z \in \mathbb{C}^n\colon |z - z_0| = r\}$, respectively. In particular, let $\mathbb{B}^n(0,1) = \mathbb{B}^n$ and $\mathbb{S}^{n-1}(0,1) = \mathbb{S}^{n-1}$. Also, we identify $\mathbb{B}^1$ with the unit disk $\mathbb{D}$.

For an $n \times n$ complex matrix A, the operator norm of A is defined by

$$\|A\| = \sup\left\{|A\xi|\colon \xi \in \mathbb{S}^{n-1}\right\}. \tag{1}$$

We use $L(\mathbb{C}^n, \mathbb{C}^m)$ to denote the space of continuous linear operators from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the standard operator norm, and let I_n be the identity operator in $L(\mathbb{C}^n, \mathbb{C}^n)$.

For two domains $\Omega_1, \Omega_2 \subset \mathbb{C}^n$, let $f = (f_1, \ldots, f_n)$ be a holomorphic mapping from Ω_1 into Ω_2. Then, the complex Jacobian matrix of f at $z \in \Omega_1$ is given by

$$Df(z) = \begin{pmatrix} \dfrac{\partial f_1(z)}{\partial z_1} & \cdots & \dfrac{\partial f_1(z)}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_n(z)}{\partial z_1} & \cdots & \dfrac{\partial f_n(z)}{\partial z_n} \end{pmatrix} = \left(\nabla f_1(z), \ldots, \nabla f_n(z)\right)^T, \tag{2}$$

where T means the matrix transpose, and ∇f_j are understood as column vectors. Furthermore, let $\overline{D}f(z)$ be the conjugate of Jacobian matrix $Df(z)$as follows:

$$\overline{D}f(z) = \begin{pmatrix} \dfrac{\partial f_1(z)}{\partial \overline{z}_1} & \cdots & \dfrac{\partial f_1(z)}{\partial \overline{z}_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_n(z)}{\partial \overline{z}_1} & \cdots & \dfrac{\partial f_n(z)}{\partial \overline{z}_n} \end{pmatrix}. \tag{3}$$

If $\det Df(z) \neq 0$ for every $z \in \Omega_1$, then we say that f is locally biholomorphic in Ω_1 (cf. [1]). If f is one-to-one, onto and locally biholomorphic, then f is said to be biholomorphic (cf. ([2], Page 55)).

A complex-valued function f of class C^2 in $\mathbb{B}^n$ is said to be pluriharmonic if its restriction to every complex line is harmonic, which is equivalent to the fact that for all $z \in \mathbb{B}^n$ and $j, k \in \{1, 2, \ldots, n\}$,

$$\frac{\partial^2}{\partial z_j \partial \overline{z}_k} f(z) \equiv 0. \quad (4)$$

Every pluriharmonic mapping $f: \mathbb{B}^n \longrightarrow \mathbb{C}^n$ can be written as $f = h + \overline{g}$, where h, g are the holomorphic mappings, and this representation is unique if $g(0) = 0$ (cf. [1–5]).

If $f = h + \overline{g}$: $\mathbb{B}^n \longrightarrow \mathbb{C}^n$ is pluriharmonic and h is locally biholomorphic, we denote by

$$\omega_f(z) = Dg(z)[Dh(z)]^{-1}, \left\|\omega_f\right\|_\infty = \sup_{z\in\mathbb{B}^n}\left\{\left\|\omega_f(z)\right\|\right\}, \quad (5)$$

and we use J_f to denote the real Jacobian matrix of f (cf. [4]). Then, for any $z \in \mathbb{B}^n$,

$$J_f(z) = \begin{pmatrix} Dh(z) & \overline{Dg(z)} \\ Dg(z) & \overline{Dh(z)} \end{pmatrix}, \quad (6)$$

and

$$\det J_f(z) = |\det Dh(z)|^2 \det\left(I_n - \omega_f(z)\overline{\omega_f(z)}\right). \quad (7)$$

Hence, f is sense-preserving, i.e., $\det J_f(z) > 0$ in $\mathbb{B}^n$, if and only if h is locally biholomorphic, and $\det(I_n - \omega_f(z)\overline{\omega_f(z)}) > 0$.

If $f = h + \overline{g}$ is a sense-preserving harmonic mapping from $\mathbb{D}$ into $\mathbb{C}$, it is known that $\det J_f(z) = |h'(z)|^2 - |g'(z)|^2$ and the dilation $\omega_f(z) = g'(z)/h'(z)$ is analytic with the property that $|\omega_f(z)| < 1$ (see [6, 7]). Especially, if f is a diffeomorphism with $\|\omega_f\|_\infty < 1$, then f is called a quasiconformal mapping.

A domain $\Omega \subset \mathbb{C}^n$ is said to be M-linearly connected if there is a constant $M > 0$ such that any two points w_1, $w_2 \in \Omega$ can be connected by a smooth curve $\gamma \subset \Omega$ with length

$$l(\gamma) \leq M\left|w_1 - w_2\right|. \quad (8)$$

It is clear that any convex domain is 1-linearly connected. For extensive discussion on linearly connected domains, see [4, 8–12]. For a biholomorphic mapping f, if f maps $\mathbb{B}^n$ onto a convex domain, then we say that f is convex in $\mathbb{B}^n$ (cf. [13]).

For sense-preserving harmonic mapping $f = h + \overline{g}$ defined on $\mathbb{D}$, Chuaqui and Hérnandez [10] showed that if $f(\mathbb{D})$ is M-linearly connected and $|\omega_f| < 1/(1 + 2M)$, then the deformation $F = h + a\overline{g}$, $|a| = 1$, is univalent. Kalaj [14] proved a more general result when $f(\mathbb{D})$ is convex that for every a with $|a| \leq 1$, F is an $|a|$-quasiconformal close-to-convex harmonic mapping.

We say that a mapping $f: \Omega \longrightarrow \mathbb{C}^n$ is in Lip_α if there exist a constant c_1 and an exponent $\alpha \in (0, 1]$ such that for all $z, w \in \Omega$,

$$|f(z) - f(w)| \leq c_1|z - w|^\alpha. \quad (9)$$

Such mappings are also called α-Hölder continuous. In particular, if $\alpha = 1$, then we say that f is Lipschitz continuous. A mapping $f: \Omega \longrightarrow \mathbb{C}^n$ is said to be co-Lipschitz continuous if there exists a constant c_2 such that for all z, $w \in \Omega$,

$$|f(z) - f(w)| \geq c_2|z - w|. \quad (10)$$

If f is both Lipschitz continuous and co-Lipschitz continuous in Ω, then f is called bi-Lipschitz.

In 2012, Partyka and Sakan established several equivalent conditions for a sense-preserving harmonic mapping $f = h + \overline{g}$ from $\mathbb{D}$ onto a bounded convex domain to be quasiconformal in terms of the relationships of two-point distortion of h, g, and f (see [15], Theorem 3.8]). Later, Partyka et al. [12] generalized the result to the case where f is a sense-preserving and locally injective harmonic mapping and h is a convex holomorphic mapping.

Theorem 1 (see [12], Theorem 3.3). *Let $f = h + \overline{g}$ be a sense-preserving harmonic mapping in $\mathbb{D}$ such that h is convex. Then, f is injective, and the following five conditions are equivalent to each other:*

(1) f is a quasiconformal mapping

(2) There exists a constant L_1 such that $L_1 \in [1, 2)$ and

$$\left|f(z_2) - f(z_1)\right| \leq L_1\left|h(z_2) - h(z_1)\right|, \quad z_1, z_2 \in \mathbb{D}. \quad (11)$$

(3) There exists a constant l_1 such that $l_1 \in [0, 1)$ and

$$\left|g(z_2) - g(z_1)\right| \leq l_1\left|h(z_2) - h(z_1)\right|, \quad z_1, z_2 \in \mathbb{D}. \quad (12)$$

(4) There exists a constant $L_2 \geq 1$ such that

$$\left|h(z_2) - h(z_1)\right| \leq L_2\left|f(z_2) - f(z_1)\right|, \quad z_1, z_2 \in \mathbb{D}. \quad (13)$$

(5) $h^\circ f^{-1}$ and $f^\circ h^{-1}$ are bi-Lipschitz mappings.

Let $f = h + \overline{g}$ be a pluriharmonic mapping in $\mathbb{B}^n$. For simplicity, here and hereafter, we always use f_A to denote the pluriharmonic mapping $h + A\overline{g}$, where $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ with $\|A\| \leq 1$. Obviously, $\omega_{f_A}(z) = \overline{A}\omega_f(z)$.

As the first aim of this study, we establish the following counterpart of ([12], Theorem 3.3) in the setting of pluriharmonic mappings.

Theorem 2. *Let $f = h + \overline{g}$: $\mathbb{B}^n \longrightarrow \mathbb{C}^n$ be a pluriharmonic mapping, where h is convex in $\mathbb{B}^n$ and $\|\omega_f\|_\infty \leq 1$. Then, the following five statements are equivalent:*

(i) There exists a constant N such that $\|\omega_f\|_\infty \leq N < 1$

(ii) There exists a constant $L_1 \in [1, 2)$ such that for any $z_1, z_2 \in \mathbb{B}^n$ and $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ with $\|A\| = 1$,

$$\left|f_A(z_1) - f_A(z_2)\right| \leq L_1\left|h(z_1) - h(z_2)\right|. \quad (14)$$

(iii) *There exists a constant* $l_1 \in [0,1)$ *such that for any* $z_1, z_2 \in \mathbb{B}^n$,

$$|g(z_1) - g(z_2)| \leq l_1 |h(z_1) - h(z_2)|. \tag{15}$$

(iv) *There exists a constant* $L_2 \geq 1$ *such that for any* $z_1, z_2 \in \mathbb{B}^n$ *and* $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ *with* $\|A\| = 1$,

$$|h(z_1) - h(z_2)| \leq L_2 |f_A(z_1) - f_A(z_2)|. \tag{16}$$

(v) *For any* $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ *with* $\|A\| = 1$, $h^\circ f_A^{-1}$ *and* $f_A \circ h^{-1}$ *are bi-Lipschitz mappings.*

As the second aim of this study, we establish a relationship of two-point distortion property between f and f_A. Our result is as follows.

Theorem 3. *Let* $f = h + \overline{g}\colon \mathbb{B}^n \longrightarrow \mathbb{C}^n$ *be a pluriharmonic mapping, where* h *is biholomorphic and* $h(\mathbb{B}^n)$ *is* M*-linearly connected with constant* $M \geq 1$. *Suppose that there exists a constant* $N \in [0, 1/M)$ *such that* $\|\omega_f\|_\infty \leq N$. *Then, for any* $z_1, z_2 \in \mathbb{B}^n$ *and* $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ *with* $\|A\| \leq 1$, *there exist two positive constants* c_1 *and* c_2 *such that*

$$c_2 |f(z_1) - f(z_2)| \leq |f_A(z_1) - f_A(z_2)| \leq c_1 |f(z_1) - f(z_2)|, \tag{17}$$

where $c_1 = M(1+N)^2/(1-N)(1-MN)$ *and* $c_2 = 1 - MN/1 + MN$. *In particular,*

$$\frac{1-MN}{(1+MN)^2} |f(z_1) - f(z_2)| \leq |h(z_1) - h(z_2)| \leq \frac{M(1+N)}{(1-N)(1-MN)} |f(z_1) - f(z_2)|. \tag{18}$$

The proofs of Theorems 2 and 3 will be given in Section 2.

2. Proofs of Main Results

The aim of this section is to prove Theorems 2 and 3. Before proving Theorem 2, we need some preparation, which consists of three lemmas.

Lemma 1. *Let* $f = h + \overline{g}\colon \mathbb{B}^n \longrightarrow \mathbb{C}^n$ *be a pluriharmonic mapping, where* h *is biholomorphic and* $h(\mathbb{B}^n)$ *is* M*-linearly connected with constant* $M \geq 1$. *Suppose that there exists a constant* $N \in [0,1]$ *such that* $\|\omega_f\|_\infty \leq N$. *Then, for any* z_1, $z_2 \in \mathbb{B}^n$,

$$|g(z_1) - g(z_2)| \leq MN |h(z_1) - h(z_2)|, \tag{19}$$

and for any $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ *with* $\|A\| \leq 1$,

$$|f_A(z_1) - f_A(z_2)| \leq (1 + MN) |h(z_1) - h(z_2)|. \tag{20}$$

Proof. The proof of (19) is based upon the ideas from Theorem 2.1 [4]. The details are as follows.

For any distinct points z_1, $z_2 \in \mathbb{B}^n$, let $w_1 = h(z_1)$ and $w_2 = h(z_2)$. Then, the M-linear connectivity of $h(\mathbb{B}^n)$ implies that there exists a smooth curve $\gamma\colon [0,1] \longrightarrow h(\mathbb{B}^n)$ between w_1 and w_2 such that $\gamma(0) = w_1$, $\gamma(1) = w_2$, and $l(\gamma) \leq M|w_1 - w_2|$. Since h is a biholomorphic mapping, we see that $\sigma = h^{-1} \circ \gamma$ is a curve in $\mathbb{B}^n$ joining z_1 and z_2. Then, the assumption $\|\omega_f\|_\infty \leq N$ implies

$$\begin{aligned} |g(z_1) - g(z_2)| &= \left| \int_0^1 D(g^\circ \sigma)(s) \mathrm{d}s \right| = \left| \int_0^1 Dg(\sigma(s)) [Dh(\sigma(s))]^{-1} \gamma\prime(s) \mathrm{d}s \right| \\ &\leq N \int_0^1 |\mathrm{d}\gamma(s)| = N l(\gamma) \\ &\leq MN |h(z_1) - h(z_2)|, \end{aligned} \tag{21}$$

which yields (19). Moreover, for any $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ with $\|A\| \leq 1$, we have that

$$|f_A(z_1) - f_A(z_2)| \leq |h(z_1) - h(z_2)| + \|A\| \cdot |g(z_1) - g(z_2)| \leq (1 + MN) |h(z_1) - h(z_2)|, \tag{22}$$

and so the proof of this lemma is complete. □

The following result is a converse of Lemma 1.

Lemma 2. *Let $f = h + \overline{g}\colon \mathbb{B}^n \longrightarrow \mathbb{C}^n$ be a pluriharmonic mapping, where h is biholomorphic. Suppose that $\|\omega_f\|_\infty \leq 1$ and there exists a constant $N \geq 1$ such that for any $z_1, z_2 \in \mathbb{B}^n$ and $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ with $\|A\| = 1$,*

$$\left|h(z_1) - h(z_2)\right| \leq N\left|f_A(z_1) - f_A(z_2)\right|. \tag{23}$$

Then, $\|\omega_f\|_\infty \leq 1 - 1/N$.

Proof. For any distinct points $w_1, w_2 \in h(\mathbb{B}^n)$, let $z_1 = h^{-1}(w_1)$ and $z_2 = h^{-1}(w_2)$, where $z_1, z_2 \in \mathbb{B}^n$ and $z_1 \neq z_2$. Then, for any $w_1 \in \mathbb{B}^n(w_2, d_{h(\mathbb{B}^n)}(w_2))$, we have

$$\begin{aligned} g\left(h^{-1}(w_1)\right) - g\left(h^{-1}(w_2)\right) &= D\left(g^\circ h^{-1}\right)(w_2)\cdot(w_1 - w_2) + o\left(|w_1 - w_2|\right) \\ &= Dg(z_2)\left[Dh(z_2)\right]^{-1}\cdot(w_1 - w_2) + o\left(|w_1 - w_2|\right), \end{aligned} \tag{24}$$

where $d_{h(\mathbb{B}^n)}(w_2)$ denotes the distance of w_2 to the boundary of $h(\mathbb{B}^n)$ and $o(|w_1 - w_2|)$ denotes a vector in $\mathbb{C}^n$ with $\lim_{w_1 \longrightarrow w_2}|o(|w_1 - w_2|)|/|w_1 - w_2| = 0$. It follows from (23) and (24) that

$$\begin{aligned} |w_1 - w_2| &\leq N\left|w_1 - w_2 + A\left(\overline{g\left(h^{-1}(w)\right) - g\left(h^{-1}(w_2)\right)}\right)\right| \\ &\leq N\left|\overline{w_1} - \overline{w_2} + \overline{A}Dg(z_2)\left[Dh(z_2)\right]^{-1}\cdot(w_1 - w_2) + o\left(|w_1 - w_2|\right)\right|. \end{aligned} \tag{25}$$

For fixed $z_2 \in \mathbb{B}^n$, we choose some $w_1 \in \mathbb{B}^n(w_2, d_{h(\mathbb{B}^n)}(w_2))\backslash\{w_2\}$ such that

$$\left|Dg(z_2)\left[Dh(z_2)\right]^{-1}\cdot\frac{w_1 - w_2}{|w_1 - w_2|}\right| = \left\|Dg(z_2)\left[Dh(z_2)\right]^{-1}\right\|. \tag{26}$$

Then, there exists $\eta \in \mathbb{S}^{n-1}$ such that

$$Dg(z_2)\left[Dh(z_2)\right]^{-1}\cdot\frac{w_1 - w_2}{|w_1 - w_2|} = \eta\cdot\left\|Dg(z_2)\left[Dh(z_2)\right]^{-1}\right\|. \tag{27}$$

Choose $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ satisfying $\|A\| = 1$ and

$$A\overline{\eta} = -\frac{w_1 - w_2}{|w_1 - w_2|}. \tag{28}$$

Then, it follows from (25), (27) and (28) that

$$\begin{aligned} \frac{1}{N} &\leq \lim_{w_1 \longrightarrow w_2}\left|\frac{\overline{w_1} - \overline{w_2}}{|w_1 - w_2|} - \frac{\overline{w_1} - \overline{w_2}}{|w_1 - w_2|}\cdot\left\|Dg(z_2)\left[Dh(z_2)\right]^{-1}\right\| + \frac{o\left(|w_1 - w_2|\right)}{|w_1 - w_2|}\right| \\ &= 1 - \left\|Dg(z_2)\left[Dh(z_2)\right]^{-1}\right\|, \end{aligned} \tag{29}$$

which, together with the arbitrary of z_2, shows that $\|\omega_f\|_\infty \leq 1 - 1/N$, as needed. □

Lemma 3. *Let $f = h + \overline{g}\colon \mathbb{B}^n \longrightarrow \mathbb{C}^n$ be a pluriharmonic mapping, where h is biholomorphic and $h(\mathbb{B}^n)$ is M-linearly connected with constant $M \geq 1$. Suppose that there exists a constant $N \in [0, 1]$ such that for any z_1, $z_2 \in \mathbb{B}^n$ and $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ with $\|A\| = 1$,*

$$\left|f_A(z_1) - f_A(z_2)\right| \leq (1 + N)\left|h(z_1) - h(z_2)\right|. \tag{30}$$

Then,

$$\left|g(z_1) - g(z_2)\right| \leq MN\left|h(z_1) - h(z_2)\right|. \tag{31}$$

Proof. By Lemma 1, we know that to prove (31), it suffices to prove

$$\left\|\omega_f\right\|_\infty \leq N. \tag{32}$$

For any distinct points $w_1, w_2 \in h(\mathbb{B}^n)$, let $z_1 = h^{-1}(w_1)$ and $z_2 = h^{-1}(w_2)$, where $z_1, z_2 \in \mathbb{B}^n$ and $z_1 \neq z_2$. Then, for any $w_1 \in \mathbb{B}^n(w_2, d_{h(\mathbb{B}^n)}(w_2))\backslash\{w_2\}$, it follows from (24) and (30) that

$$\left|\frac{\overline{w_1 - w_2}}{|w_1 - w_2|} + \overline{A}Dg(z_2)[Dh(z_2)]^{-1}\cdot\frac{w_1 - w_2}{|w_1 - w_2|} + \frac{o(|w_1 - w_2|)}{|w_1 - w_2|}\right| \le 1 + N. \tag{33}$$

For fixed $z_2 \in \mathbb{B}^n$, similar to (27), we know that there exist some $\eta \in \mathbb{S}^{n-1}$ and $w_1 \in \mathbb{B}^n(w_2, d_{h(\mathbb{B}^n)}(w_2))\backslash\{w_2\}$ such that

$$Dg(z_2)[Dh(z_2)]^{-1}\cdot\frac{w_1 - w_2}{|w_1 - w_2|} = \eta\cdot\left\|Dg(z_2)[Dh(z_2)]^{-1}\right\|. \tag{34}$$

Choose $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ satisfying $\|A\| = 1$ and

$$A\overline{\eta} = \frac{w_1 - w_2}{|w_1 - w_2|}. \tag{35}$$

Combining (33)–(35) and letting $w_1 \longrightarrow w_2$, we get $\|\omega_f\|_\infty \le N$, as required. In addition, the inequality (31) can be derived from the proof of ([4], Theorem 1.2).

Now, we are ready to prove Theorem 2. □

Proof of Theorem 2. Since every convex domain is a 1-linearly connected domain, the implication from Statement (i) to (ii) follows from Lemma 1, and the implication from Statement (ii) to (iii) follows from Lemma 3. Furthermore, the implication from Statement (iii) to (iv) follows from the triangle inequality since the assumption that $|g(z_1) - g(z_2)| \le l_1|h(z_1) - h(z_2)|$ for any $z_1, z_2 \in \mathbb{B}^n$ implies

$$|f_A(z_1) - f_A(z_2)| \ge |h(z_1) - h(z_2)| - \|A\|\cdot|g(z_1) - g(z_2)| \ge (1 - l_1)|h(z_1) - h(z_2)|, \tag{36}$$

where $l_1 \in [0, 1)$ and $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ with $\|A\| = 1$. The implication from Statement (v) to (i) follows from Lemma 2. Hence, to prove the theorem, it remains to prove the implication from Statement (iv) to (v).

Assume that Statement (iv) holds true, which means that there exists a constant $L_2 \ge 1$ such that for any $z_1, z_2 \in \mathbb{B}^n$ and $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ with $\|A\| = 1$,

$$|h(z_2) - h(z_1)| \le L_2|f_A(z_2) - f_A(z_1)|. \tag{37}$$

Then, we infer from Lemma 2 that $\|\omega_f\|_\infty \le 1 - 1/L_2$. This, together with Lemma 1, implies that

$$|f_A(z_1) - f_A(z_2)| \le \left(2 - \frac{1}{L_2}\right)|h(z_1) - h(z_2)|. \tag{38}$$

Note that the assumption "h is convex in $\mathbb{B}^n$" implies that h is a biholomorphic mapping. This, together with the above two inequalities, yields that both h and f_A are univalent in $\mathbb{B}^n$ and

$$\frac{1}{L_2}|h(z_1) - h(z_2)| \le |f_A(z_1) - f_A(z_2)| \le \left(2 - \frac{1}{L_2}\right)|h(z_1) - h(z_2)|. \tag{39}$$

Let $w_1 = f_A(z_1)$ and $w_2 = f_A(z_2)$. Obviously, we have that

$$\frac{1}{L_2}|h^\circ f_A^{-1}(w_1) - h^\circ f_A^{-1}(w_2)| \le |w_1 - w_2| \le \left(2 - \frac{1}{L_2}\right)|h^\circ f_A^{-1}(w_1) - h^\circ f_A^{-1}(w_2)|, \tag{40}$$

which means that $h^\circ f_A^{-1}$ is bi-Lipschitz continuous in $f_A(\mathbb{B}^n)$. Similarly, we see that $f_A \circ h^{-1}$ is bi-Lipschitz continuous in $h(\mathbb{B}^n)$. These show that Statement (v) is true. The proof of the theorem is complete.

In order to prove Theorem 3, we also need some preparation. First, let us recall a known result, which is useful for the Proof of Theorem 3. □

Theorem 4 (see [3], Lemma A). *Let A be an $n\times n$ complex matrix with $\|A\|<1$. Then, $I_n\pm A$ are nonsingular matrices and*

$$\left\|\left(I_n\pm A\right)^{-1}\right\|\le\frac{1}{(1-\|A\|)}. \tag{41}$$

Proof. For a complex matrix A, any kind of operator norm has the property. Let x be any nonzero vector, then

$$\left\|(I_n-A)x\right\|=\|x-Ax\|\ge\|x\|-\|Ax\|\ge\|x\|-\|A\|\|x\|=(1-\|A\|)\|x\|>0. \tag{42}$$

If $x\neq 0$, then $(I_n-A)x\neq 0$. So, for $(I_n-A)x\neq 0$, no more than zero solution, then the matrix I_n-A is nonsingular.

When I_n-A is nonsingular, we have

$$\begin{aligned}&\left(I_n-A\right)\left(I_n-A\right)^{-1}=I_n,\\&\left(I_n-A\right)^{-1}=\left[\left(I_n-A\right)+A\right]\left(I_n-A\right)^{-1}=\left(I_n-A\right)\left(I_n-A\right)^{-1}+A\left(I_n-A\right)^{-1}=I_n+A\left(I_n-A\right)^{-1}.\end{aligned} \tag{43}$$

Now,

$$\begin{aligned}\left\|\left(I_n-A\right)^{-1}\right\|&\le\left\|I_n\right\|+\|A\|\left\|\left(I_n-A\right)^{-1}\right\|=1+\|A\|\left\|\left(I_n-A\right)^{-1}\right\|,\\\left\|\left(I_n-A\right)^{-1}\right\|&\le\frac{1}{1-\|A\|}.\end{aligned} \tag{44}$$

□

Lemma 4. *Let $f=h+\overline{g}\colon\mathbb{B}^n\longrightarrow\mathbb{C}^n$ be a univalent pluriharmonic mapping, where h is locally biholomorphic and $f(\mathbb{B}^n)$ is M-linearly connected with constant $M\ge 1$. Suppose that there exists a constant $N\in[0,1)$ such that $\|\omega_f\|_\infty\le N$. Then, for any $z_1,z_2\in\mathbb{B}^n$,*

$$\left|g\left(z_1\right)-g\left(z_2\right)\right|\le\frac{MN}{1-N}\left|f\left(z_1\right)-f\left(z_2\right)\right|, \tag{45}$$

$$\left|h\left(z_1\right)-h\left(z_2\right)\right|\le\frac{M}{1-N}\left|f\left(z_1\right)-f\left(z_2\right)\right|. \tag{46}$$

Proof. The proof of the lemma is based upon the ideas from ([4], Theorem 2.3). The details are as follows.

First, we prove inequality (45). For any distinct points $w_1,w_2\in f(\mathbb{B}^n)$, it follows from the M-linear connectivity of $f(\mathbb{B}^n)$ that there exists a smooth curve $\gamma\colon[0,1]\longrightarrow f(\mathbb{B}^n)$ connecting w_1 and w_2 with $\gamma(0)=w_1$, $\gamma(1)=w_2$, and $l(\gamma)\le M|w_1-w_2|$. Since f is univalent, we assume that $z_1=f^{-1}(w_1)$ and $z_2=f^{-1}(w_2)$, and hence $\sigma=f^{-1}\circ\gamma$ is a curve in $\mathbb{B}^n$ joining z_1 and z_2.

By the assumption $\|\omega_f\|_\infty\le N<1$, the inverse mapping theorem and Theorem 4, we know that f^{-1} is differentiable (cf. [3]). Moreover, by [3], (28), we have

$$Df^{-1}=[Dh]^{-1}\left(I_n-\overline{Dg}[\overline{Dh}]^{-1}Dg\ [Dh]^{-1}\right)^{-1}, \tag{47}$$

$$\overline{D}f^{-1}=-[Dh]^{-1}\left(I_n-\overline{Dg}[\overline{Dh}]^{-1}Dg\ [Dh]^{-1}\right)^{-1}\overline{Dg}[\overline{Dh}]^{-1}. \tag{48}$$

For $s\in[0,1]$, let $z=\sigma(s)=(f^{-1}\circ\gamma)(s)\in\mathbb{B}^n$. Therefore, it follows from (47) and (48) that

$$\begin{aligned}
\left|g(z_1)-g(z_2)\right| &= \left|\int_\sigma Dg(z)\mathrm{d}z\right| = \left|\int_0^1 Dg(\sigma(s))\cdot\sigma\prime(s)\mathrm{d}s\right|\\
&= \left|\int_0^1 Dg(\sigma(s))\cdot\left(Df^{-1}(\gamma(s))\cdot\gamma\prime(s)+\overline{D}f^{-1}(\gamma(s))\cdot\overline{\gamma\prime(s)}\right)\mathrm{d}s\right|\\
&\le \left|\int_0^1 \omega_f(\sigma(s))\left(I_n-\overline{\omega_f(\sigma(s))}\cdot\omega_f(\sigma(s))\right)^{-1}\cdot\gamma'(s)\mathrm{d}s\right|\\
&\quad+\left|\int_0^1 \omega_f(\sigma(s))\left(I_n-\overline{\omega_f(\sigma(s))}\cdot\omega_f(\sigma(s))\right)^{-1}\cdot\overline{\omega_f(\sigma(s))}\cdot\overline{\gamma\prime(s)}\mathrm{d}s\right|.
\end{aligned} \tag{49}$$

Furthermore, by Theorem 4, the assumption $\|\omega_f\|_\infty\le N$ and the M-linear connectivity of $f(\mathbb{B}^n)$, we know that

$$\begin{aligned}
\left|g(z_1)-g(z_2)\right| &\le \int_0^1\left(\frac{\left\|\omega_f(\sigma(s))\right\|}{1-\left\|\omega_f(\sigma(s))\right\|^2}+\frac{\left\|\omega_f(\sigma(s))\right\|^2}{1-\left\|\omega_f(\sigma(s))\right\|^2}\right)\cdot|\gamma\prime(s)|\mathrm{d}s\\
&= \int_0^1\frac{\left\|\omega_f(\sigma(s))\right\|}{1-\left\|\omega_f(\sigma(s))\right\|}|\mathrm{d}\gamma(s)|\le\frac{N}{1-N}l(\gamma)\\
&\le\frac{MN}{1-N}\left|f(z_1)-f(z_2)\right|,
\end{aligned} \tag{50}$$

which yields the inequality (45).

Similarly, by applying (47), (48), Theorem 4, the assumption $\|\omega_f\|_\infty\le N$ and the M-linear connectivity of $f(\mathbb{B}^n)$, we get

$$\begin{aligned}
\left|h(z_1)-h(z_2)\right| &= \left|\int_\sigma Dh(z)\mathrm{d}z\right| = \left|\int_0^1 Dh(\sigma(s))\cdot\sigma\prime(s)\mathrm{d}s\right|\\
&= \left|\int_0^1 Dh(\sigma(s))\cdot\left(Df^{-1}(\gamma(s))\cdot\gamma\prime(s)+\overline{D}f^{-1}(\gamma(s))\cdot\overline{\gamma\prime(s)}\right)\mathrm{d}s\right|\\
&\le \left|\int_0^1 \left(I_n-\overline{\omega_f(\sigma(s))}\cdot\omega_f(\sigma(s))\right)^{-1}\cdot\gamma\prime(s)\mathrm{d}s\right|\\
&\quad+\left|\int_0^1 \left(I_n-\overline{\omega_f(\sigma(s))}\cdot\omega_f(\sigma(s))\right)^{-1}\cdot\overline{\omega_f(\sigma(s))}\cdot\overline{\gamma\prime(s)}\mathrm{d}s\right|\\
&\le \int_0^1\left(\frac{1}{1-\left\|\omega_f(\sigma(s))\right\|^2}+\frac{\left\|\omega_f(\sigma(s))\right\|}{1-\left\|\omega_f(\sigma(s))\right\|^2}\right)\cdot|\gamma\prime(s)|\mathrm{d}s\\
&= \int_0^1\frac{1}{1-\left\|\omega_f(\sigma(s))\right\|}|\mathrm{d}\gamma(s)|\le\frac{1}{1-N}l(\gamma)\\
&\le\frac{M}{1-N}\left|f(z_1)-f(z_2)\right|,
\end{aligned} \tag{51}$$

which leads to (46), and thus, the proof of this lemma is complete.

Based on Lemma 4, we have the following result. □

Lemma 5. *Let $f=h+\overline{g}\colon\mathbb{B}^n\longrightarrow\mathbb{C}^n$ be a pluriharmonic mapping, where h is biholomorphic and $h(\mathbb{B}^n)$ is M-linearly connected with constant $M\ge1$. Suppose that there exists a* *constant* $N \in [0, 1/M)$ *such that* $\|\omega_f\|_\infty \leq N$. *Then,* f *is a univalent and sense-preserving mapping, and for any* z_1, $z_2 \in \mathbb{B}^n$,

$$\begin{aligned}|g(z_1) - g(z_2)| &\leq \frac{N(1+N)M}{(1-N)(1-MN)}|f(z_1) - f(z_2)|,\\|h(z_1) - h(z_2)| &\leq \frac{(1+N)M}{(1-N)(1-MN)}|f(z_1) - f(z_2)|.\end{aligned} \tag{52}$$

Proof. By the proof of Theorem 2.1 [4], we know that f is a univalent and sense-preserving mapping and $f(\mathbb{B}^n)$ is a $(1+N)M/1-MN$-linearly connected domain. Then, the result of this lemma follows from Lemma 4.

Based on Lemmas 1 and 5, we can give the Proof of Theorem 3. □

Proof of Theorem 2. The inequalities in (18) follows from (17), (20), and (52). Hence, it remains to prove the inequalities in (17).

For any distinct points z_1, $z_2 \in \mathbb{B}^n$ and $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ with $\|A\| \leq 1$, we see from Lemma 5 that

$$\begin{aligned}|f_A(z_1) - f_A(z_2)| &\leq |h(z_1) - h(z_2)| + |g(z_1) - g(z_2)|\\&\leq \frac{(1+N)^2 M}{(1-N)(1-MN)}|f(z_1) - f(z_2)|.\end{aligned} \tag{53}$$

On the other hand, by Lemma 1 and the assumption "$MN < 1$," we get

$$\begin{aligned}|f_A(z_1) - f_A(z_2)| &\geq |h(z_1) - h(z_2)| - \|A\| \cdot |g(z_1) - g(z_2)|\\&\geq |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)|\\&\geq |h(z_1) - h(z_2)| - MN|h(z_1) - h(z_2)|\\&\geq \frac{1-MN}{1+MN}|f(z_1) - f(z_2)|.\end{aligned} \tag{54}$$

Hence, (17) follows, and the proof of this theorem is complete [16]. □

Data Availability

No data are available for this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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