Hindawi
Journal of Mathematics
Volume 2022, Article ID 2801582, 10 pages
https://doi.org/10.1155/2022/2801582

Research Article

Bayesian Prediction Intervals Based on Type-I Hybrid Censored Data from the Lomax Distribution under Step-Stress Model

Abdalla Rabie,[1] **Abd EL-Baset A. Ahmad**,[2] **Mohamad A. Fawzy**,[3,4] **and Tahani A. Aloafi**[5]

[1]*Department of Mathematics, Faculty of Science, Al-Azhar University, Assiut 71524, Egypt*
[2]*Department of Mathematics, Faculty of Science, Assiut University, Assiut, Egypt*
[3]*Department of Mathematics, Faculty of Science, Taibah University, Madinah, Saudi Arabia*
[4]*Department of Mathematics, Faculty of Science, Suez University, Suez, Egypt*
[5]*Department of Mathematics and Statistics, Faculty of Science, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia*

Correspondence should be addressed to Tahani A. Aloafi; t.aloafi@hotmail.com

Received 23 October 2022; Revised 6 December 2022; Accepted 7 December 2022; Published 23 December 2022

Academic Editor: Ali Sajid

The Bayesian prediction of future failures from Lomax distribution is the subject of this research. The observed data is censored using a Type-I hybrid censoring scheme under a step-stress partially accelerated life test. There are two types of sampling schemes considered: one-sample and two-sample. We create predictive intervals for failure observations in the future. Bayesian prediction intervals are constructed using MCMC algorithms. After all, two numerical examples, simulation study and a real-life example are provided for both one-sample and two-sample methods for the purpose of illustration.

1. Introduction

In highly industrial products, it is difficult to obtain sufficient information about the product to test its validity, so it is appropriate to use an acceleration life test to obtain enough information with saving time and high cost. There are two types of accelerated life tests: the fully accelerated life test, in which it is assumed that the acceleration factor is known and there must be a mathematical model that determines the relationship between the life span and stress. Another one is the partially accelerated life test which is applied when the relationship between the life span and stress is unknown and is applied throughout step stress and constant stress. Many authors have written on this topic, see for example, Saeed, and Hanieh [1]; Lone et al. [2] and Mohammad and Mohammad [3] and Nagy et al. [4].

It is important to use past data to predict future data in many experiments of life tests for units or products to improve product efficiency. One of the fascinating subjects in real-world reliability issues is prediction. Recently, there has been a lot of discussion about various prediction techniques.

In multiple fields as healthcare, finance, technology, and education, Bayesian prediction of future data plays a critical role. The Bayesian approach, according to Geisser [5]; can be used to tackle the problem of prediction. See also AL-Hussaini et al. [6, 7], Ahmadi et al. [8], Ahmad [9], Ahmad et al. [10], Ateya [11], Balakrishnan and Shafay [12], Kundu and Howlader [13], and Singh et al. [14], Ahmad et al. [15] and Corcuera and Giummole [16].

The most common type of prediction is predictive intervals. They are not the same as the commonly used confidence intervals and tolerance regions, which are intended to deal with unknown population parameters. A predictive interval is a range of values that uses the results of a previous sample to predict the outcomes of a future sample with a given probability.

Prediction concerns can be divided into two categories:

(1) One-sample prediction

Let $X_{1:n}, X_{2:n}, \ldots, X_{r:n}$ and $X_{r+1:n}, X_{r+2:n}, \ldots, X_{n:n}$ represent the known sample and a future sample, respectively. A one-sample prediction scheme entails predicting the future sample $X_{(s)}$ for $r < s \leq n$.

(2) Two-sample prediction

Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ and $Y_{1:m}, Y_{2:m}, \ldots, Y_{m:m}$ denote the informative sample of size n and a future sample of size m, respectively. The two samples are believed to be unrelated and drawn from the same distribution. The prediction of the future sample $Y_{1:m}, Y_{2:m}, \ldots, Y_{m:m}$ is part of a two-sample prediction issue.

For more details about one and two sample methods, one can refer to Prakash [17]; Wu and Gui [18]; and Mohie El-Din et al. [19].

1.1. The Model Description and Censoring Scheme. The Lomax distribution was introduced by Lomax [20] to analyse business failure data. Several authors have used Lomax distribution, for example, Howlader and Hossain [21], Abd-Ellah [22], Abd-Elfattah et al. [23], Hassan and Al-Ghamdi [24], among others. The cumulative distribution function (CDF) and the probability density function (PDF) of Lomax distribution are written, respectively, in the forms

$$\begin{aligned} F(x;\alpha,\theta) &= 1-(1+\alpha x)^{-\theta}, \quad x>0, (\alpha,\theta>0),\\ f(x;\alpha,\theta) &= \alpha\theta(1+\alpha x)^{-(\theta+1)}, \quad x>0, (\alpha,\theta>0), \end{aligned} \tag{1}$$

where, θ and α are the shape and scale parameters, respectively.

By mixing Type-I and Type-II censoring schemes, we obtain hybrid censoring scheme (HCS) in which the test is ended after obtaining a fixed number r of failures from n items or after reaching a pre-determined time T. So, we can observe the following types of HCSs.

(i) Type-I HCS: Epstein [25] considered a HCS in which the life-testing experiment is terminated at a random time $T_* = \min\{X_{r:n}, T\}$, where $1 \leq r \leq n$ and $T > 0$ are known before.

(ii) Type-II HCS: In this type the experiment is finished at a random time $T^* = \max\{X_{r:n}, T\}$, this ensures at least r of failures are obtained.

In this paper, we consider Type-I HCS which includes the following types of censored data:

Case 1: $X_{1:n} < X_{2:n} < \cdots < X_{r:n}$, if $X_{r:n} < T$, and pre-specified r number of failures occur before the censoring time T.

Case 2: $X_{1:n} < X_{2:n} < \cdots < X_{m:n}$, if $X_{m:n} \leq T < X_{r:n}$, and only m number of failures occur before the pre-specified censoring time T.

The rest of this paper is organized as follows: Section 2 gives the test method and procedure. In Section 3, the likelihood function and the posterior PDF are stated. Section 4 is devoted for Bayesian one-sample prediction scheme and MCMC method. Bayesian two-sample prediction and MCMC technique are presented in Section 5. Section 6 presents numerical computations involving numerical examples and simulation study, as well as real-life example. The findings are displayed in four tables then concluding the results and methodology in Section 7.

2. The Test Method

In this section, we explain the fundamental assumptions and test procedures for the step-stress partially accelerated life test (SSPALT) model based on the Type-I hybrid censoring method.

2.1. Fundamental Presumptions

(1) There are two stress levels: s_1 and s_2 (design and high).

(2) The life distribution of the test unit is Lomax distribution for any level of stress.

(3) An item's entire lifetime Y is calculated as follows:

$$Y = \begin{cases} T, & T \leq \tau,\\ \tau + \beta^{-1}(T-\tau), & T > \tau, \end{cases} \tag{2}$$

where, T is an item lifetime at normal use condition and $\beta > 1$, is the accelerating factor. This model discussed by Degroot and Goel [26] and referred to as *tampered random variable* (TRV).

(4) The lifetimes $Y_1, \ldots, Y_n$ of the n test items are independent and identically distributed random variables.

2.2. Test Procedure

(1) All n test items are put through a normal environment.

(2) If r number failures does not fail by a predetermined time τ in normal use, it is switched to accelerated mode and run until it fails or the censorship time is reached.

Based on the above assumptions, the CDF and PDF of a total lifetime Y of test item take the following forms, respectively,

$$F(y) = \begin{cases} 1-(1+\alpha y)^{-\theta}, & 0 \leq y \leq \tau,\\ 1-(1+\alpha\delta(y))^{-\theta}, & y > \tau, \end{cases} \tag{3}$$

where, $\delta(y) = \tau + \beta(y-\tau)$ and

$$f(y) = \begin{cases} \alpha\theta(1+\alpha y)^{-(\theta+1)}, & 0 \leq y \leq \tau,\\ \alpha\theta\beta(1+\alpha\delta(y))^{-(\theta+1)}, & y > \tau. \end{cases} \tag{4}$$

In Type-I HCS, the life-testing experiment is stopped according to the rule $T_2 = \min\{y_{r:n}, T_1\}$, that is the test is terminated at a predetermined time T_1 if the r^{th} failure obtained after T_1, or as soon as the r^{th} failure occurs, where $1 \le r \le n$. Thus,

We may obtain a random number of failures N^* given by

$$N^* = \begin{cases} \text{Case1: } r, & \text{if } y_{r:n} \le \tau \le T_1, \\ \text{Case2: } r - N_1, & \text{if } \tau < y_{r:n} \le T_1, \\ \text{Case3: } N_2, & \text{if } y_{r:n} > T_1. \end{cases} \tag{5}$$

3. The Likelihood Function and Posterior PDF

The likelihood function under Type-I HCS according to the previous assumptions can be written as:

For Case 1

$$\begin{aligned} L(\alpha, \theta \mid y) &= \frac{n!}{(n-r)!}\{1 - F(y_{r:n})\}^{n-r} \prod_{i=1}^{r} f(y_{i:n}) \\ &= \frac{n!\,(\alpha\theta)^r}{(n-r)!}\{1 + \alpha y_{r:n}\}^{\theta(r-n)} \prod_{i=1}^{r} (1 + \alpha y_{i:n})^{-(\theta+1)}, \end{aligned} \tag{6}$$

$y_{1:n} < \cdots < y_{r:n} < \tau < T_1$, in this case the prefixed number r out of n items is obtained at a normal use condition, so, this case will be discarded.

And for Cases 2 and 3, the likelihood function under Type-I HCS can be combined and written as:

$$\begin{aligned} L(\alpha, \theta, \beta | y) &= \frac{n!}{(n-r^*)!}\left\{\prod_{i=1}^{N_1} f(y_{i:n})\right\}\left\{\prod_{i=N_1+1}^{r^*} f(y_{i:n})\right\}\{1 - F(T_2)\}^{n-r*} \\ &= \frac{n!}{(n-r^*)!}(\alpha\theta)^{r^*}\beta^{r^*-N_1}\{1 + \alpha\delta(T_2)\}^{\theta(r^*-n)} \\ &\times \prod_{i=1}^{N_1}(1 + \alpha y_{i:n})^{-(\theta+1)} \prod_{i=N_1+1}^{r^*}\{1 + \alpha\delta(y_{i:n})\}^{-(\theta+1)}, \end{aligned} \tag{7}$$

where, $0 < y_{1:n} < \cdots < y_{N_1:n} \le \tau < y_{N_1+1:n} < \cdots < y_{r^*:n} \le T_1$, and $r^* = N_1 + N^*$.

We consider the likelihood function in Cases 2 and 3 only, as it runs at accelerated use condition.

We suggest the prior PDF for the parameters α and θ to be $\Gamma(a, b)$ and $\Gamma(c, d)$, respectively; because the gamma prior is wealthy enough to cover the prior belief of the experimenter, so, many authors used it, see Pak and Mohammad [27]; Okasha [28]; Abd-Elfattah et al. [23]; which expressed as follows:

$$\pi_1(\alpha) \propto \alpha^{a-1}\exp(-b\alpha), \quad a, b > 0, \tag{8}$$

$$\pi_2(\theta) \propto \theta^{c-1}\exp(-d\theta), \quad c, d > 0. \tag{9}$$

It should be pointed out that, the empirical Bayes method can be used to estimate the hyper parameters if they are unknown using previous samples, see, for example, Maritz and Lwin [29]. As an alternative, the hierarchical Bayes approach, which employs a suitable prior for the hyper parameters, could be utilized, see Bernardo and Smith [30].

The prior PDF for the acceleration factor β is a non-informative prior given by:

$$\pi_3(\beta) \propto \beta^{-1}, \quad (\beta > 1). \tag{10}$$

The joint prior PDF of α, θ and β, can be written as follows:

$$\begin{aligned} \pi(\alpha, \theta, \beta) &= \pi_1(\alpha)\pi_2(\theta)\pi_3(\beta) \\ &\propto \frac{1}{\beta}\alpha^{a-1}\theta^{c-1}e^{-(b\alpha+d\theta)}, \quad \beta > 1, a, b, c, d, \alpha, \theta > 0. \end{aligned} \tag{11}$$

The majority of published papers consider that the choice of prior distribution is the researcher's belief and the assumption of independence between parameters is the best in terms of ease of calculations and good results, and there is no way to know independence or dependence between them. Also, they are specified as independent when you do not want to assume that they are prior informative about each other. That is, knowing the value of one would not change your mind about any of the others, before seeing any data, this assumption was chosen when we determined the prior distribution and this is more understandable. In addition, choosing another prior distribution or dependent parameters will increase the complexity and difficulty of mathematical equations. Furthermore, the independent gamma priors are relatively simple and concise, which may not yield much complex inferential and computational issues. Also, in many practical situations, although the dependent prior models seem more attractive, yet the dependent property between parameters cannot be justified from a statistical perspective due to historical information and expert experience where such prior data/information may be very rare. Therefore, independent priors are more popular in statistics under the Bayesian procedure for the sake of simplicity, see EL-Sagheer et al. [31]; and Nassar et al. [32].

The joint posterior PDF of α, θ and β, given y, can be written from (7) and (11), as

$$\pi^*(\alpha,\theta,\beta|y) = K^{-1}\alpha^{r^*+a-1}\theta^{r^*+c-1}\beta^{r^*-N_1-1}\{1+\alpha\delta(T_2)\}^{\theta(r^*-n)} \times e^{-(b\alpha+d\theta)}\prod_{i=1}^{N_1}(1+\alpha y_{i:n})^{-(\theta+1)}\prod_{i=N_1+1}^{r^*}\{1+\alpha\delta(y_{i:n})\}^{-(\theta+1)}, \quad (12)$$

where, a normalizing constant K is

$$K = \int_1^\infty\int_0^\infty\int_0^\infty \pi^*(\alpha,\theta,\beta|y)\mathrm{d}\alpha\mathrm{d}\theta\mathrm{d}\beta. \quad (13)$$

The conditional posterior PDF of α, θ and β, are, respectively, given by

$$\pi^*(\alpha|\theta,\beta;y) = \alpha^{r^*+a-1}\{1+\alpha\delta(T_2)\}^{\theta(r^*-n)}e^{-b\alpha} \times\prod_{i=1}^{N_1}(1+\alpha y_{i:n})^{-(\theta+1)}\prod_{i=N_1+1}^{r^*}\{1+\alpha\delta(y_{i:n})\}^{-(\theta+1)}, \quad (14)$$

$$\pi^*(\theta|\alpha,\beta;y) = \theta^{r^*+c-1}\{1+\alpha\delta(T_2)\}^{\theta(r^*-n)}e^{-\mathrm{d}\theta} \times\prod_{i=1}^{N_1}(1+\alpha y_{i:n})^{-(\theta+1)}\prod_{i=N_1+1}^{r^*}\{1+\alpha\delta(y_{i:n})\}^{-(\theta+1)}, \quad (15)$$

$$\pi^*(\beta|\alpha,\theta;y) = \beta^{r^*-N_1-1}\{1+\alpha\delta(T_2)\}^{\theta(r^*-n)} \times\prod_{i=N_1+1}^{r^*}\{1+\alpha\delta(y_{i:n})\}^{-(\theta+1)}. \quad (16)$$

4. Bayesian One-Sample Prediction

Let the first r^{*th} order statistics, $Y_{1:n},\ldots,Y_{r^*:n}$, $1\le r^*<n$, have been observed and we want to predict the future order statistics, $Y_{s:n}$, $s=r^*+1, r^*+2,\ldots,n$. The conditional PDF of the s^{th} future order statistics given informative order statistics $y=(y_{1:n},y_{2:n},\ldots,y_{r^*:n})$, can be written as:

$$f^*(y_{s:n}|y) = \frac{(n-r^*)!}{(n-s)!(s-r^*-1)!}[F(y_{s:n})-F(T_2)]^{s-r^*-1} \times[1-F(y_{s:n})]^{n-s}[1-F(T_2)]^{-(n-r^*)}f(y_{s:n}), \quad (17)$$

substituting (3) and (4) in (17), we get

$$f^*(y_{s:n}|\alpha,\theta,\beta;y) = \mathfrak{J}\alpha\theta\beta\{1+\alpha\delta(T_2)\}^{\theta(n-r^*)}\{1+\alpha\delta(T_2)\delta(y_{s:n})\}^{\theta(s-n-1)-1} \times\left\{\{1+\alpha\delta(T_2)\}^{\theta}-\{1+\alpha\delta(y_{s:n})\}^{\theta}\right\}^{(s-r^*-1)}, \quad (18)$$

where, $\mathfrak{J} = (n-r^*)!/(n-s)!(s-r^*-1)!$.

By multiplying (12) by (18) and then integrating with respect to (α, θ, β), the predictive PDF of $Y_{s:n}$, $s = r^* + 1, r^* + 2, \ldots, n$ given the informative order statistics $y = (y_{1:n}, y_{2:n}, \ldots, y_{r^*:n})$, is given by

$$h(y_{s:n}|\alpha, \theta, \beta; y) = \int_1^\infty \int_0^\infty \int_0^\infty \pi^*(\alpha, \theta, \beta|y) f^*(y_{s:n}|\alpha, \theta, \beta; y) \mathrm{d}\alpha \mathrm{d}\theta \mathrm{d}\beta, \quad y_{s:n} > y_{r^*:n}. \tag{19}$$

By substituting (12) and (18) in (19), we obtain

$$\begin{aligned} h(y_{s:n}|\alpha, \theta, \beta; y) = {} & K^{-1}\mathfrak{J} \int_1^\infty \int_0^\infty \int_0^\infty \alpha^{r^*+a} \theta^{r^*+c} \beta^{r^*-N_1} \{1 + \alpha\delta(T_2)\}^{\theta(n-r^*)} \\ & \times \left\{\{1 + \alpha\delta(T_2)\}^{\theta} - \{1 + \alpha\delta(y_{s:n})\}^{\theta}\right\}^{(s-r^*-1)} \\ & \times \{1 + \alpha\delta(y_{s:n})\}^{\theta(s-n-1)-1} e^{-(b\alpha+d\theta)} \prod_{i=1}^{N_1} (1 + \alpha y_{i:n})^{-(\theta+1)} \\ & \times \prod_{i=N_1+1}^{r^*} \{1 + \alpha\delta(y_{i:n})\}^{-(\theta+1)} \mathrm{d}\alpha \mathrm{d}\theta \mathrm{d}\beta. \end{aligned} \tag{20}$$

A $(1 - \gamma)100\%$, two-sided Bayesian predictive interval for $Y_{s:n}$, is given by

$$P[L < Y_{s:n} < U] = 1 - \gamma, \tag{21}$$

where L and U are lower and upper bounds of Bayesian prediction for Y_s, are obtained by solving the following two equations numerically

$$\left.\begin{aligned} P(Y_s > L|y) &= 1 - \frac{\gamma}{2}, \\ P(Y_s > U|y) &= \frac{\gamma}{2}. \end{aligned}\right\} \tag{22}$$

4.1. Bayesian Prediction: One-Sample Scheme via MCMC. The predictive PDF (20), is approximated by using the MCMC method as follows:

$$h(y_{s:n}|\alpha, \theta, \beta; y) \approx \frac{\sum_{j=1}^N f^*(y_{s:n}|\alpha_j, \theta_j, \beta_j; y)}{\sum_{j=1}^N \int_{y_{r^*:n}}^\infty f^*(y_{s:n}|\alpha_j, \theta_j, \beta_j; y) \mathrm{d}y_{s:n}}, \tag{23}$$

where, $\alpha_j, \theta_j, \beta_j$, $j = 1, 2, 3, \ldots, N$, are generated from the posterior PDF (12).

A $(1 - \gamma)100\%$, two-sided Bayesian predictive interval, (L, U), of observations $Y_{s:n}$ can be calculated from solving the following nonlinear equations for a given γ

$$\left.\begin{aligned} \frac{\sum_{j=1}^N \int_L^\infty f^*(y_{s:n}|\alpha_j, \theta_j, \beta_j; y) \mathrm{d}y_{s:n}}{\sum_{j=1}^N \int_{y_{r^*:n}}^\infty f^*(y_{s:n}|\alpha_j, \theta_j, \beta_j; y) \mathrm{d}y_{s:n}} &= 1 - \frac{\gamma}{2}, \\ \frac{\sum_{j=1}^N \int_U^\infty f^*(y_{s:n}|\alpha_j, \theta_j, \beta_j; y) \mathrm{d}y_{s:n}}{\sum_{j=1}^N \int_{y_{r^*:n}}^\infty f^*(y_{s:n}|\alpha_j, \theta_j, \beta_j; y) \mathrm{d}y_{s:n}} &= \frac{\gamma}{2}. \end{aligned}\right\} \tag{24}$$

5. Bayesian Two-Sample Prediction

Let $X_{1:m} \le X_{2:m} \le \cdots \le X_{m:m}$ be the order statistics from a future random sample of size m and independent of the informative sample $Y_{1:n}, Y_{2:n}, \ldots, Y_{r^*:n}, 1 \le r^* < n$. The future sample and the informative sample are following the Lomax SSPALT model under Type-I HCS. We wish to predict the first k^{th} future order statistics of the future sample $X_{m:m}, 1 \le k < m$.

The marginal PDF of the k^{th} order statistics from $X_{k:m}$ is expressed as:

$$\begin{aligned} h^*(x_{k:m}|\alpha, \theta, \beta) &= \frac{m!}{(k-1)!(m-k)!} [F(x_{k:m})]^{k-1} [1 - F(x_{k:m})]^{m-k} f(x_{k:m}), \quad x_{k:m} > 0 \\ &= \Omega\alpha\theta\beta\left\{1 - (1 + \alpha\delta(x_{k:m}))^{-\theta}\right\}^{k-1} \\ &\quad \times \{1 + \alpha\delta(x_{k:m})\}^{\theta(k-m-1)-1}, \end{aligned} \tag{25}$$

where, $\Omega = m!/(k-1)!(m-k)!$.

Table 1: 95% Bayesian predictive intervals of future order statistics when $\theta = 2.5$, $\alpha = 1.8$, $\beta = 3.5$, $a = 0.8$, $b = 0.6$, $c = 0.6$, and $d = 0.7$ (one-sample scheme).

n	r	(τ, T_1)	Y_s	L	U	Length
20	15	(0.3, 0.8)	Y_{16}	0.394992	0.669922	0.274931
			Y_{17}	0.394926	0.61621	0.221284
			Y_{18}	0.394931	0.650672	0.255741
		(0.5, 1.6)	Y_{16}	0.524027	0.681595	0.157568
			Y_{17}	0.524205	0.713818	0.189612
			Y_{18}	0.524462	0.757883	0.233422
40	30	(0.3, 0.8)	Y_{31}	0.37462	0.545655	0.171035
			Y_{32}	0.374593	0.534925	0.160332
			Y_{33}	0.37457	0.527326	0.152756
		(0.5, 1.6)	Y_{31}	0.502775	0.634217	0.131442
			Y_{32}	0.502839	0.643297	0.140458
			Y_{33}	0.502912	0.653577	0.150665
60	45	(0.3, 0.8)	Y_{46}	0.371894	0.437151	0.0652573
			Y_{47}	0.371905	0.438426	0.0665217
			Y_{48}	0.371916	0.439731	0.0678151
		(0.5, 1.6)	Y_{46}	0.43803	0.453787	0.0157567
			Y_{47}	0.438037	0.454864	0.0168265
			Y_{48}	0.438046	0.456098	0.018052
80	60	(0.3, 0.8)	Y_{61}	0.385421	0.429208	0.0437874
			Y_{62}	0.385427	0.430039	0.0446117
			Y_{63}	0.385434	0.430895	0.0454615
		(0.5, 1.6)	Y_{61}	0.440823	0.479438	0.038615
			Y_{62}	0.440833	0.481241	0.0404073
			Y_{63}	0.440845	0.483209	0.0423641

The predictive PDF of $X_{k:m}$, $1 \le k < m$ say $g(x_{k:m}|\alpha, \theta, \beta)$ is given as:

$$g(x_{k:m}|\alpha, \theta, \beta) = \int_1^\infty \int_0^\infty \int_0^\infty \pi^*(\alpha, \theta, \beta|y) h^*(x_{k:m}|\alpha, \theta, \beta) d\alpha d\theta d\beta, \tag{26}$$

where, $\pi^*(\alpha, \theta, \beta|y)$ is the joint posterior PDF of (α, θ, β) as given in (12). By substituting from (12) and (25) in (26), we get

$$\begin{aligned} g(x_{k:m}|\theta, \beta) = K^{-1}\Omega \int_1^\infty \int_0^\infty \int_0^\infty & \alpha^{r^*+a}\theta^{r^*+c}\beta^{r^*-N_1}\{1+\alpha\delta(T_2)\}^{\theta(r^*-n)} \\ & \times\left\{1-(1+\alpha\delta(x_{k:m}))^{-\theta}\right\}^{k-1}\{1+\alpha\delta(x_{k:m})\}^{\theta(k-m-1)-1} \\ & \times e^{-(b\alpha+d\theta)}\prod_{i=1}^{N_1}(1+\alpha y_{i:n})^{-(\theta+1)}\prod_{i=N_1+1}^{r^*}\{1+\alpha\delta(y_{i:n})\}^{-(\theta+1)} d\alpha d\theta d\beta. \end{aligned} \tag{27}$$

A $(1-\gamma)100\%$, two-sided Bayesian predictive interval for $X_{k:m}$, is written as:

$$P(L < X_{k:m} < U) = 1 - \gamma, \tag{28}$$

where, bounds (L, U) of $X_{k:m}$, are computed by numerical solution of the two equations for a given γ

$$\left.\begin{aligned} P(X_{k:m} > L) &= 1 - \frac{\gamma}{2}, \\ P(X_{k:m} > U) &= \frac{\gamma}{2}. \end{aligned}\right\}. \tag{29}$$

5.1. Bayesian Two-Sample Prediction under MCMC. The predictive PDF (26), is approximated by applying the MCMC as follows:

$$g(x_{k:m}|\alpha, \theta, \beta) \approx \frac{\sum_{j=1}^N h^*(x_{k:m}|\alpha_j, \theta_j, \beta_j)}{\sum_{j=1}^N \int_0^\infty h^*(x_{k:m}|\alpha_j, \theta_j, \beta_j) dx_{k:m}}. \tag{30}$$

A $(1-\gamma)100\%$, two-sided Bayesian predictive interval, (L, U), of the future $X_{k:m}$ observations are obtained by solving the following nonlinear equations

Table 2: 95% Bayesian predictive intervals of future order statistics when $\theta = 1.4$, $\alpha = 1.8$, and $\beta = 2.5$ $a = 1.4$, $b = 0.8$, $c = 1.2$, and $d = 0.7$ (two-sample scheme).

n	r	(τ, T_1)	X_k	L	U	Length
20	15	(0.3, 0.8)	X_1	0.00129258	0.191083	0.189791
			X_2	0.00136062	0.201295	0.199935
			X_3	0.00143622	0.21266	0.211224
			X_4	0.00152071	0.225385	0.223864
		(0.5, 1.6)	X_1	0.001189	0.178083	0.176894
			X_2	0.00125159	0.187731	0.186479
			X_3	0.00132113	0.198484	0.197163
			X_4	0.00139887	0.210543	0.209145
40	30	(0.3, 0.8)	X_1	0.000878721	0.132237	0.131358
			X_2	0.000924981	0.139437	0.138512
			X_3	0.000976381	0.147465	0.146489
			X_4	0.00103383	0.156475	0.155441
		(0.5, 1.6)	X_1	0.000739947	0.109018	0.108278
			X_2	0.000778894	0.114824	0.114045
			X_3	0.00082217	0.121282	0.12046
			X_4	0.000870537	0.128511	0.12764
60	45	(0.3, 0.8)	X_1	0.000798553	0.118093	0.117294
			X_2	0.000840586	0.124406	0.123566
			X_3	0.000887291	0.131433	0.130546
			X_4	0.000939491	0.139301	0.138361
		(0.5, 1.6)	X_1	0.000621153	0.091571	0.0909499
			X_2	0.000653848	0.0964506	0.0957967
			X_3	0.000690176	0.101879	0.101189
			X_4	0.000730778	0.107956	0.107225
80	60	(0.3, 0.8)	X_1	0.000753343	0.111106	0.110352
			X_2	0.000792996	0.117029	0.116236
			X_3	0.000837055	0.123619	0.122782
			X_4	0.000886298	0.130995	0.130109
		(0.5, 1.6)	X_1	0.000577907	0.0854699	0.084892
			X_2	0.000608326	0.0900398	0.0894315
			X_3	0.000642126	0.0951259	0.0944837
			X_4	0.000679902	0.100821	0.100141

$$\left.\begin{aligned}\frac{\sum_{j=1}^{N}\int_{L}^{\infty} h^*\left(x_{k:m}|\alpha_j,\theta_j,\beta_j\right)\mathrm{d}x_{k:m}}{\sum_{j=1}^{N}\int_{0}^{\infty} h^*\left(x_{k:m}|\alpha_j,\theta_j,\beta_j\right)\mathrm{d}x_{k:m}} &= 1-\frac{\gamma}{2},\\ \frac{\sum_{j=1}^{N}\int_{U}^{\infty} h^*\left(x_{k:m}|\alpha_j,\theta_j,\beta_j\right)\mathrm{d}x_{k:m}}{\sum_{j=1}^{N}\int_{0}^{\infty} h^*\left(x_{k:m}|\alpha_j,\theta_j,\beta_j\right)\mathrm{d}x_{k:m}}, &= \frac{\gamma}{2}.\end{aligned}\right\}. \tag{31}$$

6. Numerical Computations

Here, we present two examples to illustrate one-sample and two-sample techniques, as well as a simulation study and real-life data for the discussed methodology, is stated.

6.1. Numerical Examples

6.1.1. One-Sample Scheme

(1) For known values of of hyper parameters ($a = 0.8; b = 0.6; c = 0.6; d = 0.7$), generate ($\alpha = 1.8; \theta = 2.5$ and $\beta = 3.5$) from equations (8), (9) and (10), respectively.

(2) By using generated values of $\alpha; \theta$ and β, a random sample of size 30 is simulated from (4). The first informative ordered failures $r = 20$ are obtained and listed as follows:

0.0113981, 0.0188695, 0.0638502, 0.06688, 0.0716792, 0.217725, 0.22276, 0.224412, 0.224614, 0.227522, 0.228146, 0.233232, 0.237106, 0.240437, 0.246286, 0.25205, 0.259451, 0.260386, 0.270926, 0.278369.

(3) Using these informative data in (25) with 0.95 CI, the lower and upper bounds of the next predicted failure Y_{21} are (0.27903, 0.400507) and for the last predicted failure Y_{30} are (0.279084, 0.402948).

6.1.2. Two-Sample Scheme

(1) For given values of of hyper parameters ($a = 1.4; b = 0.8; c = 1.2; d = 0.7$), generate ($\alpha = 1.8; \theta = 1.4$ and $\beta = 2.5$) from equations (8), (9) and (10), respectively.

(2) By using generated values of $\alpha; \theta$ and β, a random sample of size 20 is simulated from (3). The first

Table 3: 95% Bayesian predictive intervals of future order statistics when $\beta = 3$, $a = 0.7$, $b = 0.9$, $c = 0.9$, and $d = 0.6$ (one-sample scheme).

n	r	(τ, T_1)	Y_s	L	U	Length
46	32	(1, 3.8)	Y_{33}	3.33496	8.77466	5.4397
			Y_{34}	4.57799	13.8905	9.31248
			Y_{35}	4.84621	19.4557	14.6095
	35	(1.5, 4.6)	Y_{36}	4.51441	6.85874	2.34433
			Y_{37}	4.80035	9.17545	4.3751
			Y_{38}	5.08174	12.4674	7.38569

Table 4: 95% Bayesian predictive intervals of future order statistics when $\beta = 3$, $a = 0.7$, $b = 0.9$, $c = 0.9$, and $d = 0.6$ (two-sample scheme).

n	r	(τ, T_1)	X_k	L	U	Length
46	32	(1, 3.8)	X_1	0.00478418	0.717001	0.712217
			X_2	0.00494396	0.738654	0.73371
			X_3	0.00510012	0.761739	0.756639
			X_4	0.00521913	0.786164	0.780945
	35	(1.5, 4.6)	X_1	0.00336333	0.496913	0.493549
			X_2	0.0034289	0.511913	0.508484
			X_3	0.00350722	0.527847	0.524339
			X_4	0.00366162	0.544767	0.541106

informative ordered failures $r = 15$ are obtained and listed as follows:

0.00167013, 0.157792, 0.193577, 0.19366, 0.207637, 0.219493, 0.226598, 0.242315, 0.271873, 0.294316, 0.334488, 0.603327, 0.651955, 0.70614, 0.743701.

(3) Using these informative data in (31) with 0.95 CI, the lower and upper bounds of the first four observations of a non-informative sample X_1, X_2, X_3, X_4 are (0.00129258, 0.191083), (0.00136062, 0.201295), (0.00143622, 0.21266) and (0.00152071, 0.225385), respectively.

6.2. Simulation Study. In this part of the paper, we explain the MCMC algorithm that used for computing Bayesian bounds for future samples in the case of one-sample and two-sample approaches. We determine values of n, r and choose τ, T_1. Using values of prior parameters (a, b, c, d) we generate initial values of the parameters α, θ and β from their prior PDF. Based on the generated α, θ and β, we generate a Lomax Type-I HC sample using SSPALT model with different sizes $n = 20; 40; 60; 80$ and censoring values $r = 15; 30; 45; 60$ based on the inverse function technique given by

$$X = \frac{1}{\beta}\left\{\frac{1}{\alpha}\left\{(1-U)^{-1/\theta} - 1\right\} - \tau\right\} + \tau, \tag{32}$$

where, U denotes a number obtained from $U(0, 1)$ randomly. From Metropolis algorithm, $\alpha^{(j)}$ is generated from (14), $\theta^{(j)}$ from (15) and $\beta^{(j)}$ from (16) using normal distribution as a proposal distribution. Because the predictive PDF (21) cannot be produced in a closed form for one-sample prediction, it is approximated using the MCMC approach as described in (22). A 95% Bayesian predictive intervals for Y_s::n, s=$r*$ + 1, $r*$ + 2, . . ., n are obtained by solving the two nonlinear Eqs.(23). For two-sample prediction, the predictive PDF (26), which cannot be produced in a closed form, is estimated using the MCMC approach as described in (28). A 95% Bayesian predictive intervals for $X_{k:m}$, $k = 1, 2, \ldots m$ are obtained by solving the two nonlinear equation (29).

All results are derived by Mathematica 8 programming language software. The results of one-sample scheme with different values of n, r, τ and T_1 are listed in Table 1. The results of two-sample case with the same values of n, r, τ and T_1 are obtained and displayed in Table 2. The numerical results are computed by MATHEMATICA 8 codes, such as: (FindRoot, NMaximize, NIntegrate and RandomReal).

6.3. Illustrative Example. In this section, we present a real life example, the dataset was initially considered by Chhikara and Folks [33]. It represents 46 repair times (in hours) for an airborne communication transceiver. The Lomax distribution has been fitted on this data set by Singh et al. [34]; and they stated that the Lomax distribution can be used for analyzing this data set. The data set is ordered and listed as follows:

0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 2.7, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 22.0, 24.5.

By applying SSPALT on this data set, by considering this data represents lifetimes of units put on life testing, we can conclude the following:

(1) At $n = 46$, $r = 32$, $\tau = 1$ and $T_1 = 3.8$, we have 13 items are failed at $\tau = 1$ throughout normal usage. After changing mode to acceleration case, the test is terminated at $T_* = \min\{X_{32:46}, T_1\} = \min\{3.3, 3.8\} = 3.3$, that is, 32 items are failed at $T_* = 3.3$.

(2) At $n = 46$, $r = 35$, $\tau = 1.5$ and $T_1 = 4.6$, there have been 20 items are failed at $\tau = 1.5$ throughout normal usage. After switching to acceleration case, the test is terminated at $T_* = \min\{X_{35:46}, T_1\} = \min\{4.5, 4.6\} = 4.5$, that is, 35 items are failed at $T_* = 4.5$.

Based on SSPALT model with Type-I HCS, 95% one-sample Bayesian predictive intervals for order statistic $Y_{s:n}$, $s = r + 1, r + 2, \ldots, n$ are reported in Table 3 and from the same sample 95% two-sample Bayesian predictive intervals for order statistic $X_{k:m}$, $m = 1; 2; 3; 4$, are reported in Table 4.

7. Conclusion

Based on SSPALT under Type-I HCS of order statistics data from Lomax distribution, Bayesian prediction bounds for future observations are obtained by using one-sample and two-sample prediction techniques. The MCMC method is used to get Bayesian predictive intervals. We can state that using SSPALT with Type-I HCS, which has the advantage of having pre-determined experiment time, maximizes this advantage and speed up obtaining more information. Furthermore, one can see that SSPALT improves the disadvantage of this scheme, which is represented in the lack of failures in the specified time for the experiment. Finally, some numerical examples and simulation study are given to illustrate the results, and we observe the following remarks:

From Tables 1 and 2 (one-sample and two-sample cases), we noted that the Bayesian predictive intervals were affected by changing the values of n, r, τ and T_1 as follows:

(1) When increasing τ and T_1 with the same n, r, the length of Bayesian predictive intervals decreases.

(2) The length of Bayesian predictive intervals, for fixed values of τ and T_1, tends to be shorter when n and r increase, except for a few cases. This may be due to fluctuations in data.

From Tables 3 and 4 (one-sample and two-sample cases), we observed that:

(3) When values of τ and T_1 get larger the length of Bayesian predictive intervals becomes smaller.

Notation

T_1: A specific maximum time of the experiment
T_2: A random termination time of experiment
Y: A total lifetime of an item in a step-stress model
$y_{r:n}$: A time in which r^{th} item fail
τ: A specified time at which the stress is changed from s_1 to s_2
N_1: A number of units are failed before time τ at stress level s_1
N_2: Units number that fail before time T_1 at stress level s_2
N^*: A number of units that fail before time T_2 at the stress level where T_2 is.

Data Availability

The data are generated by simulation, and the real data was considered by Chhikara and Folks [30].

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This research was funded by Taif University Researchers Supporting Project number (TURSP-2020/321), Taif University, Taif, Saudi Arabia.

References

[1] A. Saeed and P. Hanieh, "Bayesian prediction for censored data from the kumaraswamy distribution based on constant-stress accelerated life test model and its application in ceramic materials," *Journal of Quality Engineering and Managment*, vol. 11, no. 1, pp. 15–32, 2021.

[2] S. A. Lone, H. Panahi, and I. Shah, "Bayesian prediction interval for a constant-stress partially accelerated life test model under censored data," *Journal of Taibah University for Science*, vol. 15, no. 1, pp. 1178–1187, 2021.

[3] A. Mohammad and Z. R. Mohammad, "Prediction of censored weibull lifetimes in a simple step-stress plan with khamis-higgins model," *STATISTICS, OPTIMIZATION AND INFORMATION COMPUTING*, vol. 10, no. 3, pp. 0–19, 2021.

[4] M. Nagy, M. H. Abu-Moussa, A. F Alrasheedi, and A. Rabie, "Expected Bayesian estimation for exponential model based on simple step stress with Type-I hybrid censored data." *Mathematical Biosciences and Engineering*, vol. 19, no. 10, pp. 9773–9791, 2022.

[5] S. Geisser, *Predictive Inference: An Introduction*, Champman and Hall, London, UK, 1993.

[6] E. K. Al-Hussaini and A. A. Ahmad, "On Bayesian interval prediction of future records," *Test*, vol. 12, no. 1, pp. 79–99, 2003.

[7] E. K. Al-Hussaini and A. A. Ahmad, "On Bayesian predictive distributions of generalized order statistics," *Metrika*, vol. 57, no. 2, pp. 165–176, 2003.

[8] J. Ahmadi, S. M. T. K. MirMostafaee, and N. Balakrishnan, "Bayesian prediction of k-record values based on progressively censored data from exponential distribution," *Journal of Statistical Computation and Simulation*, vol. 82, no. 1, pp. 51–62, 2012.

[9] A. A. Ahmad, "On Bayesian interval prediction of future generalized-order statistics using doubly censoring," *Statistics*, vol. 45, no. 5, pp. 413–425, 2011.

[10] A. A. Ahmad, M. A. Fawzy, and H. Ouda, "Bayesian prediction of future observations from weighted exponential distribution constant-stress model based on type-II hybrid censored data," *Communications in Statistics - Theory and Methods*, vol. 50, no. 12, pp. 2732–2746, 2021.

[11] S. F. Ateya, "Prediction under generalized exponential distribution using MCMC algorithm," *International Mathematical Forum*, vol. 6, no. 63, pp. 3111–3119, 2011.

[12] N. Balakrishnan and A. R. Shafay, "One- and two-sample Bayesian prediction intervals based on type-II hybrid censored data," *Communications in Statistics - Theory and Methods*, vol. 41, no. 9, pp. 1511–1531, 2012.

[13] D. Kundu and H. A. Howlader, "Bayesian inference and prediction of the inverse Weibull distribution for Type-II censored data," *Computational Statistics & Data Analysis*, vol. 54, no. 6, pp. 1547–1558, 2010.

[14] S. K. Singh, U. Singh, and V. K. Sharma, "Bayesian prediction of future observations from inverse Weibull distribution based on type-II hybrid censored sample," *International Journal of Advanced Statistics and Probability*, vol. 1, no. 2, pp. 32–43, 2013.

[15] K. E. Ahmad, Z. F. Jaheen, and H. S. Mohammed, "Bayesian prediction based on type-I censored data from a mixture of Burr type XII distribution and its reciprocal," *Statistics*, vol. 46, no. 6, pp. 787–797, 2012.

[16] J. M. Corcuera and F. Giummolè, "A generalized Bayes rule for prediction," *Scandinavian Journal of Statistics*, vol. 26, no. 2, pp. 265–279, 1999.

[17] G. Prakash, "Bayes prediction on optimum SS-PALT in generalized inverted exponential distribution: a two-sample approach," *Austrian Journal of Statistics*, vol. 51, pp. 1–15, 2022.

[18] M. Wu and W. Gui, "Estimation and prediction for Nadarajah-Haghighi distribution under progressive type-II censoring," *Symmetry*, vol. 13, no. 6, p. 999, 2021.

[19] M. M. Mohie El-Din, A. Sadek, and M. Nagy, "Bayesian Estimation and two-sample prediction based on unified hybrid censored sample," *J. Stat. Appl. Pro.,* vol. 5, no. 3, pp. 439–448, 2016.

[20] K. S. Lomax, "Business failures: another example of the analysis of failure data," *Journal of the American Statistical Association*, vol. 49, no. 268, pp. 847–852, 1954.

[21] H. A. Howlader and A. M. Hossain, "Bayesian survival estimation of Pareto distribution of the second kind based on failure-censored data," *Computational Statistics & Data Analysis*, vol. 38, no. 3, pp. 301–314, 2002.

[22] A. H. Abd-Ellah, "Bayesian one sample prediction bounds for the Lomax distribution," *Indian Journal of Pure and Applied Mathematics*, vol. 3, pp. 101–109, 2003.

[23] A. M. Abd-Elfattah, F. M. Alaboud, and A. H. Alharby, "On sample size estimation for Lomax ditribution," *Aust. J. Basic Appl. Sci.,* vol. 1, pp. 373–378, 2007.

[24] A. S. Hassan and A. S. Al-Ghamdi, "Optimum step stress accelerated life testing for Lomax distribution," *Journal of Applied Sciences Research*, vol. 5, pp. 2153–2164, 2009.

[25] B. Epstein, "Truncated life tests in the exponential case," *The Annals of Mathematical Statistics*, vol. 25, no. 3, pp. 555–564, 1954.

[26] M. H. Degroot and P. K. Goel, "Bayesian estimation and optimal designs in partially accelerated life testing," *Naval Research Logistics Quarterly*, vol. 26, no. 2, pp. 223–235, 1979.

[27] A. Pak and R. M. Mohammad, "Estimating the parameters of lomax distribution from imprecise information," *Journal of Statistical Theory and Applications*, vol. 17, no. 1, pp. 122–135, 2018.

[28] H. M. Okasha, "E-Bayesian estimation for the Lomax distribution based on type-II censored data," *Journal of the Egyptian Mathematical Society*, vol. 22, no. 3, pp. 489–495, 2014.

[29] J. S. Maritz and T. Lwin, *Empirical Bayes Methods*, Chapman & Hall, London, UK, 1989.

[30] J. M. Bernardo and A. F. M. Smith, *Bayesian Theory*, ATLANTISSPS, New York, NY, USA, 1994.

[31] R. M. El-Sagheer, M. S. Eliwa, K. M. Alqahtani, and M. El-Morshedy, "Asymmetric randomly censored mortality distribution: bayesian framework and parametric bootstrap with application to COVID-19 data," *Journal of Mathematics*, vol. 2022, Article ID 8300753, 14 pages, 2022.

[32] M. Nassar, S. Dey, L. Wang, and A. Elshahhat, "Estimation of Lindley constant-stress model via product of spacing with Type-II censored accelerated life data," *Communications in Statistics - Simulation and Computation*, pp. 1–27, 2021.

[33] R. S. Chhikara and J. L. Folks, "The inverse Gaussian distribution as a lifetime model," *Technometrics*, vol. 19, pp. 461–468, 1977.

[34] S. K. Singh, U. Singh, and A. S. Yadav, "Bayesian estimation of Lomax distribution under type-II hybrid censored data using Lindley's approximation method," *International Journal of Data Science*, vol. 2, no. 4, pp. 352–368, 2017.