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Research Article

Some New Ostrowski-Type Inequalities Involving σ-Fractional Integrals

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The aim of this paper is to derive some new fractional analogues of Ostrowski-type inequalities involving bounded functions using the concept of σ-Riemann–Liouville fractional integrals.

1. Introduction and Preliminaries

Inequalities play a pivotal role in modern analysis. Mathematical analysis depends upon many inequalities. In recent years, an extensive research has been carried out on obtaining various analogues of classical inequalities using different approaches, for details and applications, see [1–4]. A very interesting approach is to obtain fractional analogues of the inequalities. The fractional version of inequalities plays a significant role in the establishment of the uniqueness of solutions for certain fractional partial differential equations. Sarikaya et al. [5] were the first to introduce the concepts of fractional calculus in the theory of integral inequalities by obtaining the fractional analogues of classical Hermite–Hadamard's inequality. Dragomir [6, 7] obtained fractional versions of Ostrowski-like inequalities. Erden et al. [8] recently obtained some more new fractional analogues of Ostrowski-type inequalities using bounded functions. Sarikaya [9] introduced the notion of two-dimensional Riemann–Liouville fractional integrals and obtained some new fractional variants of Hermite–Hadamard's inequality on two dimensions. Having inspiration from the research work of Mubeen and Habibullah [10] and Sarikaya [9], Awan et al. [11] introduced the concepts of σ-Riemann–Liouville fractional integrals on two dimensions and obtained two-dimensional fractional integral inequalities. It is worth to mention here that if $\sigma \longrightarrow 1$, then σ-Riemann–Liouville fractional integrals reduces to classical Riemann–Liouville fractional integral. Note that the concept of σ-Riemann–Liouville fractional integral is a significant generalization of classical Riemann–Liouville fractional integrals; as for $\sigma \neq 1$, the properties of σ–Riemann–Liouville fractional integrals are quite different from the classical Riemann–Liouville fractional integrals.

The aim of this paper is to obtain some new fractional analogues the classical Ostrowski's inequality using the concepts of σ-fractional integrals. In order to obtain the main results of the paper, we first derive some new lemmas results, and then using these lemmas as auxiliary results, we derive our main results of the paper.

Let us first recall some previously known concepts and results. The first one is the definition of the Riemann–Liouville fractional integrals.

Definition 1 (see [12]). Let $F \in L_1[a, b]$. Then, the Riemann–Liouville integrals of order $\alpha_1 > o$ with $a > 0$ are defined as follows:

$$\mathcal{J}_{a^+}^{\alpha_1}F(x)=\frac{1}{\Gamma(\alpha_1)}\int_a^x (x-t_1)^{\alpha_1-1}F(t_1)\mathrm{d}t_1,\quad x>a,$$
$$\mathcal{J}_{a^+}^{\alpha_1}F(x)=\frac{1}{\Gamma(\alpha_1)}\int_x^b (t_1-x)^{\alpha_1-1}F(t_1)\mathrm{d}t_1,\quad x<b, \tag{1}$$

respectively. Here, $\Gamma(\alpha_1)$ is the gamma function. These integrals are motivated by the well-known Cauchy formula:

$$\int_a^x \mathrm{d}t_1\int_a^{t_1}\mathrm{d}t_2\ldots\int_a^{t_{n-1}} f(t_n)\mathrm{d}t_n=\frac{1}{\Gamma(n)}\int_a^x (x-t)^{n-1}f(t)\mathrm{d}t,\quad n\in\mathbb{N}^*. \tag{2}$$

Mubeen and Habibullah [10] introduced the σ-Riemann–Liouville fractional integrals as follows.

Definition 2 (see [10]). Let $\Xi\in L_1[a,b]$. The σ-Reimann–Liouville fractional integrals ${}_\sigma\mathcal{J}_{a^+}^{\alpha_1}\Xi$ and ${}_\sigma\mathcal{J}_{b^-}^{\alpha_1}\Xi$ of order $\alpha_1>0$ with $a\geq 0$ and $\sigma>0$ are defined as follows:

$${}_\sigma\mathcal{J}_{a^+}^{\alpha_1}\Xi(x)=\frac{1}{\sigma\Gamma_\sigma(\alpha_1)}\int_a^b (x-t_1)^{(\alpha_1/\sigma)-1}\Xi(t_1)\mathrm{d}t_1,\quad x>a,$$
$${}_\sigma\mathcal{J}_{b^-}^{\alpha_1}\Xi(x)=\frac{1}{\sigma\Gamma_\sigma(\alpha_1)}\int_x^b (t_1-x)^{(\alpha_1/\sigma)-1}\Xi(t_1)\mathrm{d}t_1,\quad x<b. \tag{3}$$

The above integrals for all functions are continuous and integrable on the interval $(0,\infty)$. Note that if $f\in L_1[a,b]$ and $a>0$, then ${}_\sigma\mathcal{J}_a^{\alpha_1}$ exists almost everywhere on $[a,b]$. If $\alpha_1\geq 1$, $\sigma>0$, and $f\in L_1[a,b]$, then ${}_\sigma\mathcal{J}_a^{\alpha_1}f\in C[a,b]$. For more details, see [13].

Awan et al. [11] defined σ-Reimannn–Liouville fractional integrals of two variable functions as follows.

Definition 3 (see [11]). Let $\Xi\in L_1([a,b]\times[c,d])$. The Riemann–Liouville σ–fractional integrals ${}_\sigma\mathcal{J}_{a^+,c^+}^{\alpha_1,\alpha_2}$, ${}_\sigma\mathcal{J}_{a^+,d^-}^{\alpha_1,\alpha_2}$, ${}_\sigma\mathcal{J}_{b^-,c^+}^{\alpha_1,\alpha_2}$, and ${}_\sigma\mathcal{J}_{b^-,d^-}^{\alpha_1,\alpha_2}$ are defined as follows:

$${}_\sigma\mathcal{J}_{a^+,c^+}^{\alpha_1,\alpha_2}\Xi(x,y)=\frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_a^x\int_c^y (x-t_1)^{(\alpha_1/\sigma)-1}(y-t_2)^{(\alpha_2/\sigma)-1}\Xi(t_1,t_2)\mathrm{d}t_2\mathrm{d}t_1,\quad x>a,y>c,$$
$${}_\sigma\mathcal{J}_{a^+,d^-}^{\alpha_1,\alpha_2}\Xi(x,y)=\frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_a^x\int_y^d (x-t_1)^{(\alpha_1/\sigma)-1}(t_2-y)^{(\alpha_2/\sigma)-1}\Xi(t_1,t_2)\mathrm{d}t_2\mathrm{d}t_1,\quad x>a,y<d,$$
$${}_\sigma\mathcal{J}_{b^-,c^+}^{\alpha_1,\alpha_2}\Xi(x,y)=\frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_x^b\int_c^y (t_1-x)^{(\alpha_1/\sigma)-1}(y-t_2)^{(\alpha_2/\sigma)-1}\Xi(t_1,t_2)\mathrm{d}t_2\mathrm{d}t_1,\quad x<b,y>c,$$
$${}_\sigma\mathcal{J}_{b^-,c^+}^{\alpha_1,\alpha_2}\Xi(x,y)=\frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_x^b\int_y^d (t_1-x)^{(\alpha_1/\sigma)-1}(t_2-y)^{(\alpha_2/\sigma)-1}\Xi(t_1,t_2)\mathrm{d}t_2\mathrm{d}t_1,\quad x<b,y<d, \tag{4}$$

for $x,y\in D:=[a,b]\times[c,d]$.

where $\alpha_1,\alpha_2>0$ and $a,b,c,d\geq 0$.

For the sake of simplicity, we define the following functions as

$${}_kM_{\alpha_1}(a,b;x):=\frac{(x-a)^{(\alpha_1/\sigma)}+(b-x)^{(\alpha_1/\sigma)}}{\Gamma_\sigma(\alpha_1+\sigma)},$$
$${}_kN_{\alpha_1}(c,d;y):=\frac{(y-c)^{(\alpha_2/\sigma)}+(d-y)^{(\alpha_2/\sigma)}}{\Gamma_\sigma(\alpha_2+\sigma)}, \tag{5}$$

2. Main Results

2.1. Key Lemmas. In this section, we prove some lemmas which will help us in obtaining the main results of the paper.

Lemma 1. *Let $\Xi\colon D\longrightarrow\mathbb{R}$ be an absolutely continuous, differentiable function such that $(\partial^2\Xi(\theta,\mu)/\partial\theta\partial\mu)$ exists and is continuous on $D\subseteq\mathbb{R}^2$. Then, for any $(x,y)\in D$, we have*

$$\begin{aligned}
&\frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_a^b\int_c^d G(x,t_1,y,t_2)\left[\int_x^{t_1}\int_y^{t_2}\frac{\partial^2\Xi(\theta,\mu)}{\partial\theta\partial\mu}\mathrm{d}\theta\mathrm{d}\mu\right]\mathrm{d}t_2\mathrm{d}t_1\\
&=\left[{}_\sigma\mathcal{J}^{\alpha_1,\alpha_2}_{a^+,c^+}\Xi(x,y)+{}_\sigma\mathcal{J}^{\alpha_1,\alpha_2}_{a^+,d^-}\Xi(x,y)+{}_\sigma\mathcal{J}^{\alpha_1,\alpha_2}_{b^-,c^+}\Xi(x,y)+{}_\sigma\mathcal{J}^{\alpha_1,\alpha_2}_{b^-,d^-}\Xi(x,y)\right]\\
&\quad-{}_kN_{\alpha_2}(c,d\colon y)\left[{}_\sigma\mathcal{J}^{\alpha_1}_{a^+}\Xi(x,y)+{}_\sigma\mathcal{J}^{\alpha_1}_{b^-}\Xi(x,y)\right]\\
&\quad-{}_kM_{\alpha_1}(a,b\colon x)\left[{}_\sigma\mathcal{J}^{\alpha_1}_{c^+}\Xi(x,y)+{}_\sigma\mathcal{J}^{\alpha_1}_{d^-}\Xi(x,y)\right]\\
&\quad+{}_kM_{\alpha_1}(a,b\colon x){}_kN_{\alpha_2}(c,d\colon y)\Xi(x,y)=F_1(x,y;a,b,c,d),
\end{aligned}\tag{6}$$

where

$$G(x,t_1,y,t_2)=\begin{cases}(x-t_1)^{(\alpha_1/\sigma)-1}(y-t_2)^{(\alpha_2/\sigma)-1}, & a\le t_1<x \text{ and } c\le t_2<y,\\ (x-t_1)^{(\alpha_1/\sigma)-1}(t_2-y)^{(\alpha_2/\sigma)-1}, & a\le t_1<x \text{ and } y\le t_2<d,\\ (t_1-x)^{(\alpha_1/\sigma)-1}(y-t_2)^{(\alpha_2/\sigma)-1}, & x\le t_1\le b \text{ and } c\le t_2<y,\\ (t_1-x)^{(\alpha_1/\sigma)-1}(t_2-y)^{(\alpha_2/\sigma)-1}, & x\le t_1\le b \text{ and } y\le t_2\le d.\end{cases}\tag{7}$$

Proof. Now,

$$\begin{aligned}
\int_x^{t_1}\int_y^{t_2}\frac{\partial^2\Xi(\theta,\mu)}{\partial\theta\partial\mu}\mathrm{d}\theta\mathrm{d}\mu&=\Xi(t_1,t_2)-\Xi(t_1,y)\\
&\quad-\Xi(x,t_2)+\Xi(x,y)\\
&=L(x,t_1,y,t_2).
\end{aligned}\tag{8}$$

This implies

$$\begin{aligned}
I&=\frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_a^b\int_c^d G(x,t_1,y,t_2)\left[\int_x^{t_1}\int_y^{t_2}\frac{\partial^2\Xi(\theta,\mu)}{\partial\theta\partial\mu}\mathrm{d}\theta\mathrm{d}\mu\right]\mathrm{d}t_2\mathrm{d}t_1\\
&=\frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_a^x\int_c^y(x-t_1)^{(\alpha_1/\sigma)-1}(y-t_2)^{(\alpha_2/\sigma)-1}L(x,t_1,y,t_2)\mathrm{d}t_2\mathrm{d}t_1\\
&\quad+\frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_a^x\int_y^d(x-t_1)^{(\alpha_1/\sigma)-1}(t_2-y)^{(\alpha_2/\sigma)-1}L(x,t_1,y,t_2)\mathrm{d}t_2\mathrm{d}t_1\\
&\quad+\frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_x^b\int_c^y(t_1-x)^{(\alpha_1/\sigma)-1}(y-t_2)^{(\alpha_2/\sigma)-1}L(x,t_1,y,t_2)\mathrm{d}t_2\mathrm{d}t_1\\
&\quad+\frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_x^b\int_y^d(t_1-x)^{(\alpha_1/\sigma)-1}(t_2-y)^{(\alpha_2/\sigma)-1}L(x,t_1,y,t_2)\mathrm{d}t_2\mathrm{d}t_1\\
&=I_1+I_2+I_3+I_4.
\end{aligned}\tag{9}$$

Now, consider

$$\begin{aligned}
I_1 &= \frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_a^x\int_c^y (x-t_1)^{(\alpha_1/\sigma)-1}(y-t_2)^{(\alpha_2/\sigma)-1}\left[\Xi(t_1,t_2)-\Xi(t_1,y)-\Xi(x,t_2)+\Xi(x,y)\right]\mathrm{d}t_2\mathrm{d}t_1\\
&= \frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_a^x\int_c^y (x-t_1)^{(\alpha_1/\sigma)-1}(y-t_2)^{(\alpha_2/\sigma)-1}\Xi(t_1,t_2)\mathrm{d}t_2\mathrm{d}t_1\\
&\quad -\frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_a^x\int_c^y (x-t_1)^{(\alpha_1/\sigma)-1}(y-t_2)^{(\alpha_2/\sigma)-1}\Xi(t_1,y)\mathrm{d}t_2\mathrm{d}t_1\\
&\quad -\frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_a^x\int_c^y (x-t_1)^{(\alpha_1/\sigma)-1}(y-t_2)^{(\alpha_2/\sigma)-1}\Xi(x,t_2)\mathrm{d}t_2\mathrm{d}t_1\\
&\quad +\frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_a^x\int_c^y (x-t_1)^{(\alpha_1/\sigma)-1}(y-t_2)^{(\alpha_2/\sigma)-1}\Xi(x,y)\mathrm{d}t_2\mathrm{d}t_1\\
&= {}_\sigma\mathcal{J}^{\alpha_1,\alpha_2}_{a^+,c^+}\Xi(x,y)-\frac{(y-c)^{(\alpha_2/\sigma)}}{\Gamma_\sigma(\alpha_2+\sigma)}{}_\sigma\mathcal{J}^{\alpha_1}_{a^+}\Xi(x,y)-\frac{(x-a)^{(\alpha_1/\sigma)}}{\Gamma_\sigma(\alpha_1+\sigma)}{}_\sigma\mathcal{J}^{\alpha_2}_{c^+}\Xi(x,y)+\frac{(x-a)^{(\alpha_1/\sigma)}(y-c)^{(\alpha_2/\sigma)}}{\Gamma_\sigma(\alpha_1+\sigma)\Gamma_\sigma(\alpha_2+\sigma)}.
\end{aligned}\tag{10}$$

Similarly,

$$\begin{aligned}
I_2 &= \frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_a^x\int_y^d (x-t_1)^{(\alpha_1/\sigma)-1}(t_2-y)^{(\alpha_2/\sigma)-1}\left[\Xi(t_1,t_2)-\Xi(t_1,y)-\Xi(x,t_2)+\Xi(x,y)\right]\mathrm{d}t_2\mathrm{d}t_1\\
&= {}_\sigma\mathcal{J}^{\alpha_1,\alpha_2}_{a^+,d^-}\Xi(x,y)-\frac{(d-y)^{(\alpha_2/\sigma)}}{\Gamma_\sigma(\alpha_2+\sigma)}{}_\sigma\mathcal{J}^{\alpha_1}_{a^+}\Xi(x,y)-\frac{(x-a)^{(\alpha_1/\sigma)}}{\Gamma_\sigma(\alpha_1+\sigma)}{}_\sigma\mathcal{J}^{\alpha_2}_{d^-}\Xi(x,y)+\frac{(x-a)^{(\alpha_1/\sigma)}(d-y)^{(\alpha_2/\sigma)}}{\Gamma_\sigma(\alpha_1+\sigma)\Gamma_\sigma(\alpha_2+\sigma)},
\end{aligned}\tag{11}$$

$$\begin{aligned}
I_3 &= \frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_x^b\int_c^y (t_1-x)^{(\alpha_1/\sigma)-1}(y-t_2)^{(\alpha_2/\sigma)-1}\left[\Xi(t_1,t_2)-\Xi(t_1,y)-\Xi(x,t_2)+\Xi(x,y)\right]\mathrm{d}t_2\mathrm{d}t_1\\
&= {}_\sigma\mathcal{J}^{\alpha_1,\alpha_2}_{b^-,c^+}\Xi(x,y)-\frac{(y-c)^{(\alpha_2/\sigma)}}{\Gamma_\sigma(\alpha_2+\sigma)}{}_\sigma\mathcal{J}^{\alpha_1}_{b^-}\Xi(x,y)-\frac{(b-x)^{(\alpha_1/\sigma)}}{\Gamma_\sigma(\alpha_1+\sigma)}{}_\sigma\mathcal{J}^{\alpha_2}_{c^+}\Xi(x,y)+\frac{(b-x)^{(\alpha_1/\sigma)}(y-c)^{(\alpha_2/\sigma)}}{\Gamma_\sigma(\alpha_1+\sigma)\Gamma_\sigma(\alpha_2+\sigma)},
\end{aligned}\tag{12}$$

$$\begin{aligned}
I_4 &= \frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_x^b\int_y^d (t_1-x)^{(\alpha_1/\sigma)-1}(y-t_2)^{(\alpha_2/\sigma)-1}\left[\Xi(t_1,t_2)-\Xi(t_1,y)-\Xi(x,t_2)+\Xi(x,y)\right]\mathrm{d}t_2\mathrm{d}t_1\\
&= {}_\sigma\mathcal{J}^{\alpha_1,\alpha_2}_{b^-,d^-}\Xi(x,y)-\frac{(d-y)^{(\alpha_2/\sigma)}}{\Gamma_\sigma(\alpha_2+\sigma)}{}_\sigma\mathcal{J}^{\alpha_1}_{b^-}\Xi(x,y)-\frac{(b-x)^{(\alpha_1/\sigma)}}{\Gamma_\sigma(\alpha_1+\sigma)}{}_\sigma\mathcal{J}^{\alpha_2}_{d^-}\Xi(x,y)+\frac{(b-x)^{(\alpha_1/\sigma)}(d-y)^{(\alpha_2/\sigma)}}{\Gamma_\sigma(\alpha_1+\sigma)\Gamma_\sigma(\alpha_2+\sigma)}.
\end{aligned}\tag{13}$$

Using (10)–(13) in (9), we get the required result. □

Lemma 2. *Let $\Xi\colon D \longrightarrow \mathbb{R}$ be an absolutely continuous, differentiable function such that $(\partial^2\Xi(\theta,\mu)/\partial\theta\partial\mu)$ exists and is continuous on $D\subseteq\mathbb{R}^2$. Then, for any $(x,y)\in D$, we have*

$$\begin{aligned}
&\frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_a^b\int_c^d H(t_1,t_2)\left[\int_x^{t_1}\int_y^{t_2}\frac{\partial^2\Xi(\theta,\mu)}{\partial\theta\partial\mu}\mathrm{d}\theta\mathrm{d}\mu\right]\mathrm{d}t_2\mathrm{d}t_1\\
&= \left[{}_\sigma\mathcal{J}^{\alpha_1,\alpha_2}_{x^+,y^+}\Xi(b,d)+{}_\sigma\mathcal{J}^{\alpha_1,\alpha_2}_{x^+,y^-}\Xi(b,c)+{}_\sigma\mathcal{J}^{\alpha_1,\alpha_2}_{x^-,y^+}\Xi(a,d)+{}_\sigma\mathcal{J}^{\alpha_1,\alpha_2}_{x^-,y^-}\Xi(a,c)\right]\\
&\quad -{}_kN_{\alpha_2}(c,d\colon y)\left[{}_\sigma\mathcal{J}^{\alpha_1}_{x^+}\Xi(b,y)+{}_\sigma\mathcal{J}^{\alpha_1}_{x^-}\Xi(a,y)\right]-{}_kM_{\alpha_1}(a,b\colon x)\left[{}_\sigma\mathcal{J}^{\alpha_1}_{y^+}\Xi(x,d)+{}_\sigma\mathcal{J}^{\alpha_1}_{y^-}\Xi(x,c)\right]\\
&\quad +-{}_kM_{\alpha_1}(a,b\colon x){}_kN_{\alpha_2}(c,d\colon y)\Xi(x,y)=F_2(x,y;a,b,c,d),
\end{aligned}\tag{14}$$

where

$$H(t_1,t_2)=\begin{cases}(t_1-a)^{(\alpha_1/\sigma)-1}(t_2-c)^{(\alpha_2/\sigma)-1} & a\le t_1<x \text{ and } c\le t_2<y,\\ (t_1-a)^{(\alpha_1/\sigma)-1}(d-t_2)^{(\alpha_2/\sigma)-1} & a\le t_1<x \text{ and } y\le t_2\le d,\\ (b-t_1)^{(\alpha_1/\sigma)-1}(t_2-c)^{(\alpha_2/\sigma)-1} & x\le t_1<b \text{ and } c\le t_2<y,\\ (b-t_1)^{(\alpha_1/\sigma)-1}(d-t_2)^{(\alpha_2/\sigma)-1} & x\le t_1\le b \text{ and } y\le t_2<d.\end{cases} \tag{15}$$

Proof. The proof is same as the proof of Lemma 1. □

Lemma 3. *Let $\Xi\colon D\longrightarrow\mathbb{R}$ be an absolutely continuous, differentiable function such that $(\partial^2\Xi(\theta,\mu)/\partial\theta\partial\mu)$ exists and is continuous on $D\subseteq\mathbb{R}^2$. Then, for any $(x,y)\in D$, we have*

$$\begin{aligned}&\frac{1}{4\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_a^b\int_c^d\left[(t_1-a)^{(\alpha_1/\sigma)-1}+(b-t_1)^{(\alpha_1/\sigma)-1}\right]\times\left[(t_2-c)^{(\alpha_2/\sigma)-1}+(d-t_2)^{(\alpha_2/\sigma)-1}\right]\\&\cdot\left[\int_x^{t_1}\int_y^{t_2}\frac{\partial^2\Xi(\theta,\mu)}{\partial\theta\partial\mu}\mathrm{d}\theta\mathrm{d}\mu\right]\mathrm{d}t_2\mathrm{d}t_1\\&=\frac{\left[{}_\sigma\mathcal{J}^{\alpha_1,\alpha_2}_{a^+,c^+}\Xi(b,d)+{}_\sigma\mathcal{J}^{\alpha_1,\alpha_2}_{x^+,d^-}\Xi(b,c)+{}_\sigma\mathcal{J}^{\alpha_1,\alpha_2}_{b^-,c^+}\Xi(a,d)+{}_\sigma\mathcal{J}^{\alpha_1,\alpha_2}_{b^-,d^-}\Xi(a,c)\right]}{4}\\&-\frac{(d-c)^{(\alpha_2/\sigma)}}{2\Gamma_\sigma(\alpha_2+\sigma)}\left[{}_\sigma\mathcal{J}^{\alpha_1}_{b^-}\Xi(a,y)+{}_\sigma\mathcal{J}^{\alpha_1}_{a^+}\Xi(b,y)\right]-\frac{(b-a)^{(\alpha_1/\sigma)}}{2\Gamma_\sigma(\alpha_1+\sigma)}\left[{}_\sigma\mathcal{J}^{\alpha_2}_{d^-}\Xi(x,c)+{}_\sigma\mathcal{J}^{\alpha_2}_{c^+}\Xi(x,d)\right]\\&+\frac{(b-a)^{(\alpha_1/\sigma)}(d-c)^{(\alpha_2/\sigma)}}{\Gamma_\sigma(\alpha_1+\sigma)\Gamma_\sigma(\alpha_2+\sigma)}\Xi(x,y)=F_3(x,y;a,b,c,d).\end{aligned} \tag{16}$$

Proof. Consider

$$\begin{aligned}I&=\frac{1}{4\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_a^b\int_c^d\left[(t_1-a)^{(\alpha_1/\sigma)-1}+(b-t_1)^{(\alpha_1/\sigma)-1}\right]\times\left[(t_2-c)^{(\alpha_2/\sigma)-1}+(d-t_2)^{(\alpha_2/\sigma)-1}\right]\\&\cdot\left[\int_x^{t_1}\int_y^{t_2}\frac{\partial^2\Xi(\theta,\mu)}{\partial\theta\partial\mu}\mathrm{d}\theta\mathrm{d}\mu\right]\mathrm{d}t_2\mathrm{d}t_1\\&=\frac{1}{4\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_a^b\int_c^d(t_1-a)^{(\alpha_1/\sigma)-1}(t_2-c)^{(\alpha_2/\sigma)-1}L(x,t_1,y,t_2)\mathrm{d}t_2\mathrm{d}t_1\\&+\frac{1}{4\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_a^b\int_c^d(t_1-a)^{(\alpha_1/\sigma)-1}(d-t_2)^{(\alpha_2/\sigma)-1}L(x,t_1,y,t_2)\mathrm{d}t_2\mathrm{d}t_1\\&+\frac{1}{4\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_a^b\int_c^d(b-t_1)^{(\alpha_1/\sigma)-1}(t_2-c)^{(\alpha_2/\sigma)-1}L(x,t_1,y,t_2)\mathrm{d}t_2\mathrm{d}t_1\\&+\frac{1}{4\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_a^b\int_c^d(b-t_1)^{(\alpha_1/\sigma)-1}(d-t_2)^{(\alpha_2/\sigma)-1}L(x,t_1,y,t_2)\mathrm{d}t_2\mathrm{d}t_1\\&=\frac{1}{4}\left[I_1+I_2+I_3+I_4\right].\end{aligned} \tag{17}$$

Now,

$$I_1 = \frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_a^b\int_c^d (t_1-a)^{(\alpha_1/\sigma)-1}(t_2-c)^{(\alpha_2/\sigma)-1}[\Xi(t_1,t_2)-\Xi(t_1,y)-\Xi(x,t_2)+\Xi(x,y)]\mathrm{d}t_2\mathrm{d}t_1$$
$$= {}_\sigma\mathcal{J}^{\alpha_1,\alpha_2}_{b^-,d^-}\Xi(a,c) - \frac{(d-c)^{(\alpha_2/\sigma)}}{\Gamma_\sigma(\alpha_2+\sigma)}{}_\sigma\mathcal{J}^{\alpha_1}_{b^-}\Xi(a,y) - \frac{(b-a)^{(\alpha_1/\sigma)}}{\Gamma_\sigma(\alpha_1+\sigma)}{}_\sigma\mathcal{J}^{\alpha_1}_{d^-}\Xi(x,c) + \frac{(b-a)^{(\alpha_1/\sigma)}(d-c)^{(\alpha_2/\sigma)}}{\Gamma_\sigma(\alpha_1+\sigma)\Gamma_\sigma(\alpha_2+\sigma)}\Xi(x,y). \tag{18}$$

Similarly,

$$I_2 = \frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_a^b\int_c^d (t_1-a)^{(\alpha_1/\sigma)-1}(d-t_2)^{(\alpha_2/\sigma)-1}[\Xi(t_1,t_2)-\Xi(t_1,y)-\Xi(x,t_2)+\Xi(x,y)]\mathrm{d}t_2\mathrm{d}t_1$$
$$= {}_\sigma\mathcal{J}^{\alpha_1,\alpha_2}_{b^-,c^+}\Xi(a,d) - \frac{(d-c)^{(\alpha_2/\sigma)}}{\Gamma_\sigma(\alpha_2+\sigma)}{}_\sigma\mathcal{J}^{\alpha_1}_{b^-}\Xi(a,y) - \frac{(b-a)^{(\alpha_1/\sigma)}}{\Gamma_\sigma(\alpha_1+\sigma)}{}_\sigma\mathcal{J}^{\alpha_1}_{c^+}\Xi(x,d) + \frac{(b-a)^{(\alpha_1/\sigma)}(d-c)^{(\alpha_2/\sigma)}}{\Gamma_\sigma(\alpha_1+\sigma)\Gamma_\sigma(\alpha_2+\sigma)}\Xi(x,y),$$
$$I_3 = \frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_a^b\int_c^d (b-t_1)^{(\alpha_1/\sigma)-1}(t_2-c)^{(\alpha_2/\sigma)-1}[\Xi(t_1,t_2)-\Xi(t_1,y)-\Xi(x,t_2)+\Xi(x,y)]\mathrm{d}t_2\mathrm{d}t_1$$
$$= {}_\sigma\mathcal{J}^{\alpha_1,\alpha_2}_{a^+,d^-}\Xi(b,c) - \frac{(d-c)^{(\alpha_2/\sigma)}}{\Gamma_\sigma(\alpha_2+\sigma)}{}_\sigma\mathcal{J}^{\alpha_1}_{a^+}\Xi(b,y) - \frac{(b-a)^{(\alpha_1/\sigma)}}{\Gamma_\sigma(\alpha_1+\sigma)}{}_\sigma\mathcal{J}^{\alpha_1}_{d^-}\Xi(x,c) + \frac{(b-a)^{(\alpha_1/\sigma)}(d-c)^{(\alpha_2/\sigma)}}{\Gamma_\sigma(\alpha_1+\sigma)\Gamma_\sigma(\alpha_2+\sigma)}\Xi(x,y), \tag{19}$$
$$I_4 = \frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_a^b\int_c^d (b-t_1)^{(\alpha_1/\sigma)-1}(d-t_2)^{(\alpha_2/\sigma)-1}[\Xi(t_1,t_2)-\Xi(t_1,y)-\Xi(x,t_2)+\Xi(x,y)]\mathrm{d}t_2\mathrm{d}t_1$$
$$= {}_\sigma\mathcal{J}^{\alpha_1,\alpha_2}_{a^+,c^+}\Xi(b,d) - \frac{(d-c)^{(\alpha_2/\sigma)}}{\Gamma_\sigma(\alpha_2+\sigma)}{}_\sigma\mathcal{J}^{\alpha_1}_{a^+}\Xi(b,y) - \frac{(b-a)^{(\alpha_1/\sigma)}}{\Gamma_\sigma(\alpha_1+\sigma)}{}_\sigma\mathcal{J}^{\alpha_1}_{c^+}\Xi(x,d) + \frac{(b-a)^{(\alpha_1/\sigma)}(d-c)^{(\alpha_2/\sigma)}}{\Gamma_\sigma(\alpha_1+\sigma)\Gamma_\sigma(\alpha_2+\sigma)}\Xi(x,y).$$

Using the values of I_1, I_2, I_3, and I_4 in (17), we get the required result. □

2.2. Results and Discussion. In this section, we discuss our main results.

Theorem 1. *Under the assumptions of Lemma 1, if Ξ is bounded, that is,*

$$\left\|\Xi_{\theta,\mu}\right\|_\infty = \sup_{(\theta,\mu)\in D}\left|\frac{\partial^2\Xi(\theta,\mu)}{\partial\theta\partial\mu}\right| < \infty, \tag{20}$$

then

$$\left|F_1(x,y\colon a,b,c,d)\right| \le \left\|\Xi_{\theta,\mu}\right\|_\infty {}_\sigma M_{\alpha_1+1}(a,b\colon x)_\sigma N_{\alpha_2+1}(c,d\colon y), \quad \forall (x,y)\in D. \tag{21}$$

Proof. Using Lemma 1 and the fact that $\Xi_{\theta,\mu}$ is bounded, we have

$$\begin{aligned}
\left|F_1(x,y\colon a,b,c,d)\right| &\leq \left\|\Xi_{\theta,\mu}\right\|_\infty \frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_a^b\int_c^d |G(x,y\colon a,b,c,d)|\left|t_1-x\right|\left|t_2-y\right|\mathrm{d}t_2\mathrm{d}t_1\\
&= \left\|\Xi_{\theta,\mu}\right\|_\infty\left[\frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_a^x\int_c^y (x-t_1)^{(\alpha_1/\sigma)}(y-t_2)^{(\alpha_2/\sigma)}\mathrm{d}t_2\mathrm{d}t_1\right.\\
&\quad+\frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_a^x\int_y^d (x-t_1)^{(\alpha_1/\sigma)}(t_2-y)^{(\alpha_2/\sigma)}\mathrm{d}t_2\mathrm{d}t_1\\
&\quad+\frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_x^b\int_c^y (t_1-x)^{(\alpha_1/\sigma)}(y-t_2)^{(\alpha_2/\sigma)}\mathrm{d}t_2\mathrm{d}t_1\\
&\quad\left.+\frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_x^b\int_y^d (t_1-x)^{(\alpha_1/\sigma)}(t_2-y)^{(\alpha_2/\sigma)}\mathrm{d}t_2\mathrm{d}t_1\right]\\
&= \left\|\Xi_{\theta,\mu}\right\|_\infty\left[I_1+I_2+I_3+I_4\right].
\end{aligned}\tag{22}$$

Now,

$$I_1=\frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_a^x\int_c^y (x-t_1)^{(\alpha_1/\sigma)}(y-t_2)^{(\alpha_2/\sigma)}\mathrm{d}t_2\mathrm{d}t_1=\frac{(x-a)^{(\alpha_1/\sigma)+1}(y-c)^{(\alpha_2/\sigma)+1}}{\Gamma_\sigma(\alpha_1+2\sigma)\Gamma_\sigma(\alpha_2+2\sigma)}.\tag{23}$$

Similarly,

$$\begin{aligned}
I_2&=\frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_a^x\int_y^d (x-t_1)^{(\alpha_1/\sigma)}(t_2-y)^{(\alpha_2/\sigma)}\mathrm{d}t_2\mathrm{d}t_1=\frac{(x-a)^{(\alpha_1/\sigma)+1}(d-y)^{(\alpha_2/\sigma)+1}}{\Gamma_\sigma(\alpha_1+2\sigma)\Gamma_\sigma(\alpha_2+2\sigma)},\\
I_3&=\frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_x^b\int_c^y (t_1-x)^{(\alpha_1/\sigma)}(y-t_2)^{(\alpha_2/\sigma)}\mathrm{d}t_2\mathrm{d}t_1=\frac{(b-x)^{(\alpha_1/\sigma)+1}(y-c)^{(\alpha_2/\sigma)+1}}{\Gamma_\sigma(\alpha_1+2\sigma)\Gamma_\sigma(\alpha_2+2\sigma)},\\
I_4&=\frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_x^b\int_y^d (t_1-x)^{(\alpha_1/\sigma)}(t_2-y)^{(\alpha_2/\sigma)}\mathrm{d}t_2\mathrm{d}t_1=\frac{(b-x)^{(\alpha_1/\sigma)+1}(d-y)^{(\alpha_2/\sigma)+1}}{\Gamma_\sigma(\alpha_1+2\sigma)\Gamma_\sigma(\alpha_2+2\sigma)}.
\end{aligned}\tag{24}$$

Substituting the values of I_1, I_2, I_3, and I_4 in (22), we get the required result. □

Corollary 1. *Considering $x=(a+b/2)$ and $y=(c+d/2)$ in Theorem 1, we have*

$$\begin{aligned}
&{}_\sigma\mathcal{J}^{\alpha_1,\alpha_2}_{a^+,c^+}\Xi\left(\frac{a+b}{2},\frac{c+d}{2}\right)+{}_\sigma\mathcal{J}^{\alpha_1,\alpha_2}_{a^+,d^-}\Xi\left(\frac{a+b}{2},\frac{c+d}{2}\right)+{}_\sigma\mathcal{J}^{\alpha_1,\alpha_2}_{b^-,c^+}\Xi\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\quad+{}_\sigma\mathcal{J}^{\alpha_1,\alpha_2}_{b^-,d^-}\Xi\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\frac{(d-c)^{(\alpha_2/\sigma)}}{2^{(\alpha_2/\sigma)-1}\Gamma_\sigma(\alpha_2+\sigma)}\left[{}_\sigma\mathcal{J}^{\alpha_1}_{a^+}\Xi\left(\frac{a+b}{2},\frac{c+d}{2}\right)+{}_\sigma\mathcal{J}^{\alpha_1}_{b^-}\Xi\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]\\
&\quad-\frac{(b-a)^{(\alpha_1/\sigma)}}{2^{(\alpha_1/\sigma)-1}\Gamma_\sigma(\alpha_1+\sigma)}\left[{}_\sigma\mathcal{J}^{\alpha_1}_{c^+}\Xi\left(\frac{a+b}{2},\frac{c+d}{2}\right)+{}_\sigma\mathcal{J}^{\alpha_1}_{d^-}\Xi\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right]\\
&\quad+\frac{(b-a)^{(\alpha_1/\sigma)}(d-c)^{(\alpha_2/\sigma)}}{2^{(\alpha_1+\alpha_2/\sigma)-2}\Gamma_\sigma(\alpha_1+\sigma)\Gamma_\sigma(\alpha_2+\sigma)}\Xi\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\quad\leq\frac{(b-a)^{(\alpha_1/\sigma)+1}(d-c)^{(\alpha_1/\sigma)+1}}{2^{(\alpha_1+\alpha_2/\sigma)}\Gamma_\sigma(\alpha_1+2\sigma)\Gamma_\sigma(\alpha_2+2\sigma)}\left\|\Xi_{\theta,\mu}\right\|_\infty.
\end{aligned}\tag{25}$$

Theorem 2. *Under the assumptions of Lemma 2, if Ξ is bounded, that is,*

$$\left\|\Xi_{\theta,\mu}\right\|_{\infty} = \sup_{(\theta,\mu)\in D}\left|\frac{\partial^2\Xi(\theta,\mu)}{\partial\theta\partial\mu}\right| < \infty, \tag{26}$$

then

$$\left|F_1(x, y: a, b, c, d)\right| \leq \left\|\Xi_{\theta,\mu}\right\|_{\infty} {}_kM_{\alpha_1+1}(a, b: x)_k N_{\alpha_2+1}(c, d: y), \quad \forall (x, y) \in D. \tag{27}$$

Proof. The proof of this theorem follows the same technique which was used in Theorem 1 by considering Lemma 2. □

Corollary 2. *By taking $x = (a + b/2)$ and $y = (c + d/2)$ in Theorem 2, we have*

$$\begin{aligned}
&\Big|{}_\sigma\mathcal{J}^{\alpha_1,\alpha_2}_{(a+b/2)^+,(c+d/2)^+}\Xi(b, d) + {}_\sigma\mathcal{J}^{\alpha_1,\alpha_2}_{(a+b/2)^+,(c+d/2)^-}\Xi(b, c) + {}_\sigma\mathcal{J}^{\alpha_1,\alpha_2}_{(a+b/2)^-,(c+d/2)^+}\Xi(a, d)\\
&+ {}_\sigma\mathcal{J}^{\alpha_1,\alpha_2}_{(a+b/2)^-,(c+d/2)^-}\Xi(a, c) - \frac{(d-c)^{(\alpha_2/\sigma)}}{2^{(\alpha_2/\sigma)-1}\Gamma_\sigma(\alpha_2+\sigma)}\left[{}_\sigma\mathcal{J}^{\alpha_1}_{(a+b/2)^+}\Xi\left(b, \frac{c+d}{2}\right) + {}_\sigma\mathcal{J}^{\alpha_1}_{(a+b/2)^-}\Xi\left(a, \frac{c+d}{2}\right)\right]\\
&- \frac{(b-a)^{(\alpha_1/\sigma)}}{2^{(\alpha_1/\sigma)-1}\Gamma_\sigma(\alpha_1+\sigma)}\left[{}_\sigma\mathcal{J}^{\alpha_1}_{(c+d/2)^+}\Xi\left(\frac{a+b}{2}, d\right) + {}_\sigma\mathcal{J}^{\alpha_1}_{(c+d/2)^-}\Xi\left(\frac{a+b}{2}, c\right)\right]\\
&+ \frac{(b-a)^{(\alpha_1/\sigma)}(d-c)^{(\alpha_2/\sigma)}}{2^{(\alpha_1+\alpha_2/\sigma)-2}\Gamma_\sigma(\alpha_1+\sigma)\Gamma_\sigma(\alpha_2+\sigma)}\Xi\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\Big| \leq \frac{(b-a)^{(\alpha_1/\sigma)+1}(d-c)^{(\alpha_2/\sigma)+1}}{2^{(\alpha_1+\alpha_2/\sigma)}\Gamma_\sigma(\alpha_1+2\sigma)\Gamma_\sigma(\alpha_2+2\sigma)}\left\|\Xi_{\theta,\mu}\right\|_{\infty}.
\end{aligned} \tag{28}$$

We now derive results for mappings whose elements are of L_p space.

Theorem 3. *Under the assumptions of Lemma 1, if $(\partial^2\Xi(\theta,\mu)/\partial\theta\partial\mu) \in L_p(D)$ for $p > 1$ with $(1/p) + (1/q) = 1$ with*

$$\left\|\Xi_{\theta,\mu}\right\|_p = \left(\int_a^b\int_c^d\left|\frac{\partial^2\Xi(\theta,\mu)}{\partial\theta\partial\mu}\right|^p\right)^{1/p} < \infty, \tag{29}$$

then

$$\left|F_1(x, y: a, b, c, d)\right| \leq \left\|\Xi_{\theta,\mu}\right\|_p \frac{\left[(x-a)^{(\alpha_1/\sigma)+(1/q)} + (b-x)^{(\alpha_1/\sigma)+(1/q)}\right]\left[(y-c)^{(\alpha_2/\sigma)+(1/q)} + (d-y)^{(\alpha_2/\sigma)+(1/q)}\right]}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)\left((\alpha_1/\sigma)+(1/q)\right)\left((\alpha_2/\sigma)+(1/q)\right)}, \tag{30}$$

for all $(x, y) \in D$.

Proof. From Lemma 1, property of the modulus, and applying definition of Ξ with the use of Holder's inequality, we have

$$\begin{aligned}
&\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)\left|F_1(x, y: a, b, c, d)\right|\\
&\leq \int_a^b\int_c^d\left|G(x, t_1, y, t_2)\right|\left|\int_x^{t_1}\int_y^{t_2}\frac{\partial^2\Xi(\theta,\mu)}{\partial\theta\partial\mu}\mathrm{d}\theta\mathrm{d}\mu\right|\mathrm{d}t_2\mathrm{d}t_1\\
&\leq \int_a^b\int_c^d\left|G(x, t_1, y, t_2)\right|\left|t_1 - x\right|^{(1/q)}\left|t_2 - y\right|^{(1/q)}\left|\int_x^{t_1}\int_y^{t_2}\left|\frac{\partial^2\Xi(\theta,\mu)}{\partial\theta\partial\mu}\right|^p\mathrm{d}\theta\mathrm{d}\mu\right|^{(1/p)}\mathrm{d}t_2\mathrm{d}t_1
\end{aligned}$$

$$\begin{aligned}
&= \left\|\Xi_{\theta,\mu}\right\|_p \int_a^b \int_c^d \left|G(x,t_1,y,t_2)\right| \left|t_1 - x\right|^{(1/q)} \left|t_2 - y\right|^{(1/q)} \mathrm{d}t_2 \mathrm{d}t_1 \\
&= \|\Xi_{\theta,\mu}\|_p \left[\int_a^x \int_c^y (x-t_1)^{(\alpha_1/\sigma)+(1/q)-1} (y-t_2)^{(\alpha_2/\sigma)+(1/q)-1} \mathrm{d}t_2 \mathrm{d}t_1 \right. \\
&\quad + \int_a^x \int_y^d (x-t_1)^{(\alpha_1/\sigma)+(1/q)-1} (t_2-y)^{(\alpha_2/\sigma)+(1/q)-1} \mathrm{d}t_2 \mathrm{d}t_1 \\
&\quad + \int_x^b \int_c^y (t_1-x)^{(\alpha_1/\sigma)+(1/q)-1} (y-t_2)^{(\alpha_2/\sigma)+(1/q)-1} \mathrm{d}t_2 \mathrm{d}t_1 \\
&\quad \left. + \int_x^b \int_y^d (t_1-x)^{(\alpha_1/\sigma)+(1/q)-1} (t_2-y)^{(\alpha_2/\sigma)+(1/q)-1} \mathrm{d}t_2 \mathrm{d}t_1 \right] \\
&= \left\|\Xi_{\theta,\mu}\right\|_p \left[I_1 + I_2 + I_3 + I_4\right].
\end{aligned} \tag{31}$$

Now,

$$\begin{aligned}
I_1 &= \int_a^x \int_c^y (x-t_1)^{(\alpha_1/\sigma)+(1/q)-1} (y-t_2)^{(\alpha_2/\sigma)+(1/q)-1} \mathrm{d}t_2 \mathrm{d}t_1 \\
&= \frac{(x-a)^{(\alpha_1/\sigma)+(1/q)} (y-c)^{(\alpha_2/\sigma)+(1/q)}}{((\alpha_1/\sigma)+(1/q))((\alpha_2/\sigma)+(1/q))}.
\end{aligned} \tag{32}$$

Similarly, we find the values of I_2, I_3, and I_4, and substituting their values in (31), we get the required result. □

Theorem 4. *Under the assumptions of Lemma 2, if* $(\partial^2 \Xi(\theta,\mu)/\partial\theta\partial\mu) \in L_p(D)$ *for* $p > 1$ *with* $(1/p) + (1/q) = 1$ *with*

$$\left\|\Xi_{\theta,\mu}\right\|_p = \left(\int_a^b \int_c^d \left| \frac{\partial^2 \Xi(\theta,\mu)}{\partial\theta\partial\mu} \right|^p \right)^{(1/p)} < \infty, \tag{33}$$

then

$$\begin{aligned}
\left|F_2(x,y\colon a,b,c,d)\right| \le \left\|\Xi_{\theta,\mu}\right\|_p & \frac{\left[\Gamma_\sigma(\sigma + (\sigma/q))\right]^2}{\Gamma_\sigma(\alpha_1 + \sigma + (\sigma/q))\Gamma_\sigma(\alpha_2 + \sigma + (\sigma/q))} \\
& \times \left[(x-a)^{(\alpha_1/\sigma)+(1/q)} + (b-x)^{(\alpha_1/\sigma)+(1/q)} \right] \left[(y-c)^{(\alpha_2/\sigma)+(1/q)} + (d-y)^{(\alpha_2/\sigma)+(1/q)} \right],
\end{aligned} \tag{34}$$

for any $(x, y) \in D$.

Proof. From Lemma 2, using the definition of Ξ and Holder's inequality, we have

$$\begin{aligned}
&\left|F_2(x,y\colon a,b,c,d)\right| \\
&\quad \le \frac{1}{\sigma^2 \Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)} \int_a^b \int_c^d \left|H(t_1,t_2)\right| \left| \int_x^{t_1} \int_y^{t_2} \frac{\partial^2 \Xi(\theta,\mu)}{\partial\theta\partial\mu} \mathrm{d}\theta \mathrm{d}\mu \right| \mathrm{d}t_2 \mathrm{d}t_1 \\
&\quad \le \frac{1}{\sigma^2 \Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)} \int_a^b \int_c^d \left|H(t_1,t_2)\right| \left|t_1 - x\right|^{(1/q)} \left|t_2 - y\right|^{(1/q)} \left| \int_x^{t_1} \int_y^{t_2} \left| \frac{\partial^2 \Xi(\theta,\mu)}{\partial\theta\partial\mu} \right|^p \mathrm{d}\theta \mathrm{d}\mu \right|^{(1/p)} \mathrm{d}t_2 \mathrm{d}t_1 \\
&\quad = \left\|\Xi_{\theta,\mu}\right\|_p \int_a^b \int_c^d \left|H(t_1,t_2)\right| \left|t_1 - x\right|^{(1/q)} \left|t_2 - y\right|^{(1/q)} \mathrm{d}t_2 \mathrm{d}t_1
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\left\|\Xi_{\theta,\mu}\right\|_p}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)} \int_a^b\int_c^d (t_1-a)^{(\alpha_1/\sigma)-1}(t_2-c)^{(\alpha_2/\sigma)-1}\left|t_1-x\right|^{(1/q)}\left|t_2-y\right|^{(1/q)}\mathrm{d}t_2\mathrm{d}t_1\\
&+ \frac{\left\|\Xi_{\theta,\mu}\right\|_p}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)} \int_a^b\int_c^d (t_1-a)^{(\alpha_1/\sigma)-1}(d-t_2)^{(\alpha_2/\sigma)-1}\left|t_1-x\right|^{(1/q)}\left|t_2-y\right|^{(1/q)}\mathrm{d}t_2\mathrm{d}t_1\\
&+ \frac{\left\|\Xi_{\theta,\mu}\right\|_p}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)} \int_a^b\int_c^d (b-t_1)^{(\alpha_1/\sigma)-1}(t_2-c)^{(\alpha_2/\sigma)-1}\left|t_1-x\right|^{(1/q)}\left|t_2-y\right|^{(1/q)}\mathrm{d}t_2\mathrm{d}t_1\\
&+ \frac{\left\|\Xi_{\theta,\mu}\right\|_p}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)} \int_a^b\int_c^d (b-t_1)^{(\alpha_1/\sigma)-1}(d-t_2)^{(\alpha_2/\sigma)-1}\left|t_1-x\right|^{(1/q)}\left|t_2-y\right|^{(1/q)}\mathrm{d}t_2\mathrm{d}t_1\\
&= \left[I_1+I_2+I_3+I_4\right].
\end{aligned}
\tag{35}
$$

Now, consider I_1:

$$
\begin{aligned}
I_1 &= \frac{\left\|\Xi_{\theta,\mu}\right\|_p}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)} \int_a^b\int_c^d (t_1-a)^{(\alpha_1/\sigma)-1}(t_2-c)^{(\alpha_2/\sigma)-1}\left|t_1-x\right|^{(1/q)}\left|t_2-y\right|^{(1/q)}\mathrm{d}t_2\mathrm{d}t_1\\
&= \frac{\left\|\Xi_{\theta,\mu}\right\|_p}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)} \int_a^b (t_1-a)^{(\alpha_1/\sigma)-1}(t_1-x)^{(1/q)}\left[\frac{\sigma(t_2-c)^{(\alpha_2/\sigma)}(y-t_2)^{(1/q)}}{\alpha_2}\Big|_c^y + \int_c^y \frac{\sigma(t_2-c)^{(\alpha_2/\sigma)}(y-t_2)^{(1/q)-1}}{q\alpha_2}\mathrm{d}t_2\right]\mathrm{d}t_1\\
&= \frac{\left\|\Xi_{\theta,\mu}\right\|_p}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\left[\frac{\sigma(t_1-a)^{(\alpha_1/\sigma)}(x-t_1)^{(1/q)}}{\alpha_1}\Big|_a^x + \int_a^x \frac{\sigma(t_1-a)^{(\alpha_1/\sigma)}(x-t_1)^{(1/q)-1}}{q\alpha_1}\mathrm{d}t_1\right]\\
&+ \left[\int_c^y \frac{\sigma(t_2-c)^{(\alpha_2/\sigma)}(y-t_2)^{(1/q)-1}}{q\alpha_2}\mathrm{d}t_2\right]\\
&= \frac{\left\|\Xi_{\theta,\mu}\right\|_p}{q^2\Gamma_\sigma(\alpha_1+\sigma)\Gamma_\sigma(\alpha_2+\sigma)}\left[\int_a^x (t_1-a)^{(\alpha_1/\sigma)}(x-t_1)^{(1/q)-1}\mathrm{d}t_1\right]\left[\int_c^y (t_2-c)^{(\alpha_2/\sigma)}(y-t_2)^{(1/q)-1}\mathrm{d}t_2\right]\\
&= \frac{\left\|\Xi_{\theta,\mu}\right\|_p\left[(x-a)^{(\alpha_1/\sigma)+(1/q)}(x-a)^{(\alpha_1/\sigma)+(1/q)}\right]}{q^2\Gamma_\sigma(\alpha_1+\sigma)\Gamma_\sigma(\alpha_2+\sigma)}\times\left[\int_0^1 u^{(\alpha_1/\sigma)}(1-u)^{(1/q)-1}\mathrm{d}u\right]\\
&\times\left[\int_0^1 v^{(\alpha_1/\sigma)}(1-v)^{(1/q)-1}\mathrm{d}v\right].
\end{aligned}
\tag{36}
$$

Thus,

$$
I_1 = \frac{\left\|\Xi_{\theta,\mu}\right\|_p\left[\Gamma_\sigma((\sigma/q)+\sigma)\right]^2\left[(x-a)^{(\alpha_1/\sigma)+(1/q)}(x-a)^{(\alpha_1/\sigma)+(1/q)}\right]}{\Gamma_\sigma(\alpha_1+\sigma+(\sigma/q))\Gamma_\sigma(\alpha_2+\sigma(\sigma/q))}.
\tag{37}
$$

Similarly, we can find the values I_2, I_3, and I_4, and substituting the values in (35), we get the required result.

We now obtain the results when Ξ is element of $L_1(D)$. □

Theorem 5. *Under the assumptions of Lemma 1, if* $(\partial^2\Xi(\theta,\mu)/\partial\theta\partial\mu) \in L_1(D)$ *for* $p>1$ *with* $(1/p)+(1/q)=1$ *with*

$$\left\|\Xi_{\theta,\mu}\right\|_1 = \left(\int_a^b\int_c^d\left|\frac{\partial^2\Xi(\theta,\mu)}{\partial\theta\partial\mu}\right|\right) < \infty, \tag{38}$$

then

$$\left|F_1(x, y\colon a, b, c, d)\right| \leq \left\|\Xi_{\theta,\mu}\right\|_1 {}_kM_{\alpha_1}(a, b\colon x)_k N_{\alpha_2}(c, d\colon y), \tag{39}$$

for all $(x, y) \in D$. *And*

$$\left|F_2(x, y\colon a, b, c, d)\right| \leq \left\|\Xi_{\theta,\mu}\right\|_1 {}_kM_{\alpha_1}(a, b\colon x)_k N_{\alpha_2}(c, d\colon y), \tag{40}$$

for any $(x, y) \in D$.

Proof. From Lemma 1, the property of modulus, and using the definition of Ξ, we have

$$\begin{aligned}\left|F_1(x, y\colon a, b, c, d)\right| &\leq \frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_a^b\int_c^d G(x, t_1, y, t_2)\left[\int_x^{t_1}\int_y^{t_2}\frac{\partial^2\Xi(\theta,\mu)}{\partial\theta\partial\mu}\mathrm{d}\theta\mathrm{d}\mu\right]\mathrm{d}t_2\mathrm{d}t_1\\ &\leq \left\|\Xi_{\theta,\mu}\right\|_1\frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_a^b\int_c^d G(x, t_1, y, t_2)\mathrm{d}t_2\mathrm{d}t_1\\ &= \left\|\Xi_{\theta,\mu}\right\|_1\left[\frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_a^x\int_c^y (x - t_1)^{(\alpha_1/\sigma)-1}(y - t_2)^{(\alpha_2/\sigma)-1}\mathrm{d}t_2\mathrm{d}t_1\right.\\ &\quad + \frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_a^x\int_y^d (x - t_1)^{(\alpha_1/\sigma)-1}(t_2 - y)^{(\alpha_2/\sigma)-1}\mathrm{d}t_2\mathrm{d}t_1\\ &\quad + \frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_x^b\int_c^y (t_1 - x)^{(\alpha_1/\sigma)-1}(y - t_2)^{(\alpha_2/\sigma)-1}\mathrm{d}t_2\mathrm{d}t_1\\ &\quad \left.+ \frac{1}{\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_x^b\int_y^d (t_1 - x)^{(\alpha_1/\sigma)-1}(t_2 - y)^{(\alpha_2/\sigma)-1}\mathrm{d}t_2\mathrm{d}t_1\right].\end{aligned} \tag{41}$$

Integrating above inequality, we get the required result. To prove the other part of the inequality, we use Lemma 2 and the same technique as used in the above part. □

Theorem 6. *Under the assumptions of Lemma 3, if* $(\partial^2\Xi(\theta,\mu)/\partial\theta\partial\mu) \in L_1(D)$ *for* $p > 1$ *with* $(1/p) + (1/q) = 1$ *with*

$$\left\|\Xi_{\theta,\mu}\right\|_1 = \left(\int_a^b\int_c^d\left|\frac{\partial^2\Xi(\theta,\mu)}{\partial\theta\partial\mu}\right|\right) < \infty, \tag{42}$$

then

$$\left|F_1(x, y\colon a, b, c, d)\right| \leq \left\|\Xi_{\theta,\mu}\right\|_1\frac{(b - a)^{\alpha_1/K}(d - c)^{\alpha_2/K}}{\Gamma_\sigma(\alpha_1 + \sigma)\Gamma_\sigma(\alpha_1 + \sigma)}, \tag{43}$$

for all $(x, y) \in D$.

Proof. From Lemma 3 and using modulus property with the definition of Ξ, we have

$$\begin{aligned}\left|F_3(x, y\colon a, b, c, d)\right| &\leq \frac{1}{4\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_a^b\int_c^d\left[(t_1 - a)^{(\alpha_1/\sigma)-1} + (b - t_1)^{(\alpha_1/\sigma)-1}\right]\\ &\quad \times\left[(t_2 - c)^{(\alpha_2/\sigma)-1} + (d - t_2)^{(\alpha_2/\sigma)-1}\right]\left|\int_x^{t_1}\int_y^{t_2}\frac{\partial^2\Xi(\theta,\mu)}{\partial\theta\partial\mu}\mathrm{d}\theta\mathrm{d}\mu\right|\mathrm{d}t_2\mathrm{d}t_1\\ &\leq \left\|\Xi_{\theta,\mu}\right\|_1\frac{1}{4\sigma^2\Gamma_\sigma(\alpha_1)\Gamma_\sigma(\alpha_2)}\int_a^b\int_c^d\left[(t_1 - a)^{(\alpha_1/\sigma)-1} + (b - t_1)^{(\alpha_1/\sigma)-1}\right]\\ &\quad \times\left[(t_2 - c)^{(\alpha_2/\sigma)-1} + (d - t_2)^{(\alpha_2/\sigma)-1}\right]\mathrm{d}t_2\mathrm{d}t_1.\end{aligned} \tag{44}$$

Calculating the above double integral, we get the required result. □

3. Conclusion

We have derived three new auxiliary results. Using these new auxiliary results, we have derived some new σ-fractional analogues of Ostrowski-type inequalities involving bounded functions in L_p, L_∞, and L_1 spaces. We have also discussed some new special cases in which we have obtained some midpoint-type inequalities. We hope that the techniques used in this paper will inspire interested readers.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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