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Research Article

Symplectic Toric Geometry and the Regular Dodecahedron

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The regular dodecahedron is the only simple polytope among the platonic solids which is not rational. Therefore, it corresponds neither to a symplectic toric manifold nor to a symplectic toric orbifold. In this paper, we associate to the regular dodecahedron a highly singular space called symplectic toric quasifold.

1. Introduction

According to a well-known theorem by Delzant [1], there exists a one-to-one correspondence between symplectic toric manifolds and simple rational convex polytopes satisfying a special integrality condition. One of the important features of Delzant's theorem is that it provides an explicit procedure for computing the symplectic manifold that corresponds to each given polytope. As it turns out, there are many important examples of simple convex polytopes that do not fall into this class, either because they do not satisfy Delzant's integrality condition, or, even worse, because they are not rational. The regular dodecahedron is one of the most remarkable examples of a simple polytope that is not rational. In this paper, we apply a generalization of Delzant's construction [2] to simple nonrational convex polytopes and we associate to the regular dodecahedron a *symplectic toric quasifold*. Quasifolds are a natural generalization of manifolds and orbifolds introduced by the author in [2]; they are not necessarily Hausdorff spaces and they are locally modeled by quotients of manifolds by the action of discrete groups.

This paper is structured as follows. In Section 2, we recall the generalized Delzant construction. In Section 3, we apply this construction to the regular dodecahedron and we describe the corresponding symplectic toric quasifold.

2. The Generalized Delzant Construction

We begin by recalling a few useful definitions.

Definition 1 (simple polytope). A dimension n convex polytope $\Delta \subset (\mathbb{R}^n)^*$ is said to be *simple* if each of its vertices is contained in exactly n facets.

Definition 2 (quasilattice). A *quasilattice* in $\mathbb{R}^n$ is the $\mathbb{Z}$-span of a set of $\mathbb{R}$-spanning vectors, $Y_1, \ldots, Y_d$, of $\mathbb{R}^n$.

Notice that $\mathrm{Span}_{\mathbb{Z}}\{Y_1, \ldots, Y_d\}$ is an ordinary lattice if and only if it admits a set of generators which is a basis of $\mathbb{R}^n$.

Consider now a dimension n convex polytope $\Delta \subset (\mathbb{R}^n)^*$ having d facets. Then, there exist elements $X_1, \ldots, X_d$ in $\mathbb{R}^n$ and $\lambda_1, \ldots, \lambda_d$ in $\mathbb{R}$ such that

$$\Delta = \bigcap_{j=1}^{d} \left\{ \mu \in (\mathbb{R}^n)^* \mid \left\langle \mu, X_j \right\rangle \geq \lambda_j \right\}. \tag{1}$$

Definition 3 (quasirational polytope). Let Q be a quasilattice in $\mathbb{R}^n$. A convex polytope $\Delta \subset (\mathbb{R}^n)^*$ is said to be *quasirational* with respect to Q if the vectors $X_1, \ldots, X_d$ in (1) can be chosen in Q.

We remark that each polytope in $(\mathbb{R}^n)^*$ is quasirational with respect to some quasilattice Q: just take the quasilattice that is generated by the elements $X_1, \ldots, X_d$ in (1). Notice that if $X_1, \ldots, X_d$ can be chosen inside an ordinary lattice, then the polytope is rational in the usual sense.

Definition 4 (quasitorus). Let $Q \subset \mathbb{R}^n$ be a quasilattice. We call *quasitorus* of dimension n the group and quasifold $D^n = \mathbb{R}^n/Q$.

For the definition and main properties of symplectic quasifolds and of Hamiltonian actions of quasitori on symplectic quasifolds, we refer the reader to [2, 3]. On the other hand, the basic facts on polytopes that are needed can be found in Ziegler's book [4]. We are now ready to recall from [2] the generalized Delzant construction. For the purposes of this paper, we will restrict our attention to the special case $n = 3$.

Theorem 5. *Let Q be a quasilattice in $\mathbb{R}^3$ and let $\Delta \subset (\mathbb{R}^3)^*$ be a simple convex polytope that is quasirational with respect to Q. Then, there exists a 6-dimensional compact connected symplectic quasifold M and an effective Hamiltonian action of the quasitorus $D^3 = \mathbb{R}^3/Q$ on M such that the image of the corresponding moment mapping is Δ.*

Proof. Let us consider the space $\mathbb{C}^d$ endowed with the standard symplectic form $\omega_0 = 1/(2\pi i)\sum_{j=1}^{d} dz_j \wedge d\bar{z}_j$ and the action of the torus $T^d = \mathbb{R}^d/\mathbb{Z}^d$ given by

$$\begin{aligned} \tau\colon\ & T^d \times \mathbb{C}^d \longrightarrow \mathbb{C}^d \\ & \left(\left(e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_d}\right), \underline{z}\right) \longmapsto \left(e^{2\pi i\theta_1} z_1, \ldots, e^{2\pi i\theta_d} z_d\right). \end{aligned} \tag{2}$$

This is an effective Hamiltonian action with moment mapping given by

$$\begin{aligned} J\colon\ & \mathbb{C}^d \longrightarrow \left(\mathbb{R}^d\right)^* \\ & \underline{z} \longmapsto \sum_{j=1}^{d} \left|z_j\right|^2 e_j^* + \lambda, \quad \lambda \in \left(\mathbb{R}^d\right)^* \text{ constant.} \end{aligned} \tag{3}$$

The mapping J is proper and its image is given by the cone $\mathscr{C}_\lambda = \lambda + \mathscr{C}$, where $\mathscr{C}$ denotes the positive orthant of $(\mathbb{R}^d)^*$. Take now vectors $X_1, \ldots, X_d \in Q$ and real numbers $\lambda_1, \ldots, \lambda_d$ as in (1). Consider the surjective linear mapping

$$\begin{aligned} \pi\colon\ & \mathbb{R}^d \longrightarrow \mathbb{R}^3, \\ & e_j \longmapsto X_j. \end{aligned} \tag{4}$$

Consider the dimension 3 quasitorus $D^3 = \mathbb{R}^3/Q$. Then, the linear mapping π induces a quasitorus epimorphism $\Pi\colon T^d \to D^3$. Define now N to be the kernel of the mapping Π and choose $\lambda = \sum_{j=1}^{d} \lambda_j e_j^*$. Denote by i the Lie algebra inclusion $\mathrm{Lie}(N) \to \mathbb{R}^d$ and notice that $\Psi = i^* \circ J$ is a moment mapping for the induced action of N on $\mathbb{C}^d$. Then, according to [2, Theorem 3.1], the quotient $M = \Psi^{-1}(0)/N$ is a symplectic quasifold which is endowed with the Hamiltonian action of the quasitorus T^d/N. Since π^* is injective and J is proper, the quasifold M is compact. If we identify the quasitori D^3 and T^d/N via the epimorphism Π, we get a Hamiltonian action of the quasitorus D^3 whose moment mapping Φ has image equal to $(\pi^*)^{-1}(\mathscr{C}_\lambda \cap \ker i^*) = (\pi^*)^{-1}(\mathscr{C}_\lambda \cap \mathrm{im}\,\pi^*) = (\pi^*)^{-1}(\pi^*(\Delta))$ which is exactly Δ. This action is effective since the level set $\Psi^{-1}(0)$ contains points of the form $\underline{z} \in \mathbb{C}^d$, $z_j \neq 0$, $j = 1, \ldots, d$, where the T^d-action is free. Notice finally that $\dim M = 2d - 2\dim N = 2d - 2(d - 3) = 6$. □

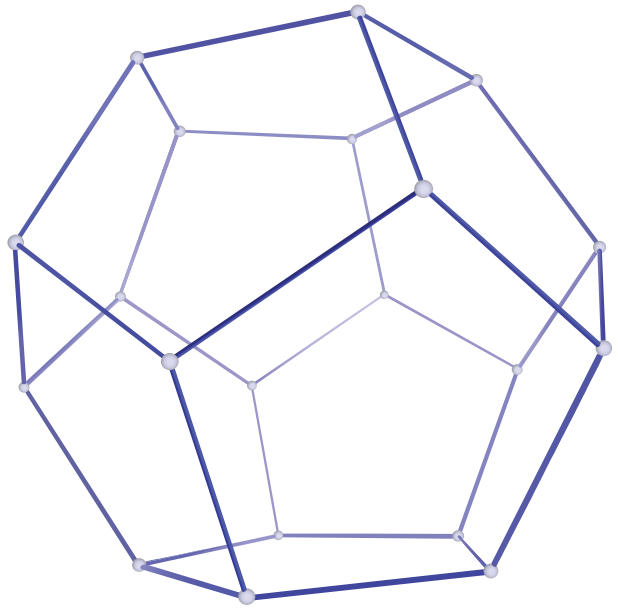

Figure 1: The regular dodecahedron.

Remark 6. We will say that M is a *symplectic toric quasifold* associated to the polytope Δ. The quasifold M depends on our choice of quasilattice Q with respect to which the polytope is quasirational and on our choice of vectors $X_1, \ldots, X_d$. Note that the case where the polytope is simple and rational, but does not necessarily satisfy Delzant's integrality condition, was treated by Lerman and Tolman in [5]. They allowed orbifold singularities and introduced the notion of symplectic toric orbifold.

3. The Regular Dodecahedron from a Symplectic Viewpoint

Let Δ be the regular dodecahedron centered at the origin and having vertices

$$\begin{gathered} (\pm 1, \pm 1, \pm 1) \\ \left(0, \pm\phi, \pm\frac{1}{\phi}\right) \\ \left(\pm\frac{1}{\phi}, 0, \pm\phi\right) \\ \left(\pm\phi, \pm\frac{1}{\phi}, 0\right), \end{gathered} \tag{5}$$

where $\phi = (1 + \sqrt{5})/2$ is the *golden ratio* and satisfies $\phi = 1 + 1/\phi$ (see Figure 1). It is a well-known fact that the polytope Δ is simple but not rational. However, consider the quasilattice P that is generated by the following vectors in $\mathbb{R}^3$:

$$\begin{aligned} Y_1 &= \left(\frac{1}{\phi}, 1, 0\right) \\ Y_2 &= \left(0, \frac{1}{\phi}, 1\right) \end{aligned}$$

$$
\begin{aligned}
Y_3 &= \left(1, 0, \frac{1}{\phi}\right)\\
Y_4 &= \left(-\frac{1}{\phi}, 1, 0\right)\\
Y_5 &= \left(0, -\frac{1}{\phi}, 1\right)\\
Y_6 &= \left(1, 0, -\frac{1}{\phi}\right).
\end{aligned}
\tag{6}
$$

We remark that these six vectors and their opposites point to the twelve vertices of a regular icosahedron that is inscribed in the sphere of radius $\sqrt{3-\phi}$ (see Figures 2 and 3). The quasilattice P is known in physics as the *simple icosahedral lattice* [6]. Now, an easy computation shows that

$$
\Delta = \bigcap_{j=1}^{12} \left\{\mu \in \left(\mathbb{R}^3\right)^* \mid \left\langle \mu, X_j \right\rangle \geq -\phi\right\}, \tag{7}
$$

where $X_i = Y_i$ and $X_{6+i} = -Y_i$, for $i = 1, \ldots, 6$. Therefore, Δ is quasirational with respect to P. Let us perform the generalized Delzant construction with respect to P and the vectors $X_1, \ldots, X_{12}$. Following the proof of Theorem 5, we consider the surjective linear mapping

$$
\begin{aligned}
\pi\colon\ \mathbb{R}^{12} &\longrightarrow \mathbb{R}^3\\
e_i &\longmapsto X_i.
\end{aligned}
\tag{8}
$$

It is easy to see that the following relations

$$
\begin{pmatrix} Y_4 \\ Y_5 \\ Y_6 \end{pmatrix} = \begin{pmatrix} \frac{1}{\phi} & \frac{1}{\phi} & -1 \\ -1 & \frac{1}{\phi} & \frac{1}{\phi} \\ \frac{1}{\phi} & -1 & \frac{1}{\phi} \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} \tag{9}
$$

imply that the kernel of π, $\mathfrak{n}$, is the 9-dimensional subspace of $\mathbb{R}^{12}$ that is spanned by the vectors

$$
\begin{aligned}
&e_1 + e_7\\
&e_2 + e_8\\
&e_3 + e_9\\
&e_4 + e_{10}\\
&e_5 + e_{11}\\
&e_6 + e_{12}\\
&e_1 + e_2 - \phi\left(e_3 + e_4\right)\\
&e_2 + e_3 - \phi\left(e_1 + e_5\right)\\
&e_1 + e_3 - \phi\left(e_2 + e_6\right).
\end{aligned}
\tag{10}
$$

Since the vectors X_i, $i = 1, \ldots, 12$, generate the quasilattice Q, the group N is connected and given by the group $\exp(\mathfrak{n})$.

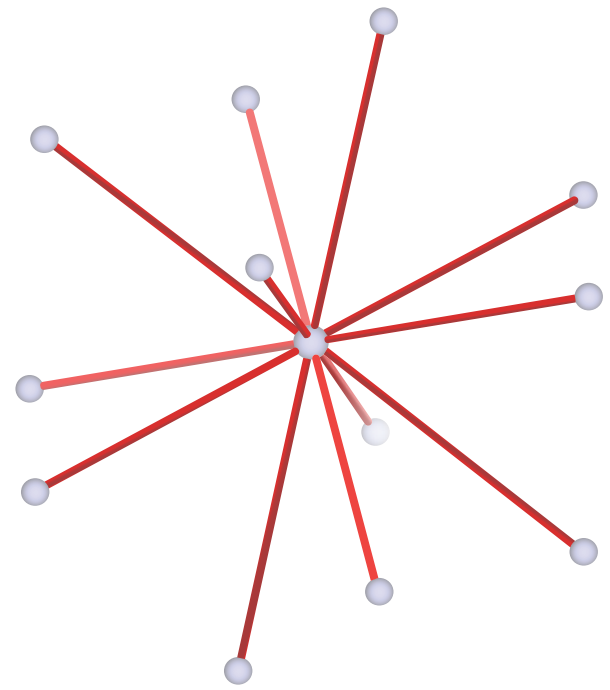

FIGURE 2: The vectors $\pm Y_1, \ldots, \pm Y_6$.

Moreover, the moment mapping for the induced N-action $\Psi\colon \mathbb{C}^{12} \to (\mathfrak{n})^*$ has the 9 components below:

$$
\begin{aligned}
&|z_1|^2 + |z_7|^2 - 2\phi\\
&|z_2|^2 + |z_8|^2 - 2\phi\\
&|z_3|^2 + |z_9|^2 - 2\phi\\
&|z_4|^2 + |z_{10}|^2 - 2\phi\\
&|z_5|^2 + |z_{11}|^2 - 2\phi\\
&|z_6|^2 + |z_{12}|^2 - 2\phi\\
&|z_1|^2 + |z_2|^2 - \phi\left(|z_3|^2 + |z_4|^2\right) + 2\\
&|z_2|^2 + |z_3|^2 - \phi\left(|z_1|^2 + |z_5|^2\right) + 2\\
&|z_1|^2 + |z_3|^2 - \phi\left(|z_2|^2 + |z_6|^2\right) + 2.
\end{aligned}
\tag{11}
$$

The level set $\Psi^{-1}(0)$ is described by the 9 equations that are obtained by setting these components to 0. Finally, the symplectic toric quasifold M is given by the compact 6-dimensional quasifold $\Psi^{-1}(0)/N$. The quasitorus $D^3 = \mathbb{R}^3/P$ acts on M in a Hamiltonian fashion, with image of the corresponding moment mapping given exactly by the dodecahedron Δ. The action of D^3 has 20 fixed points, and the moment mapping sends each of them to a different vertex of the dodecahedron. The quasifold M has an atlas made of 20 charts, each of which is centered around a different fixed point. To give an idea of the local behavior of M, we will

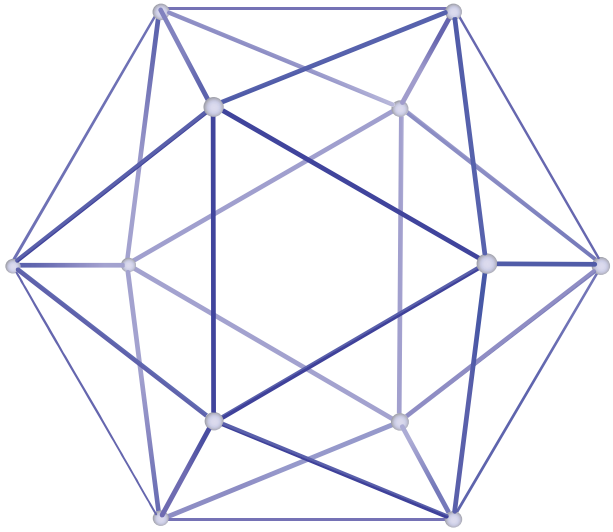

Figure 3: The regular icosahedron.

describe the chart around the fixed point that maps to the vertex $(-1,-1,-1)$. Consider the open neighborhood $\widetilde{U}$ of 0 in $\mathbb{C}^3$ defined by the inequalities below:

$$\begin{aligned}
&|z_1|^2 < 2\phi\\
&|z_2|^2 < 2\phi\\
&|z_3|^2 < 2\phi\\
&-2 < |z_1|^2 + |z_2|^2 - \phi|z_3|^2 < 2\phi\\
&-2 < |z_2|^2 + |z_3|^2 - \phi|z_1|^2 < 2\phi\\
&-2 < |z_1|^2 + |z_3|^2 - \phi|z_2|^2 < 2\phi
\end{aligned} \tag{12}$$

and consider the following slice of $\Psi^{-1}(0)$ that is transversal to the N-orbits

$$\begin{aligned}
&\widetilde{U} \xrightarrow{\tilde{\tau}} \left\{\underline{w} \in \Psi^{-1}(0) \mid w_i \neq 0,\ i = 4,\dots,12\right\}\\
&(z_1,z_2,z_3) \longmapsto (z_1,z_2,z_3,\tau_4(\underline{z}),\tau_5(\underline{z}),\tau_6(\underline{z}),\tau_7(\underline{z}),\\
&\quad \tau_8(\underline{z}),\tau_9(\underline{z}),\tau_{10}(\underline{z}),\tau_{11}(\underline{z}),\tau_{12}(\underline{z})),
\end{aligned} \tag{13}$$

where $\underline{z} = (z_1,z_2,z_3) \in \mathbb{C}^3$, $\underline{w} = (w_1,w_2,w_3,w_4,w_5,w_6,w_7,w_8,w_9,w_{10},w_{11},w_{12}) \in \mathbb{C}^{12}$, and

$$\begin{aligned}
\tau_4(\underline{z}) &= \sqrt{\frac{1}{\phi}\left(|z_1|^2 + |z_2|^2 - \phi|z_3|^2 + 2\right)}\\
\tau_5(\underline{z}) &= \sqrt{\frac{1}{\phi}\left(|z_2|^2 + |z_3|^2 - \phi|z_1|^2 + 2\right)}\\
\tau_6(\underline{z}) &= \sqrt{\frac{1}{\phi}\left(|z_1|^2 + |z_3|^2 - \phi|z_2|^2 + 2\right)}\\
\tau_7(\underline{z}) &= \sqrt{2\phi - |z_1|^2}\\
\tau_8(\underline{z}) &= \sqrt{2\phi - |z_2|^2}\\
\tau_9(\underline{z}) &= \sqrt{2\phi - |z_3|^2}\\
\tau_{10}(\underline{z}) &= \sqrt{|z_3|^2 - \frac{1}{\phi}\left(|z_1|^2 + |z_2|^2\right) + 2}\\
\tau_{11}(\underline{z}) &= \sqrt{|z_1|^2 - \frac{1}{\phi}\left(|z_2|^2 + |z_3|^2\right) + 2}\\
\tau_{12}(\underline{z}) &= \sqrt{|z_2|^2 - \frac{1}{\phi}\left(|z_1|^2 + |z_3|^2\right) + 2}.
\end{aligned} \tag{14}$$

The mapping $\tilde{\tau}$ induces a homeomorphism

$$\begin{aligned}
&\frac{\widetilde{U}}{\Gamma} \xrightarrow{\tau} U\\
&[(z_1,z_2,z_3)] \longmapsto [\tilde{\tau}(z_1,z_2,z_3)],
\end{aligned} \tag{15}$$

where the open subset U of M is the quotient

$$\frac{\left\{\underline{w} \in \Psi^{-1}(0) \mid w_i \neq 0,\ i = 4,\dots,12\right\}}{N} \tag{16}$$

and the discrete group Γ is given by

$$\left\{\left(e^{2\pi i\phi(h+l)}, e^{2\pi i\phi(h+k)}, e^{2\pi i\phi(k+l)}\right) \in T^3 \mid h,k,l \in \mathbb{Z}\right\}. \tag{17}$$

The triple $(U,\tau,\widetilde{U}/\Gamma)$ defines a chart around the fixed point corresponding to the vertex $(-1,-1,-1)$. The other charts can be described in a similar way.

Remark 7. According to joint work with Battaglia [7], to any nonrational simple polytope one can also associate a *complex toric quasifold*. If we apply the explicit construction given in [7] to the regular dodecahedron, similarly to the symplectic case, we get a complex 3-quasifold $M_{\mathbb{C}}$ endowed with an action of the complex quasitorus $D^3_{\mathbb{C}} = \mathbb{C}^3/P$. This action is holomorphic and has a dense open orbit. Moreover, by [7, Theorem 3.2], the quasifold $M_{\mathbb{C}}$ is D^3-equivariantly diffeomorphic to M and the induced symplectic form on $M_{\mathbb{C}}$ is Kähler.

Remark 8. The only other platonic solids that are simple are the cube and the regular tetrahedron. They both satisfy the hypotheses of the Delzant theorem. By applying the Delzant procedure to the cube, with respect to the standard lattice $\mathbb{Z}^3$, we get the symplectic toric manifold $S^2 \times S^2 \times S^2$; by applying it to the tetrahedron, with respect to the sublattice of $\mathbb{Z}^3$ that is generated by the 8 vectors $(\pm1,\pm1,\pm1)$, we get the symplectic toric manifold $\mathbb{CP}^3$. On the other hand, the remaining platonic solids, the regular octahedron and the regular icosahedron, are not simple. In these cases, the

Delzant procedure and our generalization will not work. The octahedron is rational, thus, the standard toric geometry applies: the toric variety associated to the octahedron is described, for example, in [8, Section 1.5]. The icosahedron, on the other hand, is not rational. However, by work of Battaglia on arbitrary convex polytopes [9, 10], one can associate to the icosahedron, both in the symplectic and complex category, a space that is stratified by quasifolds. Incidentally, Battaglia's approach can also be applied to the octahedron, yielding, in this case, a space that is stratified by manifolds.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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