Hindawi Publishing Corporation
Journal of Mathematics
Volume 2015, Article ID 950728, 11 pages
http://dx.doi.org/10.1155/2015/950728

Research Article

An Extension of Wright Function and Its Properties

Moustafa El-Shahed[1] and Ahmed Salem[2]

[1] *Unaizah Faculty of Arts and Sciences, Qassim University, P.O. Box 3771, Unaizah, Qassim 51431, Saudi Arabia*
[2] *Faculty of Information Systems & Computer Science, October 6 University, Sixth of October City, Giza 12585, Egypt*

Correspondence should be addressed to Ahmed Salem; ahmedsalem74@hotmail.com

Received 30 July 2015; Accepted 28 September 2015

Academic Editor: Peter R. Massopust

The paper is devoted to the study of the function $W_{\alpha,\beta}^{\gamma,\delta}(z)$, which is an extension of the classical Wright function and Kummer confluent hypergeometric function. The properties of $W_{\alpha,\beta}^{\gamma,\delta}(z)$ including its auxiliary functions and the integral representations are proven.

1. Introduction

The special functions of mathematical physics are found to be very useful for finding solutions of initial and/or boundary-value problems governed by partial differential equations and fractional differential equations. Special functions have widespread applications in other areas of mathematics and often new perspectives in special functions are motivated by such connections. Several special functions, called recently special functions of fractional calculus, play a very important and interesting role as solutions of fractional order differential equations, such as the Mittag-Leffler function, Wright function with its auxiliary functions, and Fox's H-function.

The Wright function is one of the special functions which plays an important role in the solution of linear partial fractional differential equations. It was introduced for the first time in [1, 2] in connection with a problem in the number theory regarding the asymptotic of the number of some special partitions of the natural numbers. Recently this function has appeared in papers related to partial differential equations of fractional order. Considering the boundary-value problems for the fractional diffusion-wave equation, that is, the linear partial integrodifferential equation obtained from the classical diffusion or wave equation by replacing the first- or second-order time derivative by a fractional derivative of order α with $0 < \alpha < 2$, it was found that the corresponding Green functions can be represented in terms of the Wright function. Furthermore, extending the methods of Lie groups in partial differential equations to the partial differential equations of fractional order, it was shown that some of the group-invariant solutions of these equations can be given in terms of the Wright and the generalized Wright functions [3]. A list of formulas concerning this function can be found in the handbook of Bateman Project, Erdélyi et al. 1953 [4]; see also [3, 5–11].

The Wright function is defined by the series representation, valid in the whole complex plane

$$W_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\alpha n + \beta)}, \quad \alpha > -1, \ \beta \in C. \tag{1}$$

It is an entire function of order $1/(1 + \alpha)$, which has been known also as generalized Bessel (or Bessel Maitland) function [12, 13]. There are two auxiliary functions of Wright function defined as

$$M_{\alpha}(z) = W_{-\alpha,1-\alpha}(-z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!\Gamma(1 - \alpha(n+1))} \quad 0 < \alpha < 1, \tag{2}$$

$$F_{\alpha}(z) = W_{-\alpha,0}(-z) = \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n!\Gamma(-\alpha n)} \quad 0 < \alpha < 1,$$

where the function $M_\alpha(z)$ is recently known as the Mainardi function [12, 13]. In a continuation of this study, we investigate the generalized Wright function $W^{\gamma,\delta}_{\alpha,\beta}(z)$ which is defined for real α and $\beta,\gamma,\delta \in C$; $\alpha > -1$, $\delta \neq 0,-1,-2,\ldots$ with $z \in C$ and $|z| < 1$ with $\alpha = -1$ as

$$W^{\gamma,\delta}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\delta)_n \Gamma(\alpha n + \beta)} \frac{z^n}{n!}, \tag{3}$$

where $(\gamma)_n = \Gamma(\gamma+n)/\Gamma(\gamma) = \gamma(\gamma+1)(\gamma+2)\cdots(\gamma+n-1)$ is a Pochhammer symbol and $\Gamma(\cdot)$ is a gamma function. The function $W^{\gamma,\delta}_{\alpha,\beta}(z)$ is an entire function of order $1/(1+\alpha)$ and generalized Wright function (1).

By analogy with (2), one can also introduce the two (generalized Wright type) auxiliary functions for any order $\alpha \in (0,1)$ and for all complex variable $z \neq 0$ by

$$\begin{aligned} M^{\gamma,\delta}_{\alpha}(z) &:= W^{\gamma,\delta}_{-\alpha,1-\alpha}(-z) \\ &= \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(\delta)_n \Gamma(1-\alpha(n+1))} \frac{(-1)^n z^n}{n!}, \end{aligned} \tag{4}$$

$$F^{\gamma,\delta}_{\alpha}(z) := W^{\gamma,\delta}_{-\alpha,0}(-z) = \sum_{n=1}^{\infty} \frac{(\gamma)_n}{(\delta)_n \Gamma(-\alpha n)} \frac{(-1)^n z^n}{n!}. \tag{5}$$

2. Basic Definitions

In this section, the definitions of some special functions and their properties necessary for our studies on generalized Wright function (3) and its auxiliary functions (4) and (5) will be presented to provide convenient references.

2.1. Wright Function. The Wright function which is defined by (1) has the Hankel contour integral representation and special case

$$W_{\alpha,\beta}(z) = \frac{1}{2\pi i} \int_{\mathrm{Ha}} t^{-\beta} e^{t+zt^{\alpha}} dt, \tag{6}$$

$$W_{-1/2,1}(-z) = \operatorname{erfc}\left(\frac{z}{2}\right), \tag{7}$$

where Ha is the Hankel contour and $\operatorname{erfc}(z)$ is the complementary error function which is defined as

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_{z}^{\infty} e^{-t^2} dt = 1 - \operatorname{erf}(z). \tag{8}$$

The important relation between two auxiliary functions (2) of Wright function (1) and some special cases are

$$F_\alpha(z) = \alpha z M_\alpha(z), \tag{9}$$

$$M_{1/2}(z) = \frac{1}{\sqrt{\pi}} e^{-z^2/4}, \tag{10}$$

$$M_{1/3}(z) = 3^{2/3} \operatorname{Ai}\left(\frac{z}{3^{1/3}}\right), \tag{11}$$

where $\operatorname{Ai}(z)$ is Airy's function which is the one of the linearly independent solutions of the ordinary differential equation $y'' - zy = 0$ and has the following properties [14, 15]:

$$\operatorname{Ai}(x) = \frac{1}{3}\sqrt{x}\left[I_{-1/3}\left(\frac{2}{3}x^{3/2}\right) - I_{1/3}\left(\frac{2}{3}x^{3/2}\right)\right] \tag{12}$$

$$\begin{aligned} &= \frac{1}{3^{2/3}\Gamma(2/3)} {}_0F_1\left(\frac{2}{3}; \frac{x^3}{9}\right) \\ &\quad - \frac{x}{3^{1/3}\Gamma(1/3)} {}_0F_1\left(\frac{4}{3}; \frac{x^3}{9}\right), \end{aligned} \tag{13}$$

$$\begin{aligned} \operatorname{Ai}(0) &= \frac{1}{3^{2/3}\Gamma(2/3)}, \\ \operatorname{Ai}'(0) &= \frac{-1}{3^{1/3}\Gamma(1/3)}. \end{aligned} \tag{14}$$

Airy's function has, from its differential equation, the following indefinite integrals:

$$\begin{aligned} &\int x \operatorname{Ai}(x)\, dx = \operatorname{Ai}'(x), \\ &\int x^2 \operatorname{Ai}(x)\, dx = x \operatorname{Ai}'(x) - \operatorname{Ai}(x), \\ &\int x^{n+3} \operatorname{Ai}(x)\, dx = x^{n+2} \operatorname{Ai}'(x) - (n+2) x^{n+1} \operatorname{Ai}(x) \\ &\qquad + (n+1)(n+2) \int x^n \operatorname{Ai}(x)\, dx, \\ &\qquad n = 0, 1, 2, \ldots. \end{aligned} \tag{15}$$

2.2. Mittag-Leffler Function. A two-parameter function of the Mittag-Leffler type is defined by the series expansion [12, 13]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad (\alpha > 0,\ \beta \in C). \tag{16}$$

It follows from the definition that

$$E_{1,m}(z) = \frac{1}{z^{m-1}}\left[e^z - \sum_{k=0}^{m-2} \frac{z^k}{k!}\right]. \tag{17}$$

Let $m > 1$ be an integer, let $\rho_1, \rho_2, \ldots, \rho_m > 0$, and let $\mu_1, \ldots, \mu_m$ be arbitrary real numbers. By means of "multi-index" (ρ_i), (μ_i), we introduce the so-called multi-index Mittag-Leffler functions [16, 17]:

$$\begin{aligned} E_{(1/\rho_i),(\mu_i)}(z) &= \sum_{k=0}^{\infty} \phi_k z^k \\ &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu_1 + k/\rho_1)\cdots\Gamma(\mu_m + k/\rho_m)}. \end{aligned} \tag{18}$$

2.3. Fox-Wright Function. The Fox-Wright ${}_p\Psi_q(z)$ function has series representation [5, 12, 18–20]:

$$ {}_p\Psi_q\left[\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p);\\ (b_1,\beta_1),\ldots,(b_q,\beta_q);\end{matrix} z\right] = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma\left(a_i+\alpha_i n\right)}{\prod_{j=1}^{q}\Gamma\left(b_j+\beta_j n\right)}\frac{z^n}{n!}. \quad (19) $$

The Fox-Wright Ψ-function is special case of Fox's H-function:

$$ \begin{aligned} &{}_p\Psi_q\left[\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p);\\ (b_1,\beta_1),\ldots,(b_q,\beta_q);\end{matrix} z\right] \\ &\quad = H^{1\,p}_{p\,q+1}\left[-z\Big|^{(1-a_1,\alpha_1)\cdots(1-a_p,\alpha_p)}_{(0,1),(1-b_1,\beta_1),\ldots,(1-b_q,\beta_q)}\right], \end{aligned} \quad (20) $$

where $H^{m\,n}_{p\,q}[z|^{(a_k,A_k)_1^p}_{(b_k,B_k)_1^q}]$ denotes the Fox H-function [21] which can be expressed as Mellin-Barnes contour integral representation in the form

$$ \begin{aligned} H^{m\,n}_{p\,q}(z) &= H^{m\,n}_{p\,q}\left[z\Big|^{(a_k,A_k)_1^p}_{(b_k,B_k)_1^q}\right] = \frac{1}{2\pi i} \\ &\cdot\int_L \frac{\prod_{k=1}^{m}\Gamma\left(b_k-B_k s\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+sA_j\right)}{\prod_{k=m+1}^{q}\Gamma\left(1-b_k+sB_k\right)\prod_{j=n+1}^{p}\Gamma\left(a_j-sA_j\right)} z^s ds, \end{aligned} \quad (21) $$

where L is a suitable contour in the complex s-plane, the orders (m,n,p,q) are integers $1\le m\le q$, $0\le n\le p$, and the parameters $a_j\in R$, $A_j>0$, $j=1,2,\ldots,p$, and $b_k\in R$, $B_k>0$, $k=1,2,\ldots,q$, are such that $A_j(b_k+i)\ne B_k(a_j-i-1)$, $i=0,1,2,\ldots$.

2.4. Meijer G-Function. In 1936, C. S. Meijer introduced the G-function as [14]

$$ \begin{aligned} G^{m\,n}_{p\,q}\left(x\Big|^{a_p}_{c_q}\right) &= G^{m\,n}_{p\,q}\left(x\Big|^{a_1,a_2,\ldots,a_p}_{c_1,c_2,\ldots,c_q}\right) = H^{m\,n}_{p\,q}\left[x\Big|^{(a_k,1)_1^p}_{(b_k,1)_1^q}\right] \\ &= \sum_{k=1}^{m}\frac{\prod_{j=1}^{m}\Gamma\left(c_j-c_k\right)\prod_{j=1}^{n}\Gamma\left(1+c_k-a_j\right)x^{c_k}}{\prod_{j=m+1}^{q}\Gamma\left(1+c_k-c_j\right)\prod_{j=n+1}^{p}\Gamma\left(a_j-c_k\right)} \\ &\cdot {}_pF_{q-1}\Big(1 \\ &+c_k-a_1,\ldots,1+c_k-a_p;1+c_k-c_1,\ldots,1+c_k \\ &-c_q;(-1)^{p-m-n}x\Big), \end{aligned} \quad (22) $$

where $1\le m\le q$, $0\le n\le p\le q-1$, and ${}_pF_q(a_1,\ldots,a_p;c_1,\ldots,c_q;x)$ is the generalized hypergeometric function which is defined as

$$ {}_pF_q\left(a_1,\ldots,a_p;c_1,\ldots,c_q;x\right) = \sum_{n=0}^{\infty}\frac{\prod_{r=1}^{p}\left(a_r\right)_n}{\prod_{r=1}^{q}\left(b_r\right)_n}\frac{x^n}{n!}. \quad (23) $$

If $p=q=1$, we have the confluent hypergeometric function which has the integral representation with the property

$$ \begin{aligned} &{}_1F_1(a;b;x) \\ &\quad = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 u^{a-1}(1-u)^{b-a-1}e^{xu}du \\ &\quad = e^x\,{}_1F_1(b-a;b;-x). \end{aligned} \quad (24) $$

2.5. Gamma Function and Incomplete Gamma Function. The contour integral representation for the reciprocal gamma function, multiplication formula, and important property [22] are

$$ \frac{1}{\Gamma(\alpha)} = \frac{1}{2\pi i}\int_{\mathrm{Ha}} e^t t^{-\alpha}dt, \quad (25) $$

$$ \Gamma(x) = m^{x-1/2}(2\pi)^{(1-m)/2}\prod_{r=1}^{m}\Gamma\left(\frac{x+r-1}{m}\right), \quad (26) $$

$$ \Gamma(a-n) = \frac{(-1)^n\Gamma(a)}{(1-a)_n}, \quad n=0,1,2,\ldots. \quad (27) $$

The upper and lower incomplete gamma function, respectively, are

$$ \Gamma(s,x) = \int_x^{\infty} t^{s-1}e^{-t}dt, \quad \Re(s)>0, \quad (28) $$

$$ \gamma(s,x) = \int_0^{x} t^{s-1}e^{-t}dt, \quad \Re(s)>0. \quad (29) $$

The two types of incomplete gamma function have the properties

$$ \begin{aligned} &\Gamma(s,x)+\gamma(s,x) = \Gamma(s), \\ &\gamma\left(\frac{1}{2},z^2\right) = \sqrt{\pi}\,\mathrm{erf}(z), \\ &\gamma(n,z) = (n-1)!\left[1-e^{-z}\sum_{k=0}^{n-1}\frac{z^k}{k!}\right], \quad n=1,2,\ldots, \\ &\gamma\left(n+\frac{1}{2},z^2\right) \\ &\quad = \left(\frac{1}{2}\right)_n\left[\sqrt{\pi}\,\mathrm{erf}(z)-e^{-z^2}\sum_{k=1}^{n}\frac{z^{2k-1}}{(1/2)_k}\right], \\ &\qquad n=1,2,\ldots. \end{aligned} \quad (30) $$

3. Integral Representations of the Generalized Wright Function

Here, we introduce the Mellin-Barnes contour integral representation and definite integral representation of function (3).

Theorem 1 (Mellin-Barnes contour integral representation). *Let $\alpha>-1$; $\gamma,\delta,\beta\in C$, and $\delta\ne 0,-1,-2,\ldots$. Then function (3) can be represented by the Mellin-Barnes contour integral as*

$$ \begin{aligned} &W^{\gamma,\delta}_{\alpha,\beta}(z) \\ &\quad = \frac{\Gamma(\delta)}{\Gamma(\gamma)}\frac{1}{2\pi i}\int_L \frac{\Gamma(\gamma+s)}{\Gamma(\delta+s)\Gamma(\beta+\alpha s)}\Gamma(-s)(-z)^s ds, \end{aligned} \quad (31) $$

where z is not equal to zero and

$$(-z)^s = \exp\left[s\left(\log|z| + i\arg(-z)\right)\right], \quad (32)$$

in which log *denotes the natural logarithm and* $|\arg(-z)|$ *is not necessarily the principal value, and L is a suitable path in the complex s-plane which runs from* $s = -i\infty$ *to* $s = i\infty$*, so the points* $s = n$, $n = 0, 1, 2, \ldots$*, lie to the right of a contour L and the points* $s = -\gamma - n$, $n = 0, 1, 2, \ldots$*, lie to its left.*

Proof. Consider the integral in (31) with the contour L replaced by the contour C_R consisting of a large clockwise-oriented semicircle of radius R and the center of the origin which lies to the right of the contour L and is bounded away from the poles. We can apply Cauchy's theorem (Residues Theorem) to the closed contour which is consisting of the contour C_R and that part of L terminated above and below by C_R as $R \to \infty$; we obtain

$$\begin{aligned}
&\frac{1}{2\pi i}\int_L \frac{\Gamma(\gamma+s)}{\Gamma(\delta+s)\Gamma(\beta+\alpha s)}\Gamma(-s)(-z)^s\,ds \\
&= \sum_{n=0}^{\infty}\operatorname{Res}_{s=n}\left[\frac{\Gamma(\gamma+s)}{\Gamma(\delta+s)\Gamma(\beta+\alpha s)}\Gamma(-s)(-z)^s\right] \\
&= \sum_{n=0}^{\infty}\lim_{s\to n}\left[\frac{\Gamma(\gamma+s)}{\Gamma(\delta+s)\Gamma(\beta+\alpha s)}(s-n)\Gamma(-s)(-z)^s\right] \\
&= \sum_{n=0}^{\infty}\frac{\Gamma(\gamma+n)}{\Gamma(\delta+n)\Gamma(\beta+\alpha n)}\frac{z^n}{n!} = \frac{\Gamma(\gamma)}{\Gamma(\delta)}W_{\alpha,\beta}^{\gamma,\delta}(z).
\end{aligned} \quad (33)$$

□

Theorem 2. *Let* $\alpha > -1$; $\gamma, \delta, \beta \in C$*, and* $\Re(\delta) > \Re(\gamma) > 0$*. Then function (3) can be represented as*

$$W_{\alpha,\beta}^{\gamma,\delta}(z) = \frac{\Gamma(\delta)}{\Gamma(\gamma)\Gamma(\delta-\gamma)}\int_0^1 u^{\gamma-1}(1-u)^{\delta-\gamma-1}W_{\alpha,\beta}(zu)\,du, \quad (34)$$

and consequently one can get

$$M_{\alpha}^{\gamma,\delta}(z) = \frac{\Gamma(\delta)}{\Gamma(\gamma)\Gamma(\delta-\gamma)}\int_0^1 u^{\gamma-1}(1-u)^{\delta-\gamma-1}M_{\alpha}(zu)\,du, \quad (35)$$

$$F_{\alpha}^{\gamma,\delta}(z) = \frac{\Gamma(\delta)}{\Gamma(\gamma)\Gamma(\delta-\gamma)}\int_0^1 u^{\gamma-1}(1-u)^{\delta-\gamma-1}F_{\alpha}(zu)\,du. \quad (36)$$

Proof. Using the contour integral representation for reciprocal gamma function (25), the integral representation of confluent hypergeometric function (23), and the Hankel contour integral representation of Wright function (6), then

$$\begin{aligned}
W_{\alpha,\beta}^{\gamma,\delta}(z) &= \sum_{n=0}^{\infty}\frac{(\gamma)_n}{(\delta)_n\Gamma(\alpha n+\beta)}\frac{z^n}{n!} = \frac{1}{2\pi i}\sum_{n=0}^{\infty}\frac{(\gamma)_n}{(\delta)_n}\frac{z^n}{n!} \\
&\cdot\int_{\mathrm{Ha}} t^{-\alpha n-\beta}e^t\,dt = \frac{1}{2\pi i}\int_{\mathrm{Ha}} t^{-\beta}e^t\sum_{n=0}^{\infty}\frac{(\gamma)_n}{(\delta)_n}\frac{t^{-\alpha n}z^n}{n!}dt \\
&= \frac{1}{2\pi i}\int_{\mathrm{Ha}} t^{-\beta}e^t\,{}_1F_1\left(\gamma;\delta;zt^{-\alpha}\right)dt = \frac{1}{2\pi i}\int_{\mathrm{Ha}} t^{-\beta}e^t \\
&\cdot\frac{\Gamma(\delta)}{\Gamma(\gamma)\Gamma(\delta-\gamma)}\int_0^1 u^{\gamma-1}(1-u)^{\delta-\gamma-1}e^{zut^{-\alpha}}\,du\,dt \\
&= \frac{\Gamma(\delta)}{\Gamma(\gamma)\Gamma(\delta-\gamma)}\int_0^1 u^{\gamma-1}(1-u)^{\delta-\gamma-1} \\
&\cdot\frac{1}{2\pi i}\int_{\mathrm{Ha}} t^{-\beta}e^{t+zut^{-\alpha}}\,dt\,du = \frac{\Gamma(\delta)}{\Gamma(\gamma)\Gamma(\delta-\gamma)} \\
&\cdot\int_0^1 u^{\gamma-1}(1-u)^{\delta-\gamma-1}W_{\alpha,\beta}(zu)\,du.
\end{aligned} \quad (37)$$

□

4. Relationship with Some Known Special Functions

We devote this section to studying the relationship between generalized Wright function (3) and some known special functions like Fox H-function, Fox-Wright function, Meijer G-function, Mittag-Leffler function, and generalized hypergeometric function.

4.1. Relation with Fox H-Function. Using (31) and the definition of Fox H-function, we obtain

$$W_{\alpha,\beta}^{\gamma,\delta}(-z) = \frac{\Gamma(\delta)}{\Gamma(\gamma)}H_{1\,3}^{1\,1}\left[z\Big|_{(0,1),(1-\beta,\alpha),(1-\delta,1)}^{(1-\gamma,1)}\right]. \quad (38)$$

4.2. Relation with Fox Wright Function. Generalized Wright function (3) can be represented by Fox Wright function (19) as

$$W_{\alpha,\beta}^{\gamma,\delta}(z) = \frac{\Gamma(\delta)}{\Gamma(\gamma)}\,{}_1\Psi_2\left[{}^{(\gamma,1);}_{(\delta,1),(\beta,\alpha);}z\right]. \quad (39)$$

4.3. Relation with Meijer G-Function. From the definitions of generalized Wright function (3) and Meijer G-function (22) with $\alpha = 1$, we obtain

$$W_{1,\beta}^{\gamma,\delta}(-z) = \frac{\Gamma(\delta)}{\Gamma(\gamma)}G_{1\,3}^{1\,1}\left[z\Big|_{0,1-\beta,1-\delta}^{1-\gamma}\right]. \quad (40)$$

Theorem 3. *Let* $\alpha = p/q$ *be a rational number with* $p, q \in N\cup\{0\}$, $q \neq 0$; $\gamma, \delta, \beta \in C$ *and* $\delta \neq 0, -1, -2, \ldots$*; then one has*

$$W^{\gamma,\delta}_{p/q,\beta}(-z) = p^{1/2-\beta} q^{\gamma-\delta+1/2} (2\pi)^{(p-q)/2} \frac{\Gamma(\delta)}{\Gamma(\gamma)} G^{q\,q}_{q\,2q+p}\left[\frac{z^q}{q^q p^p}\Big|^{1-\gamma/q,1-(\gamma+1)/q,\dots,1-(\gamma+q-1)/q}_{0,1/q,\dots,(q-1)/q;1-\delta/q,1-(\delta+1)/q,\dots,1-(\delta+q-1)/q;1-\beta/p,1-(\beta+1)/p,\dots,1-(\beta+p-1)/p}\right]. \tag{41}$$

In particular, if $\gamma = \delta$,

$$W_{p/q,\beta}(-z) = (2\pi)^{(p-q)/2} p^{1/2-\beta} q^{1/2} G^{q\,0}_{0\,p+q}\left[\frac{z^q}{q^q p^p}\Big|^{-------}_{0,1/q,\dots,(q-1)/q;1-\beta/p,1-(\beta+1)/p,\dots,1-(\beta+p-1)/p}\right]. \tag{42}$$

Proof. Using the contour integral representation of generalized Wright function (31) with change of the variable from s to qs and multiplication formula (26), we get

$$\begin{aligned} W^{\gamma,\delta}_{p/q,\beta}(-z) &= \frac{\Gamma(\delta)}{\Gamma(\gamma)}\frac{q}{2\pi i}\int_L \frac{\Gamma(\gamma+qs)}{\Gamma(\delta+qs)\Gamma(\beta+ps)}\Gamma(-qs) z^{qs} ds \\ &= p^{1/2-\beta} q^{\gamma-\delta+1/2} (2\pi)^{(p-q)/2} \frac{\Gamma(\delta)}{\Gamma(\gamma)}\frac{1}{2\pi i}\int_L \frac{\prod_{r=1}^{q}\Gamma((r-1)/q-s)\prod_{r=1}^{q}\Gamma((\gamma+r-1)/q+s)}{\prod_{r=1}^{q}\Gamma((\delta+r-1)/q+s)\prod_{r=1}^{p}\Gamma((\beta+r-1)/p+s)}\left(\frac{z^q}{q^q p^p}\right)^s ds \\ &= p^{1/2-\beta} q^{\gamma-\delta+1/2} (2\pi)^{(p-q)/2} \frac{\Gamma(\delta)}{\Gamma(\gamma)} G^{q\,q}_{q\,2q+p}\left[\frac{z^q}{q^q p^p}\Big|^{1-\gamma/q,1-(\gamma+1)/q,\dots,1-(\gamma+q-1)/q}_{0,1/q,\dots,(q-1)/q;1-\delta/q,1-(\delta+1)/q,\dots,1-(\delta+q-1)/q;1-\beta/p,1-(\beta+1)/p,\dots,1-(\beta+p-1)/p}\right]. \end{aligned} \tag{43}$$

□

4.4. Relation with Mittag-Leffler Function. Let $\alpha = 0$, $\gamma = 1$, $\beta,\delta \in C$, $\Re(\delta) > 0$; from definition (3), we have

$$W^{1,\delta}_{0,\beta}(z) = \frac{\Gamma(\delta)}{\Gamma(\beta)} E_{1,\delta}(z), \tag{44}$$

where $E_{\alpha,\beta}(z)$ is Mittag-Leffler function (16). If we put $\delta = m \in N$ with using formula (17), then we obtain

$$\begin{aligned} W^{1,m}_{0,\beta}(z) &= \frac{\Gamma(m)}{\Gamma(\beta)} E_{1,m}(z) \\ &= \frac{\Gamma(m)}{\Gamma(\beta) z^{m-1}}\left[e^z - \sum_{k=0}^{m-2}\frac{z^k}{k!}\right]. \end{aligned} \tag{45}$$

In the same way, if we put $\gamma = 1$ and $\alpha > 0$, then

$$W^{1,\delta}_{\alpha,\beta}(z) = \Gamma(\delta) E_{(\alpha,1),(\beta,\delta)}(z), \tag{46}$$

where $E_{(\alpha_i),(\beta_i)}(z)$ is the multiple Mittag-Leffler function which is defined by formula (18).

4.5. Relation with ${}_pF_q(z)$

Theorem 4. *Let $\alpha = 0$, $\beta \neq 0,-1,-2,\dots$, $\delta \neq 0,-1,-2,\dots$, where $\beta,\gamma,\delta \in C$; then one has*

$$W^{\gamma,\delta}_{0,\beta}(z) = \frac{e^z}{\Gamma(\beta)}\,{}_1F_1(\delta-\gamma;\delta;-z), \quad \Re(\delta) > \Re(\gamma) > 0. \tag{47}$$

Proof. From the definitions of generalized Wright function (3) and confluent hypergeometric function ((23); (24)), we have

$$\begin{aligned} W^{\gamma,\delta}_{0,\beta}(z) &= \frac{1}{\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{(\gamma)_n}{(\delta)_n}\frac{z^n}{n!} = \frac{1}{\Gamma(\beta)}\,{}_1F_1(\gamma;\delta;z) \\ &= \frac{e^z}{\Gamma(\beta)}\,{}_1F_1(\delta-\gamma;\delta;-z). \end{aligned} \tag{48}$$

□

Theorem 5. *Let $\alpha = 1$, $\beta \neq 0, -1, -2, \ldots$, $\delta \neq 0, -1, -2, \ldots$, where $\beta, \gamma, \delta \in C$; then one has*

$$W_{1,\beta}^{\gamma,\delta}(z) = \frac{1}{\Gamma(\beta)} {}_1F_2(\gamma; \delta, \beta; z), \quad \Re(\delta) > \Re(\gamma) > 0. \tag{49}$$

Proof. From the definitions of generalized Wright function (3) and generalized hypergeometric function, we have

$$\begin{aligned} W_{1,\beta}^{\gamma,\delta}(z) &= \frac{1}{\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{(\delta)_n (\beta)_n n!} \\ &= \frac{1}{\Gamma(\beta)} {}_1F_2(\gamma; \delta, \beta; z). \end{aligned} \tag{50}$$

□

Theorem 6. *Let $\alpha = -1$ with $|z| < 1$; $\gamma, \beta \in C$, and $\delta = 1 - \beta$, where β is not integer number; then*

$$W_{-1,\beta}^{\gamma,1-\beta}(z) = \frac{1}{\Gamma(\beta)} (1+z)^{-\gamma}, \tag{51a}$$

and, in particular, if $\gamma = 1 - \beta$,

$$W_{-1,\beta}(z) = \frac{1}{\Gamma(\beta)} (1+z)^{\beta-1}. \tag{51b}$$

Proof. From the definitions of generalized Wright function (3) and formula (27), we have

$$\begin{aligned} W_{-1,\beta}^{\gamma,1-\beta}(z) &= \sum_{n=0}^{\infty} \frac{(\gamma)_n}{(1-\beta)_n \Gamma(\beta - n)} \frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{(\gamma)_n (1-\beta)_n}{(1-\beta)_n \Gamma(\beta)} \frac{(-1)^n z^n}{n!} \\ &= \frac{1}{\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{(\gamma)_n (-1)^n z^n}{n!} \\ &= \frac{1}{\Gamma(\beta)} {}_1F_0(\gamma; -; -z) = \frac{1}{\Gamma(\beta)} (1+z)^{-\gamma}. \end{aligned} \tag{52}$$

□

Theorem 7. *Let $\alpha = -1/2$, $\beta = 1$; $\gamma, \delta \in C$; $\Re(\gamma) > 0$, $\Re(\delta) > \Re(\gamma)$, and $\Re(z) > 0$; then one can deduce that*

$$\begin{aligned} W_{-1/2,1}^{\gamma,\delta}(-z) &= 1 - \frac{\gamma z}{\delta\sqrt{\pi}} \\ &\cdot {}_3F_3\left(\frac{1}{2}, \frac{1+\gamma}{2}, \frac{2+\gamma}{2}; \frac{3}{2}, \frac{1+\delta}{2}, \frac{2+\delta}{2}; -\frac{z^2}{4}\right). \end{aligned} \tag{53}$$

Proof. From the integral representation of generalized Wright function (34) and formula (7), we can deduce that

$$\begin{aligned} W_{-1/2,1}^{\gamma,\delta}(-z) &= \frac{\Gamma(\delta)}{\Gamma(\gamma)\Gamma(\delta-\gamma)} \int_0^1 u^{\gamma-1} (1-u)^{\delta-\gamma-1} \\ &\cdot \operatorname{erfc}\left(\frac{zu}{2}\right) du = \frac{\Gamma(\delta)}{\Gamma(\gamma)\Gamma(\delta-\gamma)} \\ &\cdot \int_0^1 u^{\gamma-1} (1-u)^{\delta-\gamma-1} \\ &\cdot \left[1 - \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n (zu/2)^{2n+1}}{n!(2n+1)}\right] du = 1 - \frac{2\gamma}{\delta\sqrt{\pi}} \\ &\cdot \sum_{n=0}^{\infty} \frac{(-1)^n (1/2)_n (z/2)^{2n+1}}{n!(3/2)_n} \frac{((1+\gamma)/2)_n ((2+\gamma)/2)_n}{((1+\delta)/2)_n ((2+\delta)/2)_n} \\ &= 1 - \frac{\gamma z}{\delta\sqrt{\pi}} {}_3F_3\left(\frac{1}{2}, \frac{1+\gamma}{2}, \frac{2+\gamma}{2}; \frac{3}{2}, \frac{1+\delta}{2}, \frac{2+\delta}{2}; \right. \\ &\left. -\frac{z^2}{4}\right). \end{aligned} \tag{54}$$

□

Theorem 8. *Let $\alpha = 1/2$; $\gamma, \delta \in C$; $\Re(\gamma) > 0$, $\Re(\delta) > \Re(\gamma)$, and $\Re(z) > 0$; then one can deduce that*

$$M_{1/2}^{\gamma,\delta}(z) = \frac{1}{\sqrt{\pi}} {}_2F_2\left(\frac{\gamma}{2}, \frac{1+\gamma}{2}; \frac{\delta}{2}, \frac{1+\delta}{2}; -\frac{z^2}{4}\right). \tag{55}$$

Proof. From the integral representation of the auxiliary functions of generalized Wright function (35) and formula (10), we can deduce that

$$\begin{aligned} M_{1/2}^{\gamma,\delta}(z) &= \frac{\Gamma(\delta)}{\sqrt{\pi}\Gamma(\gamma)\Gamma(\delta-\gamma)} \\ &\cdot \int_0^1 u^{\gamma-1} (1-u)^{\delta-\gamma-1} e^{-z^2u^2/4} du \\ &= \frac{\Gamma(\delta)}{\sqrt{\pi}\Gamma(\gamma)\Gamma(\delta-\gamma)} \\ &\cdot \int_0^1 u^{\gamma-1} (1-u)^{\delta-\gamma-1} \sum_{n=0}^{\infty} \frac{(-1)^n (zu/2)^{2n}}{n!} du = \frac{1}{\sqrt{\pi}} \\ &\cdot \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n!} \frac{(\gamma)_{2n}}{(\delta)_{2n}} = \frac{1}{\sqrt{\pi}} \\ &\cdot \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n!} \frac{(\gamma/2)_n ((1+\gamma)/2)_n}{(\delta/2)_n ((1+\delta)/2)_n} = \frac{1}{\sqrt{\pi}} \\ &\cdot {}_2F_2\left(\frac{\gamma}{2}, \frac{1+\gamma}{2}; \frac{\delta}{2}, \frac{1+\delta}{2}; -\frac{z^2}{4}\right). \end{aligned} \tag{56}$$

□

Theorem 9. *Let $\alpha = 1/3$; $\gamma, \delta \in C$; $\Re(\gamma) > 0$, $\Re(\delta) > \Re(\gamma)$, and $\Re(z) > 0$; then one can deduce that*

$$M_{1/3}^{\gamma,\delta}(z) = \frac{1}{\Gamma(2/3)}\,{}_3F_4\left(\frac{\gamma}{3}, \frac{1+\gamma}{3}, \frac{2+\gamma}{3}; \frac{2}{3}, \frac{\delta}{3}, \frac{1+\delta}{3}, \frac{2+\delta}{3}; \frac{z^3}{27}\right) - \frac{\gamma z}{\delta\Gamma(1/3)}\,{}_3F_4\left(\frac{1+\gamma}{3}, \frac{2+\gamma}{3}, \frac{3+\gamma}{3}; \frac{4}{3}, \frac{1+\delta}{3}, \frac{2+\delta}{3}, \frac{3+\delta}{3}; \frac{z^3}{27}\right). \quad (57a)$$

In particular, if $\delta = \gamma + 1$,

$$M_{1/3}^{\gamma,\gamma+1}(z) = \frac{1}{\Gamma(2/3)}\,{}_1F_2\left(\frac{\gamma}{3}; \frac{2}{3}, \frac{3+\gamma}{3}; \frac{z^3}{27}\right) - \frac{\gamma z}{(\gamma+1)\Gamma(1/3)}\,{}_1F_2\left(\frac{1+\gamma}{3}; \frac{4}{3}, \frac{4+\gamma}{3}; \frac{z^3}{27}\right). \quad (57b)$$

Finally, if $\gamma = 1$,

$$M_{1/3}^{1,2}(z) = \frac{1}{\Gamma(2/3)}\,{}_1F_2\left(\frac{1}{3}; \frac{2}{3}, \frac{4}{3}; \frac{z^3}{27}\right) - \frac{z}{2\Gamma(1/3)}\,{}_1F_2\left(\frac{2}{3}; \frac{4}{3}, \frac{5}{3}; \frac{z^3}{27}\right). \quad (57c)$$

Proof. From the integral representation of the auxiliary functions of generalized Wright function (35) and formula (10), we can deduce that

$$\begin{aligned}
M_{1/3}^{\gamma,\delta}(z) &= \frac{3^{2/3}\Gamma(\delta)}{\Gamma(\gamma)\Gamma(\delta-\gamma)}\int_0^1 u^{\gamma-1}(1-u)^{\delta-\gamma-1}\operatorname{Ai}\left(\frac{zu}{3^{2/3}}\right)du \\
&= \frac{\Gamma(\delta)}{\Gamma(\gamma)\Gamma(\delta-\gamma)}\int_0^1 u^{\gamma-1}(1-u)^{\delta-\gamma-1} \cdot \left[\frac{1}{\Gamma(2/3)}\,{}_0F_1\left(\frac{2}{3}; \frac{z^3}{27}u^3\right) - \frac{zu}{\Gamma(1/3)}\,{}_0F_1\left(\frac{4}{3}; \frac{z^3}{27}u^3\right)\right]du = \frac{1}{\Gamma(2/3)} \\
&\cdot \sum_{n=0}^{\infty}\frac{(\gamma/3)_n((1+\gamma)/3)_n((2+\gamma)/3)_n}{(2/3)_n(\delta/3)_n((1+\delta)/3)_n((2+\delta)/3)_n}\frac{(z/3)^{3n}}{n!} - \frac{\gamma z}{\delta\Gamma(1/3)} \\
&\cdot \sum_{n=0}^{\infty}\frac{((1+\gamma)/3)_n((2+\gamma)/3)_n((3+\gamma)/3)_n}{(4/3)_n((1+\delta)/3)_n((2+\delta)/3)_n((3+\delta)/3)_n}\frac{(z/3)^{3n}}{n!} \\
&= \frac{1}{\Gamma(2/3)}\,{}_3F_4\left(\frac{\gamma}{3}, \frac{1+\gamma}{3}, \frac{2+\gamma}{3}; \frac{2}{3}, \frac{\delta}{3}, \frac{1+\delta}{3}, \frac{2+\delta}{3}; \frac{z^3}{27}\right) \\
&- \frac{\gamma z}{\delta\Gamma(1/3)}\,{}_3F_4\left(\frac{1+\gamma}{3}, \frac{2+\gamma}{3}, \frac{3+\gamma}{3}; \frac{4}{3}, \frac{1+\delta}{3}, \frac{2+\delta}{3}, \frac{3+\delta}{3}; \frac{z^3}{27}\right).
\end{aligned} \quad (58)$$

□

5. Basic Properties of the Generalized Wright Function

Theorem 10. *The important relation between the two auxiliary functions ((4); (5)) of generalized Wright function (3) is*

$$F_{\alpha}^{\gamma,\delta}(z) = \frac{\alpha\gamma z}{\delta}M_{\alpha}^{\gamma+1,\delta+1}(z). \quad (59)$$

Proof. By using the integral representations of two auxiliary functions ((35); (36)) and formula (9), we have

$$\begin{aligned}
F_{\alpha}^{\gamma,\delta}(z) &= \frac{\Gamma(\delta)}{\Gamma(\gamma)\Gamma(\delta-\gamma)}\int_0^1 u^{\gamma-1}(1-u)^{\delta-\gamma-1}F_{\alpha}(zu)\,du \\
&= \frac{\alpha\Gamma(\delta)z}{\Gamma(\gamma)\Gamma(\delta-\gamma)}\int_0^1 u^{\gamma}(1-u)^{\delta-\gamma-1}M_{\alpha}(zu)\,du \\
&= \frac{\alpha\Gamma(\delta)z}{\Gamma(\gamma)\Gamma(\delta-\gamma)}\frac{\Gamma(\gamma+1)\Gamma(\delta-\gamma)}{\Gamma(\delta+1)}M_{\alpha}^{\gamma+1,\delta+1}(z) \\
&= \frac{\alpha\gamma z}{\delta}M_{\alpha}^{\gamma+1,\delta+1}(z).
\end{aligned} \quad (60)$$

□

Theorem 11. *Let $\alpha = -1/2$, $\beta = 1$, and $\gamma \in C$; $\Re(\gamma) > 0$ and $\Re(z) > 0$; then one can arrive at*

$$W_{-1/2,1}^{\gamma,\gamma+1}(-z) = \operatorname{erfc}\left(\frac{z}{2}\right) + \frac{2^{\gamma}}{\sqrt{\pi}z^{\gamma}}\gamma\left(\frac{1+\gamma}{2}, \frac{z^2}{4}\right), \quad (61)$$

where $\gamma(a,x)$ is incomplete gamma function (29) and $\operatorname{erf}(x)$ is error function (8).

Proof. From the integral representation of generalized Wright function (34) and formula (7), we have

$$\begin{aligned}
W_{-1/2,1}^{\gamma,\gamma+1}(-z) &= \gamma\int_0^1 u^{\gamma-1}W_{-1/2,1}(-zu)\,du \\
&= \gamma\int_0^1 u^{\gamma-1}\operatorname{erfc}\left(\frac{zu}{2}\right)du \\
&= \frac{2\gamma}{\sqrt{\pi}}\int_0^1 u^{\gamma-1}du\int_{zu/2}^{\infty}e^{-t^2}dt \\
&= \frac{z\gamma}{\sqrt{\pi}}\int_0^1 u^{\gamma}du\int_1^{\infty}e^{-z^2u^2t^2/4}dt \\
&= \frac{\gamma 2^{\gamma}}{z^{\gamma}\sqrt{\pi}}\int_1^{\infty}t^{-\gamma-1}\gamma\left(\frac{1+\gamma}{2}, \frac{t^2z^2}{4}\right)dt \\
&= \operatorname{erfc}\left(\frac{z}{2}\right) + \frac{2^{\gamma}}{\sqrt{\pi}z^{\gamma}}\gamma\left(\frac{1+\gamma}{2}, \frac{z^2}{4}\right).
\end{aligned} \quad (62)$$

□

Remark 12. In the case $\gamma = 2n+1$, $n = 0, 1, 2, \ldots$, we have

$$\begin{aligned} &W_{-1/2,1}^{2n+1,2n+2}(-z) \\ &\quad = \operatorname{erfc}\left(\frac{z}{2}\right) + \frac{2^{2n+1}}{\sqrt{\pi} z^{2n+1}} n! \left[1 - e^{-z^2/4} \sum_{k=0}^{n} \frac{z^{2k}}{k! 2^{2k}}\right], \end{aligned} \tag{63}$$

and in the case $\gamma = 2n$, $n = 1, 2, \ldots$, we have

$$\begin{aligned} W_{-1/2,1}^{2n,2n+1}(-z) &= \operatorname{erfc}\left(\frac{z}{2}\right) + \frac{2^{2n}}{\sqrt{\pi} z^{2n}} \left(\frac{1}{2}\right)_n \\ &\quad \cdot \left[\sqrt{\pi} \operatorname{erf}\left(\frac{z}{2}\right) - e^{-z^2/4} \sum_{k=1}^{n} \frac{z^{2k-1}}{(1/2)_k 2^{2k-1}}\right]. \end{aligned} \tag{64}$$

Theorem 13. *Let $\alpha = 1/2$, $\gamma \in C$; $\Re(\gamma) > 0$ and $\Re(z) > 0$; then one has*

$$M_{1/2}^{\gamma,\gamma+1}(z) = \frac{\gamma 2^{\gamma-1}}{\sqrt{\pi} z^{\gamma}} \gamma\left(\frac{\gamma}{2}, \frac{z^2}{4}\right), \tag{65}$$

where $\gamma(a, x)$ is the incomplete gamma function which is defined by (29).

Proof. From integral representation (35) of the auxiliary function $M_{\alpha}^{\gamma,\delta}(z)$ with using relation (10) and changing the variable $z^2u^2/4 = t$, we have

$$\begin{aligned} M_{1/2}^{\gamma,\gamma+1}(z) &= \gamma \int_0^1 u^{\gamma-1} M_{1/2}(zu)\, du \\ &= \frac{\gamma}{\sqrt{\pi}} \int_0^1 u^{\gamma-1} e^{-z^2u^2/4} du \\ &= \frac{\gamma 2^{\gamma-1}}{z^{\gamma} \sqrt{\pi}} \int_0^{z^2/4} t^{\gamma/2-1} e^{-t} dt \\ &= \frac{\gamma 2^{\gamma-1}}{\sqrt{\pi} z^{\gamma}} \gamma\left(\frac{\gamma}{2}, \frac{z^2}{4}\right). \end{aligned} \tag{66}$$

□

Corollary 14. *Let $\gamma = 2n+1$, $n = 0, 1, 2, \ldots$, be an odd integer; one has*

$$\begin{aligned} M_{1/2}^{2n+1,2n+2}(z) &= \left(\frac{2}{z}\right)^{2n+1} \left(\frac{1}{2}\right)_{n+1} \\ &\quad \cdot \left[\operatorname{erf}\left(\frac{z}{2}\right) - \frac{e^{-z^2/4}}{\sqrt{\pi}} \sum_{k=1}^{n} \frac{(z/2)^{2k-1}}{(1/2)_k}\right], \end{aligned} \tag{67}$$

and if $\gamma = 2n$, $n = 1, 2, \ldots$, is an even integer, one has

$$M_{1/2}^{2n,2n+1}(z) = \frac{2^{2n} n!}{\sqrt{\pi} z^{2n}} \left[1 - e^{-z^2/4} \sum_{k=0}^{n-1} \frac{z^{2k}}{k! 2^{2k}}\right]. \tag{68}$$

Theorem 15. *Let $\alpha = 1/3$ and $\Re(z) > 0$; then first auxiliary function (4) of the generalized Wright function has the forms*

$$\begin{aligned} M_{1/3}^{2,3}(z) &= \frac{6}{z^2} \left[3^{1/3} \mathrm{Ai}'\left(\frac{z}{3^{1/3}}\right) + \frac{1}{\Gamma(1/3)}\right], \\ M_{1/3}^{3,4}&(z) \\ &= \frac{9}{z^3} \left[3^{1/3} z \mathrm{Ai}'\left(\frac{z}{3^{1/3}}\right) - 3^{2/3} \mathrm{Ai}\left(\frac{z}{3^{1/3}}\right) + \frac{1}{\Gamma(2/3)}\right], \\ M_{1/3}^{n+4,n+5}&(z) \\ &= \frac{3^{2/3}(n+4)}{z^2} \mathrm{Ai}'\left(\frac{z}{3^{1/3}}\right) \\ &\quad - \frac{3(n+2)(n+3)}{z^3} \mathrm{Ai}\left(\frac{z}{3^{1/3}}\right) \\ &\quad + \frac{3(n+1)(n+2)(n+4)}{z^3} \int_0^1 t^n \mathrm{Ai}\left(\frac{z}{3^{1/3}} t\right) dt, \\ &\qquad n = 0, 1, 2, \ldots. \end{aligned} \tag{69}$$

Proof. From the integral representation of $M_{1/3}^{\gamma,\delta}(z)$ (35) and relation (11), we can deduce that

$$\begin{aligned} M_{1/3}^{\gamma,\gamma+1}(z) &= \gamma \int_0^1 u^{\gamma-1} M_{1/3}(zu)\, du \\ &= \gamma 3^{2/3} \int_0^1 u^{\gamma-1} \mathrm{Ai}\left(\frac{zu}{3^{1/3}}\right) du. \end{aligned} \tag{70}$$

In the above relation if we change the variable $zu/3^{1/3} = v$ and using the relations ((14)-(15)) and integral of Airy's function with putting $\gamma = 2$, $\gamma = 3$, and $\gamma = n+4$, $n = 0, 1, 2, \ldots$, then the proof is completed. □

6. Recurrence Relations

Theorem 16. *Let $\alpha > -1$; $z, \beta, \gamma, \delta \in C$; $\Re(\gamma) > 0$, $\Re(\delta) > \Re(\gamma)$; $\Re(z) > 0$ and $n = 0, 1, 2, \ldots$, then one can deduce the following recurrence relation:*

$$\begin{aligned} &W_{\alpha,\beta}^{\gamma,\gamma+n+1}(z) \\ &\quad = (\gamma)_{n+1} \sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!(\gamma+k)} W_{\alpha,\beta}^{\gamma+k,\gamma+k+1}(z), \end{aligned} \tag{71}$$

and, in particular, if $n = 1$,

$$W_{\alpha,\beta}^{\gamma,\gamma+2}(z) + \gamma W_{\alpha,\beta}^{\gamma+1,\gamma+2}(z) = (\gamma+1) W_{\alpha,\beta}^{\gamma,\gamma+1}(z). \tag{72}$$

Proof. Consider

$$W_{\alpha,\beta}^{\gamma,\gamma+n+1}(z)=\frac{\Gamma(\gamma+n+1)}{\Gamma(\gamma)\Gamma(n+1)}\cdot\int_0^1 u^{\gamma-1}(1-u)^n W_{\alpha,\beta}(zu)\,du=\frac{\Gamma(\gamma+n+1)}{\Gamma(\gamma)\Gamma(n+1)}\cdot\sum_{k=0}^{n}\binom{n}{k}(-1)^k\int_0^1 u^{\gamma+k-1}W_{\alpha,\beta}(zu)\,du=(\gamma)_{n+1}\cdot\sum_{k=0}^{n}\frac{(-1)^k}{k!(n-k)!(\gamma+k)}W_{\alpha,\beta}^{\gamma+k,\gamma+k+1}(z). \quad (73)$$

□

Theorem 17. *Let $\alpha>-1$; $z,\beta,\gamma,\delta\in C$; $\Re(\gamma)>0$, $\Re(\delta)>\Re(\gamma)$, and $\Re(z)>0$; then one can deduce that*

$$W_{\alpha,\beta-1}^{\gamma,\delta}(z)+(1-\beta)W_{\alpha,\beta}^{\gamma,\delta}(z)=\frac{\alpha\gamma z}{\delta}W_{\alpha,\alpha+\beta}^{\gamma+1,\delta+1}(z). \quad (74)$$

Remark 18. The above theorem can be returned to Theorem 10 if we substitute $\alpha=-\alpha$, $\beta=1$, and $z=-z$.

7. Integral Transforms of the Generalized Wright Function

In this section, we will introduce the Euler transform, Laplace transform, and Mellin transform of the function $W_{\alpha,\beta}^{\gamma,\delta}(z)$.

7.1. Euler Transform. The Euler transform of the function $W_{\alpha,\beta}^{\gamma,\delta}(z)$ follows from the beta function

$$\int_0^1 t^{a-1}(1-t)^{b-1}W_{\alpha,\beta}^{\gamma,\delta}(xt^{\sigma})\,dt=\frac{\Gamma(\delta)\Gamma(b)}{\Gamma(\gamma)}\,{}_2\Psi_3\left[{}^{(\gamma,1),(a,\sigma);}_{(\delta,1),(\beta,\alpha),(a+b,\sigma);}x\right]. \quad (75)$$

7.2. Laplace Transform. Consider

$$\int_0^{\infty} t^{a-1}e^{-st}W_{\alpha,\beta}^{\gamma,\delta}(xt^{\sigma})\,dt=\frac{\Gamma(\delta)}{s^a\Gamma(\gamma)}\,{}_2\Psi_2\left[{}^{(\gamma,1),(a,\sigma);}_{(\delta,1),(\beta,\alpha);}\frac{x}{s^{\sigma}}\right]. \quad (76)$$

7.3. Mellin Transform. Consider

$$\int_0^{\infty} t^{s-1}W_{\alpha,\beta}^{\gamma,\delta}(-t)\,dt=\frac{\Gamma(s)\Gamma(\gamma-s)}{\Gamma(\delta-s)\Gamma(\beta-\alpha s)}. \quad (77)$$

8. Derivative of the Generalized Wright Function

Theorem 19. *Let $z,\beta,\gamma,\delta\in C$; $\alpha>-1$ and $\delta\neq 0,-1,-2,\ldots$; then the first derivative of generalized Wright function (3) has the form*

$$\frac{d}{dz}\left[W_{\alpha,\beta}^{\gamma,\delta}(z)\right]=\frac{\gamma}{\delta}W_{\alpha,\alpha+\beta}^{\gamma+1,\delta+1}(z). \quad (78)$$

And the higher derivative is

$$\frac{d^m}{dz^m}\left[W_{\alpha,\beta}^{\gamma,\delta}(z)\right]=\frac{(\gamma)_m}{(\delta)_m}W_{\alpha,m\alpha+\beta}^{\gamma+m,\delta+m}(z),\quad m=0,1,2,\ldots. \quad (79)$$

Proof. From definition (3), we have

$$\frac{d}{dz}\left[W_{\alpha,\beta}^{\gamma,\delta}(z)\right]=\frac{d}{dz}\sum_{n=0}^{\infty}\frac{(\gamma)_n}{(\delta)_n\Gamma(\alpha n+\beta)}\frac{z^n}{n!}=\sum_{n=1}^{\infty}\frac{(\gamma)_n}{(\delta)_n\Gamma(\alpha n+\beta)}\frac{z^{n-1}}{(n-1)!}=\frac{\gamma}{\delta}\sum_{n=0}^{\infty}\frac{(\gamma+1)_n}{(\delta+1)_n\Gamma(\alpha n+\alpha+\beta)}\frac{z^n}{n!}=\frac{\gamma}{\delta}W_{\alpha,\alpha+\beta}^{\gamma+1,\delta+1}(z). \quad (80)$$

By repeating this process m-times, we arrive at the second requisition. □

Theorem 20. *Let $z,\beta,\gamma,\delta\in C$; $q\in N$ and $\Re(\gamma)>0$, $\Re(\delta)>\Re(\gamma)$, and $\Re(z)>0$; then the $(q-1)$th derivative of auxiliary function (4) of generalized Wright function (3) has the form*

$$\frac{d^{q-1}}{dz^{q-1}}\left[M_{1/q}^{\gamma,\delta}(z)\right]=\frac{(-1)^{q-1}(\gamma)_q}{q(\delta)_q}zM_{1/q}^{\gamma+q,\delta+q}(z),\quad q=1,2,\ldots, \quad (81)$$

and, for $h=0,1,2,\ldots,q-2$ at $z=0$, one obtains

$$\frac{d^h}{dz^h}\left[M_{1/q}^{\gamma,\delta}(z)\right]_{z=0}=\frac{(-1)^h(\gamma)_h\Gamma((h+1)/q)}{\pi(\delta)_h}\sin\frac{\pi(h+1)}{q}. \quad (82)$$

Proof. From integral representation (35), we have

$$\frac{d^{q-1}}{dz^{q-1}}\left[M_{1/q}^{\gamma,\delta}(z)\right]=\frac{d^{q-1}}{dz^{q-1}}\left[\frac{\Gamma(\delta)}{\Gamma(\gamma)\Gamma(\delta-\gamma)}\cdot\int_0^1 u^{\gamma-1}(1-u)^{\delta-\gamma-1}M_{1/q}(zu)\,du\right]$$
$$=\frac{(-1)^{q-1}\Gamma(\delta)}{\Gamma(\gamma)\Gamma(\delta-\gamma)}\int_0^1 u^{\gamma+q-2}(1-u)^{\delta-\gamma-1}\cdot\frac{1}{2\pi i}\int_{\mathrm{Ha}} e^{t-zut^{1/q}}dt\,du=\frac{(-1)^{q-1}\Gamma(\delta)}{\Gamma(\gamma)\Gamma(\delta-\gamma)}\cdot\int_0^1 u^{\gamma+q-2}(1-u)^{\delta-\gamma-1}F_{1/q}(zu)\,du$$
$$=\frac{(-1)^{q-1}(\gamma)_{q-1}}{(\delta)_{q-1}}F_{1/q}^{\gamma+q-1,\delta+q-1}(z),\tag{83}$$

and one can use the relation between two auxiliary functions of generalized Wright function (59) to complete the proof of the first requisition. To obtain the second requisition, we start with definition (4) and reflection formula of gamma function, as

$$\frac{d^h}{dz^h}\left[M_{1/q}^{\gamma,\delta}(z)\right]=\frac{d^h}{dz^h}\sum_{n=0}^{\infty}\frac{(\gamma)_n}{(\delta)_n\Gamma(1-(n+1)/q)}\frac{(-1)^n z^n}{n!}=\sum_{n=h}^{\infty}\frac{(\gamma)_n}{(\delta)_n\Gamma(1-(n+1)/q)}\frac{(-1)^n z^{n-h}}{(n-h)!},\tag{84}$$

when $z=0$; then we get

$$\frac{d^h}{dz^h}\left[M_{1/q}^{\gamma,\delta}(z)\right]_{z=0}=\frac{(-1)^h(\gamma)_h}{(\delta)_h\Gamma(1-(h+1)/q)}=\frac{(-1)^h(\gamma)_h\Gamma((h+1)/q)}{\pi(\delta)_h}\sin\frac{\pi(h+1)}{q}.\tag{85}$$

□

Theorem 21. *Let $\gamma,\delta\in C$; $n\in N\cup\{0\}$, $0<\alpha<1$, $\Re(\gamma)>0$, and $\Re(\delta)>\Re(\gamma)$; then one has*

$$\int_0^{\infty}r^n M_{\alpha}^{\gamma,\delta}(r)\,dr=\frac{\Gamma(n+1)}{\Gamma(\alpha n+1)}\frac{(1-\delta)_{n+1}}{(1-\gamma)_{n+1}}.\tag{86}$$

In particular, if $n=0$, then one obtains

$$\int_0^{\infty}M_{\alpha}^{\gamma,\delta}(r)\,dr=\frac{1-\delta}{1-\gamma}.\tag{87}$$

Proof. Consider

$$\int_0^{\infty}r^n M_{\alpha}^{\gamma,\delta}(r)\,dr=\frac{\Gamma(\delta)}{\Gamma(\gamma)\Gamma(\delta-\gamma)}\cdot\int_0^1 u^{\gamma-1}(1-u)^{\delta-\gamma-1}\int_0^{\infty}r^n M_{\alpha}(ru)\,dr\,du$$
$$=\frac{n!\Gamma(\delta)}{\Gamma(\gamma)\Gamma(\delta-\gamma)}\cdot\int_0^1 u^{\gamma-n-2}(1-u)^{\delta-\gamma-1}\frac{1}{2\pi i}\int_{\mathrm{Ha}}t^{-\alpha n-1}e^t\,dt\,du$$
$$=\frac{n!\Gamma(\delta)}{\Gamma(\gamma)\Gamma(\delta-\gamma)\Gamma(\alpha n+1)}\cdot\int_0^1 u^{\gamma-n-2}(1-u)^{\delta-\gamma-1}du$$
$$=\frac{n!\Gamma(\delta)}{\Gamma(\gamma)\Gamma(\delta-\gamma)\Gamma(\alpha n+1)}\cdot\frac{\Gamma(\gamma-n-1)\Gamma(\delta-\gamma)}{\Gamma(\delta-n-1)}=\frac{\Gamma(n+1)}{\Gamma(\alpha n+1)}\frac{(1-\delta)_{n+1}}{(1-\gamma)_{n+1}}.\tag{88}$$

□

9. Conclusion

In this paper, we generalize the definition of Wright function (1) and its auxiliary functions (2) to be the function $W_{\alpha,\beta}^{\gamma,\delta}(z)$ defined as in (3) and its auxiliary functions (4) and (5). The properties of $W_{\alpha,\beta}^{\gamma,\delta}(z)$ including its auxiliary functions and the integral representations are provided and proven. The relationship with some known special functions like Fox H-function, Fox-Wright function, Meijer G-function, Mittag-Leffler function, and generalized hypergeometric function are given. The Euler transform, Laplace transform, and Mellin transform of function (3) are introduced.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] E. M. Wright, "The generalized bessel function of order greater than one," *Quarterly Journal of Mathematics*, vol. 11, no. 1, pp. 36–48, 1940.

[2] E. M. Wright, "On the coefficients of power series having exponential singularities," *Journal of the London Mathematical Society*, vol. s1–8, no. 1, pp. 71–79, 1933.

[3] R. Goreno, Y. Luchko, and F. Mainardi, "Analytical properties and applications of the Wright function," *Fractional Calculus and Applied Analysis*, vol. 2, pp. 383–414, 1999.

[4] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions*, vol. 1–3, McGraw-Hill, New York, NY, USA, 1953.

[5] Y. Luchko, "Maximum principle for the generalized time-fractional diffusion equation," *Journal of Mathematical Analysis and Applications*, vol. 351, no. 1, pp. 218–223, 2009.

[6] Y. Luchko, H. Matrinez, and J. J. Trujillo, "Fractional Fourier transform and some of its applications," *Fractional Calculus and Applied Analysis*, vol. 11, pp. 457–470, 2008.

[7] Y. Luchko, "Integral transforms of the Mellin convolution type and their generating operators," *Integral Transforms and Special Functions*, vol. 19, no. 11, pp. 809–851, 2008.

[8] Y. Luchko, "Algorithms for evaluation of the Wright function for the real arguments' values," *Fractional Calculus and Applied Analysis*, vol. 11, pp. 57–75, 2008.

[9] Y. Luchko and J. Trujillo, "Caputo-type modification of the Erdelyi-Kober fractional derivative," *Fractional Calculus and Applied Analysis*, vol. 10, pp. 249–267, 2007.

[10] F. Mainardi and G. Pagnini, "The role of the Fox-Wright functions in fractional sub-diffusion of distributed order," *Journal of Computational and Applied Mathematics*, vol. 207, no. 2, pp. 245–257, 2007.

[11] F. Mainardi, "Fractional calculus: some basic problems in continuum and statistical mechanics," in *Fractals and Fractional Calculus in Continuum Mechanics*, A. Carpinteri and F. Mainardi, Eds., vol. 378 of *International Centre for Mechanical Sciences*, pp. 291–348, Springer, Vienna, Austria, 1997.

[12] V. Kiryakova, "Some special functions related to fractional calculus and fractional (non-integer) order control systems and equations," *Facta Universitatis Series: Automatic Control and Robotics*, vol. 7, no. 1, pp. 79–98, 2008.

[13] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, NY, USA, 1999.

[14] L. C. Andrews, *Special Functions for Engineers Mathematicians*, MacMillan, New York, NY, USA, 1985.

[15] F. W. J. Olver, *Asymptotics and Special Functions*, Academic Press, New York, NY, USA, 1974.

[16] R. Goreno, A. A. Kilbas, F. Mainardi, and S. V. Rogosin, *Mittag-Leffler Functions. Related Topics and Applications*, Springer, Heidelberg, Germany, 2014.

[17] V. S. Kiryakova, "Multiple (multiindex) Mittag–Leffler functions and relations to generalized fractional calculus," *Journal of Computational and Applied Mathematics*, vol. 118, no. 1-2, pp. 241–259, 2000.

[18] A. A. Kilbas, "Fractional calculus of the generalized Wright function," *Fractional Calculus and Applied Analysis*, vol. 8, pp. 113–126, 2005.

[19] A. A. Kilbas and M. Saigo, *H-Transforms: Theory and Applications*, Chapman & Hall, CRC Press, Boca Raton, Fla, USA, 2004.

[20] A. A. Kilbas, M. Saigo, and J. Trujillo, "On the generalized Wright function," *Fractional Calculus and Applied Analysis*, vol. 5, pp. 437–460, 2002.

[21] A. M. Mathai, R. K. Saxena, and J. J. Haubold, *The H-Function: Theory and Applications*, Springer, Dordrecht, The Netherlands, 2010.

[22] G. Arfken and H. Weber, *Mathematical Methods for Physicists*, Harcourt, Academic Press, 2000.