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Research Article

Irreducible Modular Representations of the Reflection Group $G(m, 1, n)$

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In an article published in 1980, Farahat and Peel realized the irreducible modular representations of the symmetric group. One year later, Al-Aamily, Morris, and Peel constructed the irreducible modular representations for a Weyl group of type B_n. In both cases, combinatorial methods were used. Almost twenty years later, using a geometric construction based on the ideas of Macdonald, first Aguado and Araujo and then Araujo, Bigeón, and Gamondi also realized the irreducible modular representations for the Weyl groups of types A_n and B_n. In this paper, we extend the geometric construction based on the ideas of Macdonald to realize the irreducible modular representations of the complex reflection group of type $G(m, 1, n)$.

1. Introduction

The irreducible modular representations for the symmetric group were first realized by Farahat and Peel in the manuscript [1] and one year later Al-Aamily et al. [2], using similar methods, constructed the irreducible modular representations for a Weyl group of type B_n.

In an article published in 1972 [3], Macdonald introduced a geometric construction of some irreducible representations for the Weyl groups built from the action of a Weyl group on the associated root systems. Every irreducible representation of a Weyl group of type A_n or B_n can be realized as Macdonald's representation (see Carter [4] or Lusztig [5]). Following Macdonald's ideas, the irreducible modular representations for the symmetric group were constructed in [6] and in [7]; a similar circle of ideas produces the irreducible modular representations for a Weyl group of type B_n. Along these lines, a construction of a Gelfand model for the complex reflection group $G(m, 1, n)$ was given in [8], also based on Macdonald's ideas (see [9]).

The main result of this study is an extension of the construction given in [6] to obtain the irreducible modular representations of the group $G(m, 1, n)$ (see Theorem 25).

2. Notation and Preliminary Results

In this section, we introduce the main ingredients used in our construction and their respective notations. In particular, $\mathfrak{S}_n^m$ will denote the group $G(m, 1, n)$, which is also known as the generalized symmetric group. K will denote a field of characteristic $p \neq 2$. $\mathscr{A}$ will denote the polynomial ring $K[x_1, \dots, x_n]$, and, for every $j \geq 0$, $\mathscr{A}_j$ denotes the subspace of homogenous polynomials of degree j. We let $\mathbb{I}_n$ denote the set $\{1, 2, \dots, n\}$ and $\mathfrak{S}_n$ indicates the permutation group of $\mathbb{I}_n$.

Initially we assume that K contains all mth roots of the unity; in particular, we assume that m and p are coprime. In order to deal with the general case, this hypothesis will be reconsidered at the end of the paper.

For convenience, the group $\mathfrak{S}_n^m$ will be presented as the semidirect product:

$$\mathfrak{S}_n^m = \mathscr{C}_m^n \ltimes \mathfrak{S}_n, \quad (1)$$

where $\mathscr{C}_m \subset K$ is the group of the mth roots of the unity.

Each element $\sigma \in \mathfrak{S}_n^m$ has a unique decomposition

$$\sigma = (d, \tau), \quad (2)$$

where $d = (d_1, \dots, d_n) \in \mathscr{C}_m^n$ and $\tau \in \mathfrak{S}_n$.

In order to simplify notation, in certain cases, $\mathscr{C}_m^n$ will be identified with $(\mathscr{C}_m^n, 1)$ and $\mathfrak{S}_n$ with $(1, \mathfrak{S}_n)$, that is, $d \in \mathscr{C}_m^n$ with $(d, 1)$ and $\tau \in \mathfrak{S}_n$ with $(1, \tau)$.

2.1. The Character. The following character will be utilized for defining projectors associated with subgroups of $\mathfrak{S}_n^m$. Let $\chi : \mathfrak{S}_n^m \to K$ be the linear character given by

$$\chi(\sigma) = \left(\prod_{i=1}^{n} d_i\right) \operatorname{sgn}(\tau), \tag{3}$$

where sgn is the sign map.

Remark 1. In the case where $K = \mathbb{C}$, the field of complex numbers, this character is, precisely, the determinant of the geometric representation of $\mathfrak{S}_m^n$, realized as a group generated by unitary reflections (see [10]). Thus, we have

$$\begin{aligned} \chi(d) &= \prod_{i=1}^{n} d_i \quad \text{if } d \in \mathscr{C}_m^n, \\ \chi(\tau) &= \operatorname{sgn}(\tau) \quad \text{if } \tau \in S_n. \end{aligned} \tag{4}$$

2.2. Subgroups. As in the case of characteristic zero, the irreducible representations are associated with subgroups of $\mathfrak{S}_n^m$. We now introduce the type of subgroup that will be used for building irreducible modules.

If $J \subset \mathbb{I}_n$, let $\mathfrak{S}_n^m(J)$ denote the subgroup of $\mathfrak{S}_n^m$ given by

$$\begin{aligned} &\mathfrak{S}_n^m(J) \\ &\quad = \left\{(d, \tau) \in \mathfrak{S}_n^m : d_k = 1,\ \tau(k) = k,\ \forall k \in \mathbb{I}_n - J\right\}. \end{aligned} \tag{5}$$

Note that

$$\mathfrak{S}_n^m(J) = \mathscr{C}_m^n(J) \times_s \mathfrak{S}_n(J), \tag{6}$$

where $\mathfrak{S}_n(J)$ is the group of all permutations which fix all elements in $\mathbb{I}_n - J$ and $\mathscr{C}_m^n(J)$ is the subgroup of $\mathscr{C}_m^n$ given by

$$\mathscr{C}_m^n(J) = \left\{d \in \mathscr{C}_m^n : d_k = 1,\ \forall k \in \mathbb{I}_n - J\right\}. \tag{7}$$

2.3. The Action of the $\mathfrak{S}_n^m$ in $\mathscr{A}$ and the Bilinear Form. From now on, $\mathscr{M} = \{\alpha : \mathbb{I}_n \to \mathbb{N}_0\}$ will denote the set of multi-indexes and we use the notation

$$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \quad \text{if } \alpha \in \mathscr{M}. \tag{8}$$

There is a natural action of $\mathfrak{S}_n^m$ in the polynomial ring $\mathscr{A}$ as well as in its dual space $\mathscr{A}^*$, which gives them $\mathfrak{S}_n^m$-module structure. This action is described as follows: consider $\mathfrak{S}_n$ acting on the set multi-indexes $\mathscr{M}$ by

$$\tau \cdot \alpha = \alpha \circ \tau^{-1} \quad \text{if } \tau \in \mathfrak{S}_n. \tag{9}$$

This action, in turn, naturally induces an action of $\mathfrak{S}_n^m$ on the polynomial ring $\mathscr{A}$ given by

$$\sigma \cdot x^\alpha = \prod_{i=1}^{n} \left(d_i x_i\right)^{(\tau\cdot\alpha)_i}, \tag{10}$$

where $\alpha \in \mathscr{M}$. On the other hand, for $\varphi \in \mathscr{A}^*$, we have

$$(\sigma \cdot \varphi)(P) = \varphi\left(\sigma^{-1} P\right) \quad \forall P \in \mathscr{A}. \tag{11}$$

If $\mathfrak{S}_n$ is identified with the subgroup $\{1\} \times \mathfrak{S}_n$ of $\mathfrak{S}_n^m$, it is clear that the defined action extends the natural action of the symmetric group on the polynomial ring $\mathscr{A}$.

We will use the K-bilinear form, $\langle -, - \rangle$, defined on $\mathscr{A}$ determined by

$$\left\langle x^\alpha, x^\beta \right\rangle = \delta_{\alpha,\beta}, \tag{12}$$

where $\alpha, \beta \in \mathscr{M}$ and δ is the Kronecker function.

2.4. Multipartitions. The multipartitions are introduced as the natural parameters for enumerating the conjugacy classes of $\mathfrak{S}_n^m$ (see Proposition 4).

If λ is a partition of n, it can be denoted as $\lambda = (\lambda_1, \ldots, \lambda_l)$ with $\lambda_1 \geq \cdots \geq \lambda_l$ or as $\lambda = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$, where

$$m_i = \left|\left\{j : \lambda_j = i\right\}\right|. \tag{13}$$

A partition $\lambda = (1^{m_1}, 2^{m_2}, \ldots, n^{m_n})$ of n is called p-regular if

$$m_j < p \quad \text{for } j = 1, \ldots, n. \tag{14}$$

A conjugacy class of a finite group is called p-regular if p does not divide the order of an element of the class.

Remark 2. For the symmetric group $\mathfrak{S}_n$, the number of nonequivalent irreducible modular representations coincides with the number of p-regular conjugacy classes of $\mathfrak{S}_n$ [11, Theorem 11.5]. For a general finite group this is true if K is a decomposition field for G (see [12, Theorem 21.25 and p. 492]).

Definition 3. m-partition Λ of n is m-uple $(\mu^0, \ldots, \mu^{m-1})$ where each μ^i is a partition and

$$\sum_{i=0}^{m-1} \left|\mu^i\right| = n. \tag{15}$$

The m-partition is called p-regular if all partitions μ^i are p-regular.

The following results will be used later on. They are well known and will be presented without proof.

Proposition 4. *There is bijective correspondence among the conjugacy classes of $\mathfrak{S}_n^m$ and the m-partitions of n.*

Proof. See [13]. □

Lemma 5. *The number of p-regular partitions of n coincides with the number of p-regular conjugacy classes of $\mathfrak{S}_n$.*

Proof. See [11]. □

Remark 6. Each element in $\mathfrak{S}_n^m$ can be uniquely factorized, except for reordering, as a product of disjoint cycles, where there are m distinct classes of these cycles. This is the reason for which the conjugacy classes of $\mathfrak{S}_n^m$ are indexed by m-partitions of n and, consequently, the p-regular conjugacy classes are indexed by p-regular m-partitions; see [13].

3. The Modules $\mathfrak{N}_\lambda$ and $\mathfrak{N}_\Lambda$

As has been indicated above, it will be assumed that p does not divide m until the last section. After that, the extension to the general case does not present a major difficulty and will be treated at the end.

In this section two modules will be introduced: $\mathfrak{M}_\lambda$, associated with the symmetric group $\mathfrak{S}_n$, and $\mathfrak{M}_\Lambda$, associated with the group $\mathfrak{S}_n^m$. Then, we will define spaces $\mathfrak{N}_\lambda$ and $\mathfrak{N}_\Lambda$ of linear functionals in $\mathfrak{M}_\lambda$ and $\mathfrak{M}_\Lambda$, respectively, on which we will realize the irreducible modular representations of $\mathfrak{S}_n$ and $\mathfrak{S}_n^m$. It is convenient to have in account what happens in the case of the symmetric group, as it can be extended to the construction for the group $\mathfrak{S}_n^m$, from the key fact given in Theorem 12(i).

When K is field of characteristic 0, it is not difficult to show that $\mathfrak{M}_\lambda \simeq \mathfrak{N}_\lambda$ as $\mathfrak{S}_n$-modules and $\mathfrak{M}_\Lambda \simeq \mathfrak{N}_\Lambda$ as $\mathfrak{S}_n^m$-modules.

For each nonempty set $C = \{c_1, \dots, c_k\}$ of $\mathbb{I}_n$, where $c_1 < \cdots < c_k$, $V_C(x_1, \dots, x_n)$ will denote the usual Vandermonde determinant

$$V_C = \prod_{1 \le i < j \le k} \left(x_{c_j} - x_{c_i} \right). \tag{16}$$

To each partition $\lambda = (\lambda_1, \dots, \lambda_r)$ of n, we associate the subset $\mathcal{O}_\lambda$ of $\mathcal{M}$ given by

$$\begin{gathered} \alpha \in \mathcal{O}_\lambda \Longleftrightarrow \\ \left|\alpha^{-1}(0)\right| = \lambda_1, \left|\alpha^{-1}(1)\right| = \lambda_2, \dots, \left|\alpha^{-1}(r-1)\right| = \lambda_r. \end{gathered} \tag{17}$$

It is clear that $\mathcal{O}_\lambda$ is $\mathfrak{S}_n$-orbit in $\mathcal{M}$ with the action given in (9).

For each l, $1 \le l \le \lambda_1 = u$, we consider the subset $C_{\lambda,l}$ given by

$$C_{\lambda,l} = \left\{ i \in \mathbb{I}_n : \sum_{j=1}^{l-1} \lambda'_j + 1 \le i \le \sum_{j=1}^{l} \lambda'_j \right\}, \tag{18}$$

where $\lambda' = (\lambda'_1, \dots, \lambda'_u)$ is the *conjugate* partition of λ.

Now, we define $e_\lambda \in \mathcal{A}$ and $\alpha_\lambda \in \mathcal{M}$ as follows:

$$e_\lambda = \prod_{l=1}^{\lambda_1} V_{C_{\lambda,l}}(x_1, \dots, x_n), \tag{19}$$

$$\alpha_\lambda(i) = i - 1 - \sum_{j=1}^{l-1} \lambda'_j \quad \text{if } i \in C_{\lambda,l}. \tag{20}$$

Proposition 7. *Let $\mathfrak{S}_\lambda$ be the subgroup of $\mathfrak{S}_n$ defined as*

$$\mathfrak{S}_\lambda = \mathfrak{S}_{\lambda'_1}\left(C_{\lambda_1,1}\right) \times \cdots \times \mathfrak{S}_{\lambda'_u}\left(C_{\lambda_u,u}\right); \tag{21}$$

then

$$\left(\sum_{\tau \in \mathfrak{S}_\lambda} \operatorname{sgn}(\tau)\, \tau \right) x^{\alpha_\lambda} = e_\lambda. \tag{22}$$

Proof. The result is implied by following fact: given a nonempty subset $C = (c_1, \dots, c_k)$ of $\mathbb{I}_n$, where $c_1 < \cdots < c_k$, $\alpha_C \in \mathcal{M}$ is defined by

$$\alpha_C(j) = \begin{cases} l-1 & \text{if } j = c_l \\ 0 & \text{if } j \ne c_l; \end{cases} \tag{23}$$

then

$$\left(\sum_{\tau \in \mathfrak{S}_\lambda} \operatorname{sgn}(\tau)\, \tau \right) x^{\alpha_C} = V_C(x_1, \dots, x_n). \tag{24}$$

□

To each $\mathfrak{S}_n$-orbit γ in $\mathcal{M}$, we associate the subspace S_γ of $\mathcal{A}$ defined by

$$S_\gamma = \left\{ \sum_{\alpha \in \gamma} a_\alpha x^\alpha,\ a_\alpha \in K \right\}. \tag{25}$$

Definition 8. If λ is partition of n, one defines $\mathfrak{M}_\lambda$ to be the subspace of $\mathcal{A}$ generated by the $\mathfrak{S}_n$-orbit of e_λ. This means

$$\mathfrak{M}_\lambda = \left\langle \tau e_\lambda : \tau \in \mathfrak{S}_n \right\rangle. \tag{26}$$

Remark 9. If K is a field of characteristic 0 and $\lambda' = (\lambda'_1, \dots, \lambda'_u)$, then e_λ coincides with the product of linear functionals associated with a set of positive roots of a subgroup of Gelfand of type

$$A_{\lambda'_1 - 1} \times \cdots \times A_{\lambda'_u - 1}. \tag{27}$$

In this case, according to [3], $\mathfrak{M}_\lambda$ gives a Macdonald representation for $\mathfrak{S}_n$, considered as a Weyl group of type A_{n-1}, which is associated with a subgroup of the type given in (27) (see [4] or [5]). The $\mathfrak{S}_n$-module $\mathfrak{M}_\lambda$ can also be realized in the following way. Let δ be the differential operator defined by

$$\delta = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}; \tag{28}$$

then

$$\mathfrak{M}_\lambda = \left\{ P \in S_\gamma \mid \delta(P) = 0 \right\} \tag{29}$$

conforming with the statement of Theorem 4.2 of [14].

Let $\Lambda = (\mu^0, \dots, \mu^{m-1})$ be m-partition of n and, for each i, $0 \le i < m$, $q_i = |\mu^i|$ and suppose that

$$\begin{gathered} p_0 = 0, \\ p_i = \sum_{j<i} q_j. \end{gathered} \tag{30}$$

Define $e_\Lambda \in \mathcal{A}$ as

$$\begin{aligned} & e_\Lambda(x_1, \dots, x_n) \\ & \quad = \prod_{i=0}^{m-1} \left(x_{p_i+1} \cdots x_{p_i+q_i} \right)^i e_{\mu^i}\left(x_{p_i+1}^m \cdots x_{p_i+q_i}^m \right) \end{aligned} \tag{31}$$

and define $\alpha_\Lambda \in \mathscr{M}$ by

$$\alpha_{\Lambda_j} = m\alpha_{\mu^i}(j - p_i) + i \quad \text{if } p_i + 1 \le j \le p_i + q_i, \tag{32}$$

where α_{μ^i} is as in (20).

Now, if $\Gamma = \mathscr{O}_{\alpha_\Lambda}$ is the $\mathfrak{S}_n$-orbit of α_Λ in $\mathscr{M}$, it is clear that $e_\Lambda \in S_\Gamma$, where S_Γ is as in (25).

Remark 10. Provided that the mapping $\lambda \to \alpha_\lambda$ which sends a partition to a multi-index is injective, the mapping $\Lambda \to \alpha_\Lambda$ which sends m-partition to a multi-index is also injective.

Definition 11. If Λ is m-partition of n, one defines $\mathfrak{M}_\Lambda$ as the subspace of $\mathscr{A}$ generated by the $\mathfrak{S}_n$-orbit of e_Λ. This means that

$$\mathfrak{M}_\Lambda = \langle \sigma e_\Lambda : \sigma \in \mathfrak{S}_n \rangle. \tag{33}$$

Let Δ be the differential operator

$$\Delta = \sum_{i=1}^{n} \frac{\partial^m}{\partial x_i^m}. \tag{34}$$

As in the case of the symmetric group, by Theorem 2.5 in [8] it turns out that

$$\mathfrak{M}_\Lambda = \{P \in S_\Gamma \mid \Delta(P) = 0\}. \tag{35}$$

3.1. The Subspaces $\mathfrak{N}_\lambda$ and $\mathfrak{N}_\Lambda$. We identify the symmetric group $\mathfrak{S}_n$ with $G(1,1,n) = \mathfrak{S}_n^1$.

For each partition λ of n, consider $f_\lambda \in \mathfrak{M}_\lambda^*$ (the dual subspace of $\mathfrak{M}_\lambda$) defined by

$$f_\lambda(P) = \langle P, e_\lambda \rangle \quad (P \in \mathfrak{M}_\lambda). \tag{36}$$

Using the canonical action of $\mathfrak{S}_n$ on $\mathfrak{M}_\lambda^*$, we define $\mathfrak{N}_\lambda$ as the subspace of $\mathfrak{M}_\lambda^*$ generated by the $\mathfrak{S}_n$-orbit of f_λ. This means that

$$\mathfrak{N}_\lambda = \langle \tau f_\lambda : \tau \in \mathfrak{S}_n \rangle. \tag{37}$$

Analogously, for each m-partition $\Lambda = (\mu^0, \ldots, \mu^{m-1})$ of n, we consider $f_\Lambda \in \mathfrak{M}_\Lambda^*$ given by

$$f_\Lambda(P) = \langle P, e_\Lambda \rangle \quad (P \in \mathfrak{M}_\Lambda). \tag{38}$$

With respect to the canonical action of $\mathfrak{S}_n^m$ on $\mathfrak{M}_\Lambda^*$, we define $\mathfrak{N}_\Lambda$ as the subspace of $\mathfrak{M}_\Lambda^*$ generated by the $\mathfrak{S}_n$-orbit of f_Λ. This means that

$$\mathfrak{N}_\Lambda = \langle \sigma f_\Lambda : \sigma \in \mathfrak{S}_n \rangle. \tag{39}$$

With a different notation, the following result can be found in [6] Lemma 2.2 and Theorem 2.3.

Theorem 12. *Keeping the previous notation one has the following:*

(i) *If λ is p-regular, then $\mathfrak{N}_\lambda \neq 0$.*

(ii) *If $\mathfrak{N}_\lambda \neq \{0\}$, $\mathfrak{N}_\lambda$ is an irreducible $\mathfrak{S}_n$-module.*

(iii) *If λ and μ are p-regular partitions and $\lambda \neq \mu$, then $\mathfrak{N}_\lambda \neq \mathfrak{N}_\mu$.*

(iv) *Every simple $\mathfrak{S}_n$-module is isomorphic to $\mathfrak{N}_\lambda$ for some p-regular partition λ of n.*

In the proof of Theorem 12, an idempotent in $K\mathfrak{S}_n$ plays a central role. Analogously, in order to establish a similar result in the case of $\mathfrak{S}_n^m$, an idempotent in $K\mathfrak{S}_n^m$ is defined as follows.

Given $\Lambda = (\mu^0, \ldots, \mu^{m-1})$, m-partition of n, with

$$\begin{aligned} \mu^i &= (\mu_1^i, \ldots, \mu_r^i), \\ |\mu^i| &= q_i, \\ p_0 &= 0, \\ p_i &= \sum_{j<i} q_j \end{aligned} \tag{40}$$

and as in (18), we define for each l, $1 \le l \le \mu_1^i = r_i$, the subset $C_{\Lambda,i,l}$ given by

$$\begin{aligned} C_{\Lambda,i,l} = \Bigg\{ k \in \{p_i + 1, \ldots, p_i + q_i\} : \sum_{j=1}^{l-1} (\mu^i)'_j + 1 \le k \\ - p_i \le \sum_{j=1}^{l} (\mu^i)'_j \Bigg\}. \end{aligned} \tag{41}$$

Then, we have

$$\begin{aligned} e_\Lambda(x_1, \ldots, x_n) = \prod_{i=0}^{m-1} \Bigg[\left(x_{p_i+1} \cdots x_{p_i+q_i}\right)^i \\ \times \prod_{l=1}^{r_i} V_{C_{\Lambda,i,l}}\left(x_{p_i+1}^m, \ldots, x_{p_i+q_i}^m\right) \Bigg]. \end{aligned} \tag{42}$$

Let $\mathscr{H}_\Lambda \subseteq \mathfrak{S}_n^m$ be the subgroup defined by

$$\mathscr{H}_\Lambda = \mathop{\times}_{i=0}^{m-1} \left(\mathop{\times}_{l=1}^{r_i} \mathfrak{S}_m^n (C_{\Lambda,i,l}) \right), \tag{43}$$

where $\mathfrak{S}_n^m(C_{\Lambda,i,l})$ is defined in (5). Since each element $\sigma \in \mathfrak{S}_n^m$ has a unique decomposition as $\sigma = (d, \tau)$ with $d \in \mathscr{C}_m^n$ and $\tau \in \mathfrak{S}_n$, the subgroup $\mathscr{H}_\Lambda$ also has a decomposition as

$$\mathscr{H}_\Lambda = \mathscr{C}_\Lambda \times_s \mathfrak{S}_\Lambda, \tag{44}$$

where

$$\begin{aligned} \mathfrak{S}_\Lambda &= \prod_{i=0}^{m-1} \left(\prod_{l=1}^{r_i} \mathfrak{S}_n^1 (C_{\Lambda,i,l}) \right), \\ \mathscr{C}_\Lambda &= \prod_{i=0}^{m-1} \left(\prod_{l=1}^{r_i} \mathscr{C}_m^n (C_{\Lambda,i,l}) \right). \end{aligned} \tag{45}$$

Each $d \in \mathscr{C}_\Lambda$ can be factorized uniquely as

$$d = d_0 d_1 \cdots d_{m-1}, \tag{46}$$

where d_i is in $\prod_{l=1}^{r_i}\mathscr{C}_m^n(C_{\Lambda,i,l})$. We let χ_Λ denote the linear character of $\mathscr{H}_\Lambda$ given by

$$\chi_\Lambda(d,\tau)=\left(\prod_{i=0}^{m-1}\chi(d_i)^i\right)\operatorname{sgn}(\tau). \quad (47)$$

Finally we define the operators Ω_Λ and Ω_Λ^* in $K\mathfrak{S}_n^m$ by

$$\Omega_\Lambda=\frac{1}{|\mathscr{C}_\Lambda|}\sum_{\sigma\in\mathscr{H}_\Lambda}\chi_\Lambda(\sigma)^{-1}\sigma,$$
$$\Omega_\Lambda^*=\frac{1}{|\mathscr{C}_\Lambda|}\sum_{\sigma\in\mathscr{H}_\Lambda}\chi_\Lambda(\sigma)\sigma. \quad (48)$$

Remark 13. A reflection s in a vector space is a diagonalizable endomorphism such that the space of fixed points of s is a hyperplane. Acting on $\mathscr{A}_1$, $\mathscr{H}_\Lambda$ is realized as a group of reflections (this means it is generated by reflections). If K is the field of complex numbers, from (42) and (43), the polynomial e_Λ can be factorized as a product of linear functionals associated with a system of roots of the subgroup $\mathscr{H}_\Lambda$ of $\mathfrak{S}_n^m$ where some of them, precisely the ones associated with the roots e_i, are taken with certain multiplicities, in a way such that

$$e_\Lambda(x_1,\ldots,x_n)=\prod_{i=0}^{m-1}\left\{\left(x_{p_i+1}\cdots x_{p_i+q_i}\right)^i\right.$$
$$\left.\times\prod_{l=1}^{r_i}\left[\prod_{j,k\in C_{\Lambda,i,l},j<k}\left(x_k^m-x_j^m\right)\right]\right\}. \quad (49)$$

Proposition 14. *With the previous notations, for each m-partition $\Lambda=(\mu^0,\ldots,\mu^{m-1})$ of n, we have*

(i)

$$\tau\Omega_\Lambda=\Omega_\Lambda\tau=\chi_\Lambda(\tau)\Omega_\Lambda\quad\forall\tau\in\mathscr{H}_\Lambda, \quad (50)$$

(ii)

$$\Omega_\Lambda=\frac{1}{|\mathscr{C}_\Lambda|}\left(\sum_{d\in\mathscr{C}_\Lambda}\chi_\Lambda(d)^{-1}d\right)\left(\sum_{\tau\in\mathfrak{S}_\Lambda}\operatorname{sgn}(\tau)\tau\right),$$
$$\Omega_\Lambda^*=\frac{1}{|\mathscr{C}_\Lambda|}\left(\sum_{d\in\mathscr{C}_\Lambda}\chi_\Lambda(d)d\right)\left(\sum_{\tau\in\mathfrak{S}_\Lambda}\operatorname{sgn}(\tau)\tau\right), \quad (51)$$

(iii)

$$\Omega_\Lambda\left(x^{\alpha_\Lambda}\right)=e_\Lambda. \quad (52)$$

Proof. (i) This identity is clear from the definition of Ω_Λ.

(ii) It is a consequence of the decomposition (44).

(iii) By (22), we have

$$\left(\sum_{t\in\mathfrak{S}_\Lambda}\operatorname{sgn}(\tau)\tau\right)\left(x^{\alpha_\Lambda}\right)=e_\Lambda. \quad (53)$$

Since each element $d\in\mathscr{C}_\Lambda$ can be factorized as $d=(d_0,\ldots,d_{m-1})$, where

$$d_i\in\left(\prod_{l=1}^{r_i}\mathscr{C}_{m,i}^n(C_{\Lambda,i,l})\right), \quad (54)$$

we have

$$d(e_\Lambda)=\left(\prod_{i=0}^{m-1}\chi(d_i)\right)e_\Lambda=\chi(d)e_\Lambda. \quad (55)$$

Hence,

$$\left(\sum_{d\in\mathscr{C}_\Lambda}\chi(d)^{-1}d\right)e_\Lambda=|\mathscr{C}_\Lambda|e_\Lambda \quad (56)$$

and (iii) is proved. □

Using the same notation as in (42) we have the following lemma.

Lemma 15. *Let τ be a reflection of order k in $\mathscr{A}_1$ and r a root of τ. If $P\in\mathscr{A}$ is such that*

$$\tau(P)=\det(\tau)^jP \quad (57)$$

for some $j\in\mathbb{Z}$, then r^k is a factor of P.

Proof. From the hypothesis, we have $\tau(r)=\zeta r$ with ζ a primitive kth root of unity. Fix $\varphi_1=r,\varphi_2,\ldots,\varphi_n$, a base of $\mathscr{A}_1$, where $\varphi_2,\ldots,\varphi_n$ is a base of the reflective hyperplane of τ. Thus, $r(\varphi_i)=\varphi_i$ for $i\geq 2$. We can express P as

$$P=\sum_{\alpha\in\mathscr{M}}\lambda_\alpha\varphi^\alpha, \quad (58)$$

where $\varphi^\alpha=\varphi_1^{\alpha_1}\cdots\varphi_n^{\alpha_n}$.

As $\det(\tau)=\zeta$, in the identity

$$\tau P=\det(\tau)^jP, \quad (59)$$

it can be assumed that $j\in\{0,1,\ldots,k-1\}$. It follows that

$$\sum_{\alpha\in\mathscr{M}}\zeta^{\alpha_1}\lambda_\alpha\varphi^\alpha=\zeta^j\sum_{\alpha\in\mathscr{M}}\lambda_\alpha\varphi^\alpha, \quad (60)$$

where

$$\alpha_1\equiv j \mod(k),\ \forall\lambda_\alpha\neq 0. \quad (61)$$

If α_1 has the form $j+hk$, with $h\geq 0$, so that $\alpha_1\geq j$ whenever $\lambda_\alpha\neq 0$, then $r^j=\varphi_1^j$ is a factor of P. □

Corollary 16. *If $P\in\mathscr{M}$, then e_Λ is a factor of $\Omega_\Lambda(P)$.*

Proof. Because of the result established in Proposition 14(i), it follows that

$$\sigma\Omega_\Lambda(P)=\chi_\Lambda(\sigma)\Omega_\Lambda(P)\quad\forall\sigma\in\mathscr{H}_\Lambda. \quad (62)$$

Taking into account the reflections in $\mathscr{H}_\Lambda$ and the expression of e_Λ in (49), from Lemma 15 it follows that all distinct factors of e_Λ in the set

$$\left\{\left(x_j - \zeta x_k\right)\right\}_{j,k\in C_{\Lambda,i,l}, j<k} \cup \left\{x_j^i\right\}_{p_i+1\le j\le p_i+q_i} \tag{63}$$

are factors of P. But as they are irreducible factors not associated with $\mathscr{A}$, the fact that e_Λ is factor of P results. □

We let $gr(P)$ denote the degree of the polynomial P.

Proposition 17. *Let $\Lambda = (\mu^0, \dots, \mu^{m-1})$ be m-partition of n and $P \in \mathscr{A}$. Considering the action of Ω_Λ on $\mathscr{A}$ given by (10) and on $\mathfrak{N}_\Lambda$ given by the restriction of the usual action on $\mathfrak{M}_\Lambda^*$, we have the following:*

(i) *$\Omega_\Lambda(P) = e_\Lambda Q$, where Q is a polynomial $\mathscr{H}_\Lambda$-invariant.*

(ii) *If $gr(P) = gr(e_\Lambda)$, then*

$$\Omega_\Lambda(P) = \langle P, e_\Lambda\rangle e_\Lambda. \tag{64}$$

(iii) *For each $\varphi \in \mathfrak{N}_\Lambda$*

$$\Omega_\Lambda^*(\varphi) = \varphi(e_\Lambda) f_\Lambda. \tag{65}$$

Proof. (i) By Proposition 14, for each $\tau \in \mathscr{H}_\Lambda$ we have

$$\begin{aligned} \tau\Omega_\Lambda &= \chi_\Lambda(\tau)\Omega_\Lambda, \\ \Omega_\Lambda\left(x^{\alpha_\Lambda}\right) &= e_\Lambda. \end{aligned} \tag{66}$$

Therefore,

$$\tau\Omega_\Lambda(P) = \chi_\Lambda(\tau)\Omega_\Lambda(P). \tag{67}$$

In particular, this identity is valid for a reflection in $\mathscr{H}_\Lambda$, so that by Lemma 15 it follows that

$$\Omega_\Lambda(P) = e_\Lambda Q. \tag{68}$$

On the other hand, if we apply $\tau \in \mathscr{H}_\Lambda$ to both sides of the former identity, we have

$$\begin{aligned} \chi_\Lambda(\tau) e_\Lambda Q &= \chi_\Lambda(\tau)\Omega_\Lambda(P) = \tau\Omega_\Lambda(P) = \tau\left(e_\Lambda Q\right) \\ &= \tau\left(e_\Lambda\right)\tau(Q) = \chi_\Lambda(\tau) e_\Lambda \tau(Q) \end{aligned} \tag{69}$$

so that Q is $\mathscr{H}_\Lambda$-invariant.

(ii) By the linearity of Ω_Λ, it is sufficient to prove that for each $\alpha \in \mathscr{M}$ such that $|\alpha| = gr(e_\Lambda)$ we have

$$\Omega_\Lambda\left(x^\alpha\right) = \left\langle x^\alpha, e_\Lambda\right\rangle e_\Lambda. \tag{70}$$

By the identity given in Proposition 14,

$$\Omega_\Lambda\left(x^{\alpha_\Lambda}\right) = e_\Lambda. \tag{71}$$

Thus, if α does not belong to the $\mathfrak{S}_n$-orbit of α_Λ, this means that

$$\left\langle x^\alpha, e_\Lambda\right\rangle = 0. \tag{72}$$

Moreover, given that $gr(\Omega_\Lambda(x^\alpha)) = gr(e_\Lambda)$, by (i) there exists $\lambda \in K$ such that

$$\Omega_\Lambda\left(x^\alpha\right) = \lambda e_\Lambda. \tag{73}$$

As x^α and x^{α_Λ} are in different $\mathfrak{S}_n$-orbits, it must be the case that $\lambda = 0$ and so (70) is proved when α does not belong to $\mathfrak{S}_\Lambda$-orbit of α_Λ.

If α is in $\mathfrak{S}_n$-orbit of α_Λ, there is $\tau \in \mathfrak{S}_n$ such that

$$x^\alpha = \tau x^{\alpha_\Lambda}. \tag{74}$$

Then,

$$\Omega_\Lambda\left(x^\alpha\right) = \Omega_\Lambda\left(\tau x^{\alpha_\Lambda}\right) = \operatorname{sgn}(\tau) e_\Lambda. \tag{75}$$

Since the coefficient of τx^{α_Λ} in the monomial decomposition of e_Λ is precisely $\operatorname{sgn}(\tau)$, this completes the proof of the identity (70).

(iii) If $\varphi \in \mathfrak{N}_\Lambda$ and $P \in \mathfrak{M}_\Lambda$, we have

$$\begin{aligned} \Omega_\Lambda^*(\varphi)(P) &= \left(\frac{1}{\left|\mathscr{C}_\Lambda\right|}\sum_{\sigma\in\mathscr{H}_\Lambda}\chi_\Lambda(\sigma)(\sigma\varphi)\right)P \\ &= \frac{1}{\left|\mathscr{C}_\Lambda\right|}\sum_{\sigma\in\mathscr{H}_\Lambda}\chi_\Lambda(\sigma)\varphi\left(\sigma^{-1}P\right) \\ &= \varphi\left(\frac{1}{\left|\mathscr{C}_\Lambda\right|}\sum_{\sigma\in\mathscr{H}_\Lambda}\chi_\Lambda(\sigma)\sigma^{-1}P\right) \\ &= \varphi\left(\Omega_\Lambda(P)\right) = \varphi\left(\left\langle P, e_\Lambda\right\rangle e_\Lambda\right) \\ &= \varphi\left(e_\Lambda\right) f_\Lambda(P). \end{aligned} \tag{76}$$

□

4. The Bilinear Form

In this section, some properties of the linear form introduced in $\mathscr{A}$ will be proven, in order to extend the results in [6] to the group $\mathfrak{S}_n^m$.

Definition 18. If $\mathscr{J}$ is a subset of $\mathbb{I}_n$, and $\alpha \in \mathscr{M}$ one says that α is supported in $\mathscr{J}$ if $\alpha_i = 0$ for each $i \in \mathbb{I}_n - J$. One also says that the monomial x^α is supported in $\mathscr{J}$ if α is supported in $\mathscr{J}$. If $P \in \mathscr{A}$, one says that P is supported in $\mathscr{J}$ if each monomial in P is.

If $\alpha \in \mathscr{M}$ we associate $\alpha^{\mathscr{J}} \in \mathscr{M}$ supported in $\mathscr{J}$ given by

$$\alpha_i^{\mathscr{J}} = \begin{cases} \alpha_i, & \text{if } i \in \mathscr{J}, \\ 0, & \text{if } i \notin \mathscr{J}. \end{cases} \tag{77}$$

It is clear that α is supported in $\mathscr{J}$ if, and only if, $\alpha^{\mathscr{J}} = \alpha$.

If $\mathbb{I}_n = \bigcup_{l=0}^h \mathscr{J}_l$ is a partition of $\mathbb{I}_n$ and $\alpha \in \mathscr{M}$, α can be decomposed in unique way as

$$\alpha = \sum_{l=0}^h \alpha^{\mathscr{J}_l} \tag{78}$$

such that if $\alpha, \beta \in \mathcal{M}$, then

$$\left\langle x^{\alpha}, x^{\beta} \right\rangle = \prod_{l=0}^{h} \left\langle x^{\alpha^{\mathcal{J}_l}}, x^{\beta^{\mathcal{J}_l}} \right\rangle. \tag{79}$$

Proposition 19. *Let $P, Q \in \mathcal{A}$ and $\mathbb{I}_n = \bigcup_{l=0}^{h} \mathcal{J}_l$ is a partition of $\mathbb{I}_n$. If P and Q are factorized as*

$$P = \prod_{l=0}^{h} P^{\mathcal{J}_l}, \qquad Q = \prod_{l=0}^{h} Q^{\mathcal{J}_l}, \tag{80}$$

where $P^{\mathcal{J}_l}$ and $Q^{\mathcal{J}_l}$ are both supported in $\mathcal{J}_l$ for each l, then

$$\langle P, Q \rangle = \prod_{l=0}^{h} \left\langle P^{\mathcal{J}_l}, Q^{\mathcal{J}_l} \right\rangle. \tag{81}$$

Proof. The proof can be obtained using induction in h, so it is sufficient to treat the case of a partition of two terms. For simplicity, we express $\mathbb{I}_n = \mathcal{I} \cup \mathcal{J}$. If we write

$$P = \sum a_{\alpha} x^{\alpha}, \qquad Q = \sum b_{\alpha} x^{\alpha}, \tag{82}$$

then

$$\langle P, Q \rangle = \sum_{\alpha,\beta} \delta_{\alpha,\beta} a_{\alpha} b_{\beta}, \tag{83}$$

where δ is Kronecker's function. On the other hand, it follows from the factorization that

$$P = P^{\mathcal{I}} P^{\mathcal{J}}, \qquad Q = Q^{\mathcal{I}} Q^{\mathcal{J}} \tag{84}$$

and this decomposition is unique as was observed in (78); therefore,

$$a_{\alpha} = a_{\alpha^{\mathcal{I}}} a_{\alpha^{\mathcal{J}}}, \qquad b_{\beta} = b_{\beta^{\mathcal{I}}} b_{\beta^{\mathcal{J}}}, \qquad \delta_{\alpha,\beta} = \delta_{\alpha^{\mathcal{I}},\beta^{\mathcal{I}}} \delta_{\alpha^{\mathcal{J}},\beta^{\mathcal{J}}}, \tag{85}$$

where $a_{\alpha^{\mathcal{I}}} a_{\alpha^{\mathcal{J}}}$ are the coefficients of $x^{\alpha^{\mathcal{I}}}$ and $x^{\alpha^{\mathcal{J}}}$ in $P^{\mathcal{I}}$ and $P^{\mathcal{J}}$, respectively, and, analogously, $b_{\beta^{\mathcal{I}}} b_{\beta^{\mathcal{J}}}$ are the coefficient of $x^{\beta^{\mathcal{I}}}$ and $x^{\beta^{\mathcal{J}}}$ in $Q^{\mathcal{I}}$ and $Q^{\mathcal{J}}$. Then

$$\begin{aligned} &\sum_{\alpha,\beta} \delta_{\alpha,\beta} a_{\alpha} b_{\beta} \\ &= \left(\sum_{\alpha^{\mathcal{I}},\beta^{\mathcal{I}}} \delta_{\alpha^{\mathcal{I}},\beta^{\mathcal{I}}} a_{\alpha^{\mathcal{I}}} b_{\beta^{\mathcal{I}}} \right) \left(\sum_{\delta_{\alpha^{\mathcal{J}},\beta^{\mathcal{J}}}} \delta_{\alpha^{\mathcal{J}},\beta^{\mathcal{J}}} a_{\alpha^{\mathcal{J}}} b_{\beta^{\mathcal{J}}} \right) \\ &= \left\langle P^{\mathcal{I}}, Q^{\mathcal{I}} \right\rangle \left\langle P^{\mathcal{J}}, Q^{\mathcal{J}} \right\rangle \end{aligned} \tag{86}$$

from which the following proposition holds. □

Proposition 20. *If $P, Q \in \mathcal{A}$ and k is a natural number, then*

$$\begin{aligned} &\left\langle P\left(x_1^k, \dots, x_n^k\right), Q\left(x_1^k, \dots, x_n^k\right) \right\rangle \\ &\quad = \langle P(x_1, \dots, x_n), Q(x_1, \dots, x_n) \rangle. \end{aligned} \tag{87}$$

Proof. This is a consequence of the identity

$$\left\langle x^{k\alpha}, x^{k\beta} \right\rangle = \left\langle x^{\alpha}, x^{\beta} \right\rangle \tag{88}$$

for each pair $\alpha, \beta \in \mathcal{M}$. □

Given a decomposition $\mathcal{P} = (\mathcal{P}_0, \dots, \mathcal{P}_{m-1})$ of $\mathbb{I}_n$, we have

$$\mathbb{I}_n = \bigcup_{l=0}^{m-1} \mathcal{P}_l, \tag{89}$$

where the subsets $\mathcal{P}_l$ are pairwise disjoint. We indicate by $\mathcal{N}_{\mathcal{P}}$ the set of all $\alpha \in \mathcal{M}$ such that, for each l, $0 \le l < m$, $\alpha_{|\mathcal{P}_l}$ is congruent to l module m.

Definition 21. If $P \in \mathcal{A}$, one will say that P has m-type $\mathcal{P}$ if

$$P = \sum_{\alpha \in \mathcal{N}_{\mathcal{P}}} \lambda_{\alpha} x^{\alpha}. \tag{90}$$

Proposition 22. *Let P and $Q \in \mathcal{A}$ be P with m-type $\mathcal{P}$ and Q with m-type $\mathcal{Q}$. If $\mathcal{P} \neq \mathcal{Q}$, then $\langle P, Q \rangle = 0$.*

Proof. If $\mathcal{P}$ and $\mathcal{Q}$ are different decompositions of $\mathbb{I}_n$, it is clear that $\mathcal{N}_{\mathcal{P}} \cap \mathcal{N}_{\mathcal{Q}} = \emptyset$ and consequently $\langle P, Q \rangle = 0$. □

Lemma 23. *If Λ is a p-regular m-partition, the bilinear form $\langle , \rangle$ when restricted to $\mathfrak{M}_{\Lambda} \times \mathfrak{M}_{\Lambda}$ is nonzero.*

Proof. Let $\Lambda = (\mu^0, \dots, \mu^{m-1})$ be the m-partition of n. For each i, $0 \le i \le m-1$, let $q_i = |\mu^i|$ and suppose that

$$p_0 = 0, \qquad p_i = \sum_{j<i} q_j. \tag{91}$$

Since Λ is p-regular, each μ^i is a p-regular partition. By Lemma 2.2 (iv) in [6], there are permutations σ_i in $\mathfrak{S}_{q_i}(\{p_i + 1, \dots, p_i + q_i\})$ such that for each i

$$\left\langle e_{\alpha_{\mu^i}}, \sigma_i e_{\alpha_{\mu^i}} \right\rangle \neq 0. \tag{92}$$

If we put $\tau = \sigma_0 \cdots \sigma_{m-1} \in \mathfrak{S}_n$ from (31) and Propositions 19 and 20 we have

$$\langle e_{\Lambda}, \tau e_{\Lambda} \rangle = \prod_{i=0}^{m-1} \left\langle e_{\alpha_{\mu^i}}, \sigma e_{\alpha_{\mu^i}} \right\rangle \neq 0. \tag{93}$$

□

5. The Modular Representations

Theorem 24. *Using the previous notation, we have the following:*

(i) *If Λ is a p-regular m-partition of n, then $\mathfrak{N}_\Lambda \neq \{0\}$.*

(ii) *If $\mathfrak{N}_\Lambda \neq \{0\}$, $\mathfrak{N}_\Lambda$ is an irreducible $\mathfrak{S}_n^m$-module.*

(iii) *If Λ and Υ are p-regular m-partitions of n and $\Lambda \neq \Upsilon$, then $\mathfrak{N}_\Lambda \neq \mathfrak{N}_\Omega$.*

(iv) *Every simple $\mathfrak{S}_n^m$-module is isomorphic to $\mathfrak{N}_\Lambda$ for some p-regular m-partition Λ of n.*

Proof. (i) This is a consequence of Lemma 23.

(ii) Let $\Lambda = (\mu^0, \ldots, \mu^{m-1})$ be an m-partition of n, $\mathfrak{L}$ a $\mathfrak{S}_n^m$-submodule of $\mathfrak{N}_\Lambda$, and $\varphi \in \mathfrak{L}$, φ nonzero. Given that $\varphi \in \mathfrak{M}_\Lambda^*$, there exists $P \in \mathfrak{M}_\Lambda$ such that

$$\varphi(Q) = \langle P, Q\rangle \quad \forall Q \in \mathfrak{M}_\Lambda. \tag{94}$$

If $\varphi(\sigma e_\Lambda) = 0$ for every $\sigma \in \mathfrak{S}_n$, the linear functional φ should be zero, but we have assumed that $\varphi \neq 0$. Hence, there exists $\pi \in \mathfrak{S}_n$ such that $g(\pi e_\Lambda) \neq 0$. We can suppose, without lack of generality, that $\varphi(e_\Lambda) \neq 0$, changing φ by $\pi^{-1}\varphi$ if that was the case. Now, if $Q \in \mathfrak{M}_\Lambda$, $gr(Q) = gr(e_\Lambda)$ and by Proposition 17

$$\varphi(e_\Lambda) f_\Lambda = \Omega_\Lambda^*(\varphi) \in \mathfrak{L}. \tag{95}$$

Since $\varphi(e_\Lambda) \neq 0$, it follows that $f_\Lambda \in \mathfrak{L}$, which means that $\mathfrak{L} = \mathfrak{N}_\Lambda$.

(iii) Let Λ and Υ be p-regular m-partitions of n such that $\mathfrak{N}_\Lambda \simeq \mathfrak{N}_\Upsilon$. We prove in Lemma 23 that there exists $\tau \in \mathfrak{S}_n$ such that

$$\langle e_\Lambda, \tau e_\Lambda\rangle \neq 0. \tag{96}$$

If in Proposition 17(iii) we change φ by $\tau^{-1} f_\Lambda$, we have

$$\Omega_\Lambda^*(\tau^{-1} f_\Lambda) = ((\tau^{-1} f_\Lambda)(e_\Lambda)) f_\Lambda = \langle e_\Lambda, \tau e_\Lambda\rangle f_\Lambda \neq 0. \tag{97}$$

Since the operator Ω_Λ^* is nonzero in $\mathfrak{N}_\Lambda$ and it is a linear combination of elements in $\mathfrak{S}_n^m$, Ω_Λ^*, it must be nonzero in $\mathfrak{N}_\Upsilon$. Hence, there exists $\pi \in \mathfrak{S}_n$ such that

$$\Omega_\Lambda^*(\pi^{-1} f_\Upsilon) \neq 0. \tag{98}$$

Thus,

$$0 \neq \Omega_\Lambda^*(\pi^{-1} f_\Upsilon) = \langle e_\Upsilon, \pi e_\Lambda\rangle f_\Upsilon. \tag{99}$$

Hence,

$$\langle e_\Upsilon, \pi e_\Lambda\rangle \neq 0 \tag{100}$$

but this only occurs if α_Υ and α_Λ belong to the same $\mathfrak{S}_n$-orbit, and, from Remark 10, it must be $\Upsilon = \Lambda$.

(iv) It is sufficient to prove that the number of p-regular m-partitions agrees with the number of p-regular conjugacy classes in $\mathfrak{S}_n^m$. But this is a consequence of the fact that there is a bijection between the m-partitions and the conjugacy classes of $\mathfrak{S}_n^m$ which put in correspondence p-regular m-partitions with the p-regular conjugacy classes of $\mathfrak{S}_n^m$; see [13]. □

Finally, considering the generic case where p and m may not be coprime, write $m = p^\alpha k$, where $\alpha \in \mathbb{N}_0$ and p does not divide k. Suppose that K contains the kth roots of the unity. Let us consider $\tilde{\psi} : \mathscr{C}_m^n \to \mathscr{C}_k^n$, the canonical projection, given by the decomposition

$$\mathscr{C}_m^n = \mathscr{C}_{p^\alpha}^n \oplus \mathscr{C}_k^n. \tag{101}$$

This morphism induces the projection $\psi : \mathfrak{S}_n^m \to \mathfrak{S}_n^k$ given by

$$\pi(d, \tau) = (\pi(d), \tau) \quad (d, \tau) \in \mathfrak{S}_n^m. \tag{102}$$

By the previous theorem, for each p-regular k-partition Λ of n we have an irreducible representation of $\mathfrak{S}_n^k$

$$\rho_\Lambda : \mathfrak{S}_n^k \longrightarrow \mathrm{Aut}_K(\mathfrak{N}_\Lambda) \tag{103}$$

and thus $\rho_\Lambda \circ \psi$ is an irreducible representation of $\mathfrak{S}_n^m$.

Keeping the previous notation, we have following theorem which establishes the irreducible modular representations of $\mathfrak{S}_n^m$ in the general case.

Theorem 25. *If K contains the kth roots of the unity, the following statements hold:*

(i) *If Λ is a p-regular k-partition of n, then $\rho_\Lambda \circ \pi$ is an irreducible representation of $\mathfrak{S}_n^m$.*

(ii) *If Λ and Υ are p-regular k-partitions of n and $\Lambda \neq \Upsilon$, then $\rho_\Lambda \circ \pi$ and $\rho_\Upsilon \circ \pi$ are nonequivalent.*

(iii) *Every irreducible representation of $\mathfrak{S}_n^m$ is equivalent to $\rho_\Lambda \circ \pi$ for some p-regular k-partition Λ of n.*

Proof. (i) and (ii) follow from Theorem 24.

(iii) By (i) and (ii) we have as many nonequivalent irreducible representations of $\mathfrak{S}_n^m$ as p-regular k-partitions of n. On the other hand, it follows from [13] that every p-regular conjugacy class of $\mathfrak{S}_n^m$ has a representative $\mathfrak{S}_n^k$; in consequence $\mathfrak{S}_n^m$ and $\mathfrak{S}_n^k$ have the same number of p-regular classes. Since the number of p-regular classes of $\mathfrak{S}_n^m$ agrees with the number of p-regular k-partitions of n it follows that the representations $\rho_\Lambda \circ \pi$ are all irreducible presentations of $\mathfrak{S}_n^m$. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] H. K. Farahat and M. H. Peel, "On the representation theory of the symmetric groups," *Journal of Algebra*, vol. 67, no. 2, pp. 280–304, 1980.

[2] E. Al-Aamily, A. O. Morris, and M. H. Peel, "The representations of the Weyl groups of type B_n," *Journal of Algebra*, vol. 68, no. 2, pp. 298–305, 1981.

[3] I. G. Macdonald, "Some irreducible representations of Weyl groups," *Bulletin of the London Mathematical Society*, vol. 4, no. 2, pp. 148–150, 1972.

[4] R. W. Carter, "Conjugacy classes in the Weyl group," *Compositio Mathematica*, vol. 25, pp. 1–59, 1972.

[5] G. Lusztig, "A class of irreducible representations of a Weyl group," *Indagationes Mathematicae*, vol. 41, no. 3, pp. 323–335, 1979.

[6] J. Aguado and J. Araujo, "Representations of the symmetric group Sn on $K[x_1; \ldots ; x_n]$," *Revista de la Unión Matemática Argentina*, vol. 41, no. 2, 1998.

[7] J. O. Araujo, J. J. Bigeón, and R. M. Gamondi, "Modular representations of Weyl groups of type B_n," *Communications in Algebra*, vol. 38, no. 7, 2010.

[8] J. O. Araujo and J. J. Bigeón, "On Gelfand model for the symmetric generalized group," *Communications in Algebra*, vol. 37, no. 5, pp. 1808–1830, 2009.

[9] A. O. Morris and P. Mwamba, "Macdonald representations of complex reflection groups," *Séminaire Lotharingien de Combinatoire*, vol. 52, 17 pages, 2004.

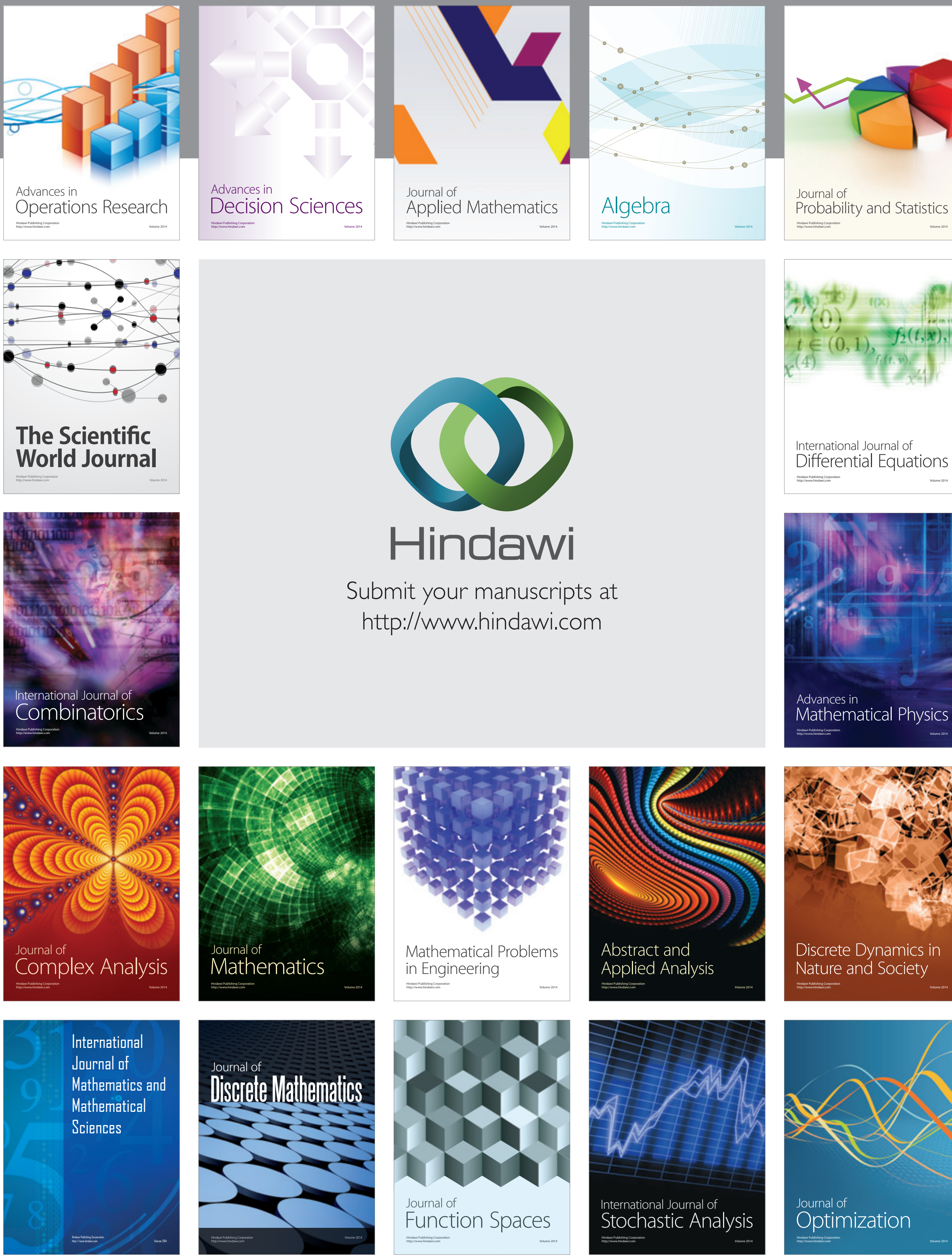
[10] R. Kane, *Reflection Groups and Invariant Theory*, Springer, 2001.

[11] G. D. James, *The Representation Theory of the Symmetric Groups*, Springer, Berlin, Germany, 1978.

[12] C. Curtis and I. Reiner, *Methods of Representation theory with Applications to Finite Groups and Orders, Volume I*, John Wiley & Sons, 1981.

[13] M. C. Hughes, "The conjugacy classes of complex imprimitive reflection groups," *Indagationes Mathematicae (Proceedings)*, vol. 88, no. 3, pp. 313–330, 1985.

[14] J. Aguado and J. Araujo, "A Gelfand model for the symmetric group," *Communications in Algebra*, vol. 29, no. 4, pp. 1841–1851, 2001.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