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Research Article

On ϕ-Symmetric $N(k)$-Paracontact Metric Manifolds

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The notions of ϕ-symmetric, 3-dimensional locally ϕ-symmetric, ϕ-Ricci symmetric, and 3-dimensional locally ϕ-Ricci symmetric $N(k)$-paracontact metric manifolds have been introduced and properties of these structures have been discussed.

1. Introduction

The study of paracontact geometry was initiated by Kaneyuki and Williams [1]. A systematic study of paracontact metric manifolds and their subclasses were started out by Zamkovoy [2]. Since then, several geometers studied paracontact metric manifolds and obtained various important properties of these manifolds ([3–10], etc.). The geometry of paracontact metric manifolds can be related to the theory of Legendre foliations. In [11], the authors introduced the class of paracontact metric manifolds for which the characteristic vector field ξ belongs to the (k,μ)-nullity condition (or distribution) for some real constants k and μ. Such manifolds are known as (k,μ)-paracontact metric manifolds. If $\mu = 0$, then the notion of (k,μ)-nullity distribution reduces to k-nullity distribution. A paracontact metric manifold with ξ belonging to k-nullity distribution is called $N(k)$-paracontact metric manifold.

In [12], Takahashi introduced the notion of locally ϕ-symmetric Sasakian manifold as a weaker version of local symmetry of such manifolds. In the context of contact geometry, the notion of ϕ-symmetry was introduced and studied by Boeckx et al. [13] with examples. In ([14, 15]), they studied the notion of ϕ-symmetry and discussed several examples for Kenmotsu manifolds and almost contact metric manifolds of dimension 3. In [16, 17], S. S. Shukla and M. K. Shukla, studied ϕ-Ricci symmetric Kenmotsu manifolds and ϕ-symmetric para-Sasakian manifolds.

In the present work, we study ϕ-symmetry and Ricci ϕ-symmetry on $N(k)$-paracontact metric manifolds. In Section 2, we give a brief account of the $N(k)$-paracontact metric manifolds. In Section 3, we study the properties of ϕ-symmetric $N(k)$-paracontact metric manifolds. Section 4 deals with 3-dimensional locally ϕ-symmetric $N(k)$-paracontact metric manifolds. In this section, we prove that scalar curvature r is constant. Section 5 is devoted to studying the Ricci ϕ-symmetric $N(k)$-paracontact metric manifolds. Finally, we study the properties of 3-dimensional locally Ricci ϕ-symmetric $N(k)$-paracontact metric manifolds in Section 6.

2. Preliminaries

A $(2n+1)$-dimensional smooth manifold M^{2n+1} has an almost paracontact structure (ϕ,ξ,η,g) if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ, a 1-form η, and a Riemannian metric g satisfying the following conditions ([2, 18]):

$$\begin{gathered}\eta(X) = g(X,\xi),\\ \eta(\xi) = 1,\\ \eta\circ\phi = 0,\\ \phi(\xi) = 0,\\ \phi^2 X = X - \eta(X)\xi,\end{gathered} \tag{1}$$

$$\begin{gathered}g(\phi X,\phi Y) = -g(X,Y) + \eta(X)\eta(Y),\\ d\eta(X,Y) = g(X,\phi Y),\end{gathered} \tag{2}$$

for every vector field X, Y on M^{2n+1}.

In a paracontact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, we define a $(1,1)$ tensor field h by $h = (1/2)\pounds_{\xi}\phi$, where $\pounds$ denotes the operator of Lie differentiation. Then, h is symmetric and satisfies

$$\begin{aligned} h\xi &= 0, \\ h\phi &= -\phi h, \\ Tr \cdot h &= Tr \cdot \phi h = 0. \end{aligned} \tag{3}$$

If ∇ denotes the Levi-Civita connection of g, then we have the following relation:

$$\nabla_X \xi = -\phi X + \phi h X. \tag{4}$$

A paracontact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is said to be a (k, μ)-space if its curvature tensor R satisfies

$$\begin{aligned} R(X,Y)\xi = {} & k[\eta(Y)X - \eta(X)Y] \\ & + \mu[\eta(Y)hX - \eta(X)hY], \end{aligned} \tag{5}$$

for all tangent vector fields X, Y, where k, μ are smooth functions on M^{2n+1}.

Here, the characteristic vector field ξ belongs to the (k,μ)-nullity distribution. A paracontact metric manifold with ξ belonging to (k,μ)-nullity distribution is called a (k,μ)-paracontact metric manifold. In particular, if $\mu = 0$, then the notion of (k,μ)-nullity distribution reduces to k-nullity distribution. A paracontact metric manifold such that ξ belongs to k-nullity distribution is called $N(k)$-paracontact metric manifold. Then, curvature tensor R reduces to the following form:

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y]. \tag{6}$$

For $N(k)$-paracontact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ $(n > 1)$, the following identities hold:

$$h^2 = (1+k)\phi^2, \tag{7}$$

$$(\nabla_X \phi)Y = -g(X - hX, Y)\xi + \eta(Y)(X - hX), \tag{8}$$

$$(\nabla_X \eta)Y = g(X - hX, \phi Y), \tag{9}$$

$$S(X,\xi) = 2nk\eta(X), \tag{10}$$

$$Q\xi = 2nk\xi, \tag{11}$$

for any vector fields X, Y on M^{2n+1}, where Q and S denote the Ricci operator and Ricci tensor of (M^{2n+1}, g), respectively.

$N(k)$-paracontact metric manifold is called an Einstein manifold if it satisfies

$$S(X,Y) = \lambda g(X,Y), \tag{12}$$

where λ is any scalar.

Definition 1. $N(k)$-paracontact metric manifold is said to be ϕ-symmetric if

$$\phi^2((\nabla_W R)(X,Y)Z) = 0, \tag{13}$$

for arbitrary vector fields X, Y, Z, W.

Definition 2. $N(k)$-paracontact metric manifold is said to be locally ϕ-symmetric if

$$\phi^2((\nabla_W R)(X,Y)Z) = 0, \tag{14}$$

for all vector fields X, Y, Z, W orthogonal to ξ.

3. ϕ-Symmetric $N(k)$-Paracontact Metric Manifolds

Let us consider ϕ-symmetric $N(k)$-paracontact metric manifold. Then, by virtue of (1) and (13), we have

$$(\nabla_W R)(X,Y)Z - \eta((\nabla_W R)(X,Y)Z)\xi = 0. \tag{15}$$

Taking the inner product of (15) by U, we have

$$\begin{aligned} & g((\nabla_W R)(X,Y)Z, U) - \eta((\nabla_W R)(X,Y)Z)\eta(U) \\ & = 0. \end{aligned} \tag{16}$$

Let $\{e_i\}$, $i = 1, 2, \ldots, (2n+1)$, be an orthonormal basis of the tangent space at any point p of the manifold. Then, putting $X = U = \xi$ in (16) and taking summation over i, $1 \le i \le (2n+1)$, we have

$$(\nabla_W S)(Y,Z) - \sum_{i=1}^{2n+1} \eta((\nabla_W R)(e_i, Y)Z)\eta(e_i) = 0. \tag{17}$$

Considering the second term of (17) and setting $Z = \xi$, we have

$$\begin{aligned} & \sum_{i=1}^{2n+1} \eta((\nabla_W R)(e_i, Y)\xi)\eta(e_i) \\ & = \sum_{i=1}^{2n+1} g((\nabla_W R)(e_i, Y)\xi, \xi) g(e_i, \xi). \end{aligned} \tag{18}$$

Next,

$$\begin{aligned} g((\nabla_W R)(e_i, Y)\xi, \xi) = {} & g(\nabla_W R(e_i, Y)\xi, \xi) \\ & - g(R(\nabla_W e_i, Y)\xi, \xi) \\ & - g(R(e_i, \nabla_W Y)\xi, \xi) \\ & - g(R(e_i, Y)\nabla_W \xi, \xi). \end{aligned} \tag{19}$$

Since $\{e_i\}$ is an orthonormal basis, $\nabla_X e_i = 0$. Using (6), we have

$$\begin{aligned} & g(R(e_i, \nabla_W Y)\xi, \xi) \\ & = g(k(\eta(\nabla_W Y)e_i - \eta(e_i)\nabla_W Y), \xi) \\ & = k[g(\nabla_W Y, \xi)g(e_i, \xi) - g(e_i, \xi)g(\nabla_W Y, \xi)] \\ & = 0. \end{aligned} \tag{20}$$

Using (20) in (19), we have

$$\begin{aligned} g((\nabla_W R)(e_i, Y)\xi, \xi) = {} & g(\nabla_W R(e_i, Y)\xi, \xi) \\ & - g(R(e_i, Y)\nabla_W \xi, \xi). \end{aligned} \tag{21}$$

Since $g(R(e_i, Y)\xi, \xi) = -g(R(\xi, \xi)Y, e_i) = 0$, we have

$$g(\nabla_W R(e_i, Y)\xi, \xi) + g(R(e_i, Y)\xi, \nabla_W \xi) = 0. \quad (22)$$

Using (22) in (21), we have

$$g((\nabla_W R)(e_i, Y)\xi, \xi) = -g(R(e_i, Y)\xi, \nabla_W \xi) - g(R(e_i, Y)\nabla_W \xi, \xi). \quad (23)$$

Using (4) in (23), we have

$$\begin{aligned} g((\nabla_W R)(e_i, Y)\xi, \xi) &= -g(R(e_i, Y)\xi, -\phi X + \phi h X) \\ &\quad - g(R(e_i, Y)(\phi X + \phi h X), \xi) \\ &= g(R(e_i, Y)\xi, \phi X) - g(R(e_i, Y)\xi, \phi h X) \\ &\quad + g(R(e_i, Y)\phi X, \xi) - g(R(e_i, Y)\phi h X, \xi) = 0. \end{aligned} \quad (24)$$

Putting $Z = \xi$ in (17) and using (24), it follows that

$$(\nabla_W S)(Y, \xi) = 0. \quad (25)$$

We know that

$$(\nabla_W S)(Y, \xi) = \nabla_W S(Y, \xi) - S(\nabla_W Y, \xi) - S(Y, \nabla_W \xi). \quad (26)$$

Using (4), (9), (10), and (25) in (26), we have

$$2nkg(W - hW, \phi Y) - S(Y, -\phi W + \phi h W) = 0. \quad (27)$$

Putting $W = \phi W$ in (27) and using (1), (2), (3), and (10), we have

$$S(Y, W) = 2nkg(Y, W) + 2nkg(Y, hW) - S(Y, hW). \quad (28)$$

Again, putting $W = hW$ in (28) and using (1) and (7), we obtain

$$2nkg(Y, hW) - S(Y, hW) = (1 + k)S(Y, W) - 2nk(1 + k)g(Y, W). \quad (29)$$

By virtue of (28) and (29), we have

$$S(Y, W) = 2nkg(Y, W). \quad (30)$$

Thus, we can state the following theorem.

Theorem 3. *A $(2n+1)$-dimensional ϕ-symmetric $N(k)$-paracontact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is an Einstein manifold.*

4. Three-Dimensional Locally ϕ-Symmetric $N(k)$-Paracontact Metric Manifolds

For a three-dimensional semi-Riemannian manifold, the conformal curvature tensor C is given by

$$\begin{aligned} C(X, Y)Z &= R(X, Y)Z - [S(Y, Z)X - S(X, Z)Y \\ &\quad + g(Y, Z)QX - g(X, Z)QY] + \frac{r}{2}[g(Y, Z)X \\ &\quad - g(X, Z)Y], \end{aligned} \quad (31)$$

for arbitrary vector fields X, Y, Z.

If $C = 0$, then (31) reduces to the following form:

$$\begin{aligned} R(X, Y)Z &= [S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX \\ &\quad - g(X, Z)QY] - \frac{r}{2}[g(Y, Z)X - g(X, Z)Y]. \end{aligned} \quad (32)$$

Putting $Z = \xi$ in (32) and using (6), we get

$$\left(\frac{r}{2} - k\right)[\eta(Y)\xi - \eta(X)\xi] = \eta(Y)QX - \eta(X)QY. \quad (33)$$

Again, putting $Y = \xi$ in (33) and using (11), we get

$$QX = \left(\frac{r}{2} - k\right)X + \left(3k - \frac{r}{2}\right)\eta(X)\xi. \quad (34)$$

Taking the inner product of (34) with Y, we obtain

$$S(X, Y) = \left(\frac{r}{2} - k\right)g(X, Y) + \left(3k - \frac{r}{2}\right)\eta(X)\eta(Y). \quad (35)$$

Using (34) and (35) in (32), we have

$$\begin{aligned} R(X, Y)Z &= \left(\frac{r}{2} - 2k\right)[g(Y, Z)X - g(X, Z)Y] \\ &\quad + \left(3k - \frac{r}{2}\right)[g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi \\ &\quad + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y], \end{aligned} \quad (36)$$

where R is Riemannian curvature tensor on the 3-dimensional $N(k)$-paracontact metric manifold.

Taking the covariant differentiation of (36) with respect to W, we have

$$\begin{aligned} (\nabla_W R)(X, Y)Z &= \frac{dr(W)}{2}[g(Y, Z)X - g(X, Z)Y \\ &\quad - g(Y, Z)\eta(X)\xi + g(X, Z)\eta(Y)\xi \\ &\quad - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y] + \left(3k - \frac{r}{2}\right) \\ &\quad \cdot [g(Y, Z)(\nabla_W \eta)(X)\xi \\ &\quad - g(Y, Z)\eta(X)\phi W + g(Y, Z)\eta(X)\phi h W \\ &\quad - g(X, Z)(\nabla_W \eta)(Y)\xi + g(X, Z)\eta(Y)\phi W \\ &\quad - g(X, Z)\eta(Y)\phi h W + (\nabla_W \eta)(Y)\eta(Z)X \\ &\quad + \eta(Y)(\nabla_W \eta)(Z)X - (\nabla_W \eta)(X)\eta(Z)Y \\ &\quad - \eta(X)(\nabla_W \eta)(Z)Y]. \end{aligned} \quad (37)$$

Applying ϕ^2 to both sides of (37), we have

$$\begin{aligned}\phi^2 (\nabla_W R)(X,Y)Z = {} & \frac{dr(W)}{2} [g(Y,Z)X \\ & - g(X,Z)Y - g(Y,Z)\eta(X)\xi + g(X,Z)\eta(Y)\xi \\ & - \eta(Y)\eta(Z)X + \eta(X)\eta(Z)Y] + \left(3k - \frac{r}{2}\right) \\ & \cdot [g(Y,Z)\eta(X)\phi hW \\ & - g(X,Z)\eta(Y)\phi hW - g(Y,Z)\eta(X)\phi W \\ & + g(X,Z)\eta(Y)\phi W + (\nabla_W \eta)(Y)\eta(Z)X \\ & - (\nabla_W \eta)(Y)\eta(Z)\eta(X)\xi + \eta(Y)(\nabla_W \eta)(Z)X \\ & - (\nabla_W \eta)(X)\eta(Z)Y + (\nabla_W \eta)(X)\eta(Z)\eta(Y)\xi \\ & - \eta(X)(\nabla_W \eta)(Z)Y].\end{aligned} \tag{38}$$

Now, taking X, Y, Z orthogonal to ξ and using (14), we get

$$\frac{dr(W)}{2}[g(Y,Z)X - g(X,Z)Y] = 0. \tag{39}$$

Hence, we can state the following theorem.

Theorem 4. *A 3-dimensional $N(k)$-paracontact metric manifold $(M^3, \phi, \xi, \eta, g)$ is locally ϕ-symmetric if the scalar curvature tensor r of g is constant.*

5. ϕ-Ricci Symmetric $N(k)$-Paracontact Metric Manifolds

Definition 5. $N(k)$-paracontact metric manifold is said to be ϕ-Ricci symmetric if the Ricci operator Q satisfies

$$\phi^2((\nabla_X Q)Y) = 0, \tag{40}$$

for all vector fields X, Y on M.

If X, Y are orthogonal to ξ, then manifold is said to be locally ϕ-Ricci symmetric.

Using (1) in (40), we have

$$(\nabla_X Q)Y - \eta((\nabla_X Q)Y)\xi = 0. \tag{41}$$

Taking the inner product of (41) with Z, we have

$$g((\nabla_X Q)Y, Z) - \eta((\nabla_X Q)Y)\eta(Z) = 0. \tag{42}$$

Further simplification of (42) gives the following:

$$\begin{aligned} g(\nabla_X Q(Y), Z) - S(\nabla_X Y, Z) - \eta((\nabla_X Q)Y)\eta(Z) \\ = 0. \end{aligned} \tag{43}$$

Putting $Y = \xi$ in (43), we have

$$g(\nabla_X Q(\xi), Z) - S(\nabla_X \xi, Z) - \eta((\nabla_X Q)\xi)\eta(Z) = 0. \tag{44}$$

Using (4), (10), and (11) in (44), we have

$$\begin{aligned} S(\phi X, Z) = {} & 2nk[g(\phi X, Z) - g(\phi hX, Z)] \\ & + S(\phi hX, Z) + \eta((\nabla_X Q)\xi)\eta(Z). \end{aligned} \tag{45}$$

Putting $Z = \phi Z$ in (45), we have

$$\begin{aligned} S(\phi X, \phi Z) = {} & 2nk[g(\phi X, \phi Z) - g(\phi hX, \phi Z)] \\ & + S(\phi hX, \phi Z). \end{aligned} \tag{46}$$

Again, putting $X = \phi X$, $Z = \phi Z$ in (46) and then using (1), (3), and (10), we have

$$S(X,Z) = 2nkg(X,Z) + 2nkg(hX,Z) - S(hX,Z). \tag{47}$$

Replace $Z = hZ$ in (47), and using (1), (7), and symmetric property of h, we have

$$\begin{aligned} & 2nkg(hX,Z) - S(hX,Z) \\ & \quad = (k+1)[S(X,Z) - 2nkg(X,Z)]. \end{aligned} \tag{48}$$

By virtue of (47) and (48), we have

$$S(Y,W) = 2nkg(Y,W). \tag{49}$$

Hence, we can state the following theorem.

Theorem 6. *A $(2n+1)$-dimensional $N(k)$-paracontact metric manifold (M, ϕ, ξ, η, g) is ϕ-Ricci symmetric if g is an Einstein manifold.*

6. Three-Dimensional ϕ-Ricci Symmetric $N(k)$-Paracontact Metric Manifolds

On a 3-dimensional $N(k)$-paracontact metric manifold, the Ricci operator Q is given by (34).

Now, taking the covariant differentiation of (34) with respect to W, we have

$$\begin{aligned} (\nabla_W Q)X = {} & \frac{dr(W)}{2}[X - \eta(X)\xi] \\ & - \left(3k - \frac{r}{2}\right)\eta(X)\phi W \\ & + \left(3k - \frac{r}{2}\right)\eta(X)\eta(W)\xi \\ & + \left(3k - \frac{r}{2}\right)g(W, \phi X)\xi \\ & + \left(3k - \frac{r}{2}\right)g(hW, \phi X)\xi. \end{aligned} \tag{50}$$

Applying ϕ^2 to both sides of (50), we have

$$\begin{aligned} \phi^2((\nabla_W Q)X) = {} & \frac{dr(W)}{2}[X - \eta(X)\xi] \\ & - \left(3k - \frac{r}{2}\right)\eta(X)\phi W. \end{aligned} \tag{51}$$

Taking X orthogonal to ξ in (51), we get the following form:

$$\phi^2\left(\left(\nabla_W Q\right)X\right)=\frac{dr\left(W\right)}{2}X. \tag{52}$$

In view of the above equation, we are able to state the following theorem.

Theorem 7. *A 3-dimensional $N(k)$-paracontact metric manifold (M^3,ϕ,ξ,η,g) is locally ϕ-Ricci symmetric if the scalar curvature tensor r of g is constant.*

7. Example of 3-Dimensional Locally ϕ-Symmetric $N(k)$-Paracontact Metric Manifolds with $k=-1$

We consider the manifold $M=\mathbb{R}^3$ with the usual cartesian coordinates (x,y,z). The vector fields

$$\begin{aligned} e_1&=\frac{\partial}{\partial x}+\frac{x}{z^2}\frac{\partial}{\partial y}-2y\frac{\partial}{\partial z},\\ e_2&=\frac{\partial}{\partial y},\\ e_3&=\frac{\partial}{\partial z}\end{aligned} \tag{53}$$

are linearly independent at each point of M. We can compute

$$\begin{aligned}\left[e_1,e_2\right]&=2e_3,\\ \left[e_1,e_3\right]&=\frac{2x}{z^3}e_2,\\ \left[e_2,e_3\right]&=0.\end{aligned} \tag{54}$$

We define the semi-Riemannian metric g as the nondegenerate one, whose only nonvanishing components are $g(e_1,e_2)=g(e_3,e_3)=1$, and the 1-form η as $\eta=2ydx+dz$, which satisfies $\eta(e_1)=\eta(e_2)=0,\eta(e_3)=1$. Let ϕ be the $(1,1)$-tensor field defined by $\phi e_1=e_1$, $\phi e_2=-e_2$, and $\phi\xi=0$. Then,

$$\begin{aligned} d\eta\left(e_1,e_2\right)&=\frac{1}{2}\left[e_1\left(\eta\left(e_2\right)\right)-e_2\left(\eta\left(e_1\right)\right)-\eta\left(\left[e_1,e_2\right]\right)\right]\\ &=-1=-g\left(e_1,e_2\right)=g\left(e_1,\phi e_2\right),\\ d\eta\left(e_1,e_3\right)&=\frac{1}{2}\left[e_1\left(\eta\left(e_3\right)\right)-e_3\left(\eta\left(e_1\right)\right)-\eta\left(\left[e_1,e_3\right]\right)\right]\\ &=0=g\left(e_1,\phi e_3\right),\\ d\eta\left(e_2,e_3\right)&=\frac{1}{2}\left[e_2\left(\eta\left(e_3\right)\right)-e_3\left(\eta\left(e_2\right)\right)-\eta\left(\left[e_2,e_3\right]\right)\right]\\ &=0=g\left(e_2,\phi e_3\right).\end{aligned} \tag{55}$$

Therefore, (ϕ,ξ,η,g) is a paracontact metric structure on M.

Moreover, $he_1=-(2x/z^3)e_2$, $he_2=0$, and $h\xi=0$. Hence, $h^2=0$ and, given $p=(x,y,z)\in\mathbb{R}^3$, $\operatorname{rank}(h_p)=0$ if $x=0$ and $\operatorname{rank}(h_p)=1$ if $x\neq 0$.

Let ∇ be the Levi-Civita connection. Using the properties of paracontact metric structure and Koszul's formula

$$\begin{aligned}2g\left(\nabla_X Y,Z\right)&=Xg\left(Y,Z\right)+Yg\left(Z,X\right)-Zg\left(X,Y\right)\\ &\quad-g\left(X,\left[Y,Z\right]\right)-g\left(Y,\left[X,Z\right]\right)\\ &\quad+g\left(Z,\left[X,Y\right]\right),\end{aligned} \tag{56}$$

we can compute

$$\begin{aligned}\nabla_{e_1}e_3&=-e_1+\frac{2x}{z^3}e_2,\\ \nabla_{e_2}e_3&=e_2,\\ \nabla_{e_3}e_3&=0,\\ \nabla_{e_1}e_1&=-\frac{2x}{z^3}e_3,\\ \nabla_{e_2}e_1&=-e_3,\\ \nabla_{e_3}e_1&=-e_1,\\ \nabla_{e_1}e_2&=e_3,\\ \nabla_{e_2}e_2&=0,\\ \nabla_{e_3}e_2&=e_2.\end{aligned} \tag{57}$$

Hence, (M,ϕ,ξ,η,g) is $N(k)$-paracontact metric manifold with $k=-1$.

Using the following definition of Riemannian curvature tensor

$$R\left(X,Y\right)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z, \tag{58}$$

we obtain

$$\begin{aligned}R\left(e_1,e_2\right)e_3&=0,\\ R\left(e_1,e_3\right)e_3&=-e_1,\\ R\left(e_2,e_3\right)e_3&=-e_2,\\ R\left(e_1,e_2\right)e_2&=-3e_2,\\ R\left(e_2,e_3\right)e_2&=0,\\ R\left(e_1,e_3\right)e_2&=e_3,\\ R\left(e_1,e_2\right)e_1&=3e_1,\\ R\left(e_2,e_3\right)e_1&=e_3,\\ R\left(e_1,e_3\right)e_1&=\frac{4x}{z^3}e_3.\end{aligned} \tag{59}$$

From this, it follows that $\phi^2((\nabla_W R)(X,Y)Z)=0$ for all vector fields X, Y, and Z are orthogonal to ξ. Thus, the three-dimensional $N(k)$-paracontact metric manifold with $k=-1$ is locally ϕ-symmetric.

Also from the above expressions for the curvature tensor, we obtain that the scalar curvature tensor is constant. Therefore, from Theorem 4, it follows that the manifold under consideration is locally ϕ-symmetric.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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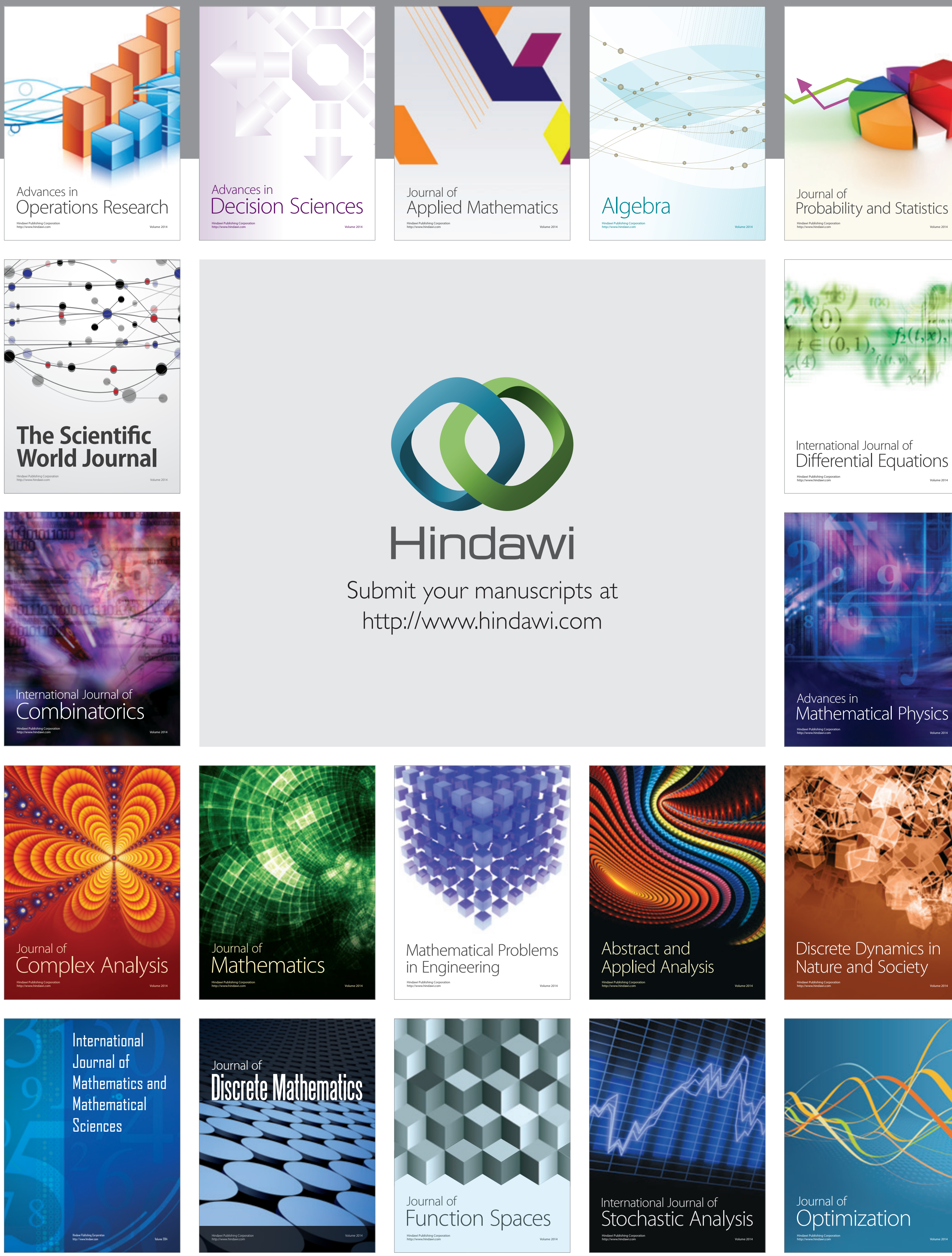
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