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Research Article

The Distortion Theorems for Harmonic Mappings with Analytic Parts Convex or Starlike Functions of Order β

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Some sharp estimates of coefficients, distortion, and growth for harmonic mappings with analytic parts convex or starlike functions of order β are obtained. We also give area estimates and covering theorems. Our main results generalise those of Klimek and Michalski.

1. Introduction

Let S denote the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are analytic and univalent in the unit disk $\mathbb{D} := \{z : |z| < 1\}$. An analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is said to be convex of order β if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \beta > 0, \quad z \in \mathbb{D} \tag{1}$$

for which we write $f(z) \in C_\beta \subset S$. An analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is said to be starlike of order β if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq \beta > 0, \quad z \in \mathbb{D} \tag{2}$$

for which we write $f(z) \in S_\beta^* \subset S$, where $\beta \in (0, 1]$. Moreover, when the positive constant β vanishes in (1), (2), the function f turns to be starlike or convex function. That is to say, when an analytic function f only satisfies $\operatorname{Re}\{zf'(z)/f(z)\} > 0$ or $\operatorname{Re}\{1 + zf''(z)/f'(z)\} > 0$, we call f belongs to starlike or convex function, for which we write $f \in S_0^*$ or $f \in C_0$ for convenience.

A complex-valued harmonic function f in the open unit disk $\mathbb{D} \subset \mathbb{C}$ has a canonical decomposition

$$f(z) = h(z) + \overline{g(z)}, \tag{3}$$

where h and g are analytic in $\mathbb{D}$. Generally, we call $h(z)$ the analytic part and $g(z)$ the coanalytic part of $f(z) = h(z) + \overline{g(z)}$. An elegant and complete account of the theory of planar harmonic mappings is given in Duren's monograph [1].

Lewy [2] proved in 1936 that a necessary and sufficient condition for $f(z) = h(z) + \overline{g(z)}$ to be locally univalent and sense-preserving in $\mathbb{D}$ is $J_f(z) > 0$, where

$$J_f(z) = \left|h'(z)\right|^2 - \left|g'(z)\right|^2, \quad z \in \mathbb{D}. \tag{4}$$

To such a function f, h' does not vanish in $\mathbb{D}$; let

$$\omega(z) = \frac{g'(z)}{h'(z)}, \quad z \in \mathbb{D}, \tag{5}$$

and then the second complex dilatation $\omega(z)$ is analytic with $|\omega(z)| < 1$.

Clunie and Sheil-Small introduced the class of all sense-preserving univalent harmonic mappings of $\mathbb{D}$ with $h(0) = g(0) = h'(0) - 1 = 0$ that is denoted by S_H [3]. In [4], Hotta and Michalski denoted the class L_H of all locally univalent and sense-preserving harmonic functions in the unit disk with $h(0) = g(0) = h'(0) - 1 = 0$. Obviously, $S_H \subset L_H$.

Every function $f \in L_H$ is uniquely determined by coefficients of the following power series expansions:

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n, \qquad z \in \mathbb{D}, \tag{6}$$

where $a_n, b_n \in \mathbb{C}$, $n = 2, 3, 4, \ldots$.

In [5], the classes of starlike and convex functions of order β were first introduced by Robertson. Then, such functions have been studied and used in [6–9], and so forth. In [10, 11], Klimek and Michalski studied the cases when the analytic part h is the identity mapping (convex of order 1) and convex mapping (convex of order 0), respectively. In [4], Hotta and Michalski considered the case when the analytic part h is a starlike analytic mapping (starlike of order 0). The main idea of this paper is to characterize the subclasses of S_H when $h \in C_\beta$ and the subclasses of L_H when $h \in S^*_\beta$, where $\beta \in [0, 1]$.

In order to establish our main results, we need the following lemma.

Lemma 1 (see [12]). *If $f(z) = a_0 + a_1 z + \cdots + a_n z^n + \cdots$ is analytic and $|f(z)| \le 1$ on $\mathbb{D}$, then*

$$|a_n| \le 1 - |a_0|^2, \quad n = 1, 2, \ldots. \tag{7}$$

2. Main Results and Their Proofs

In what follows, the harmonic mappings that we consider are all normalized locally univalent and sense-preserving.

Definition 2. For $\alpha \in [0, 1)$, let

$$S_H^\alpha(C_\beta) := \left\{ f(z) = h(z) + \overline{g(z)} : h(z) \in C_\beta,\ |b_1| = \alpha,\ 0 \le \beta \le 1 \right\} \subset S_H. \tag{8}$$

By [3, Theorem 5.7], if $h(z) \in C_\beta$ with $|\omega(z)| = |g'(z)/h'(z)| < 1$, then $f(z) = h(z) + \overline{g(z)} \in S_H$; hence, the class $S_H^\alpha(C_\beta)$ is well-defined.

Definition 3. For $\alpha \in [0, 1)$, let

$$L_H^\alpha(S^*_\beta) := \left\{ f(z) = h(z) + \overline{g(z)} : h(z) \in S^*_\beta,\ |b_1| = \alpha,\ 0 \le \beta \le 1 \right\} \subset L_H. \tag{9}$$

In particular, we establish a smaller subclass of S_H,

$$S_H^\alpha(S^*_\beta) := \left\{ f(z) = h(z) + \overline{g(z)} \in S_H : h(z) \in S^*_\beta,\ |b_1| = \alpha,\ 0 \le \beta \le 1 \right\}. \tag{10}$$

Lemma 4. *If $f(z) = h(z) + \overline{g(z)} \in L_H^\alpha(S^*_\beta)$, then $F(z) = H(z) + \overline{G(z)} \in S_H^\alpha(C_\beta)$, where $h(z)$, $g(z)$, and $H(z)$, $G(z)$ are related by $zH'(z) = h(z)$, $zG'(z) = g(z)$, $z \in \mathbb{D}$.*

Proof. By the definition of $L_H^\alpha(S^*_\beta)$, $h(z) \in S^*_\beta$. Using classical Alexander's theorem [13, page 43], the function $H(z) \in C_\beta$. Also, $H(0) = 0$, $H'(0) = \lim_{z\to 0} h(z)/z = h'(0) = 1$, and $|G'(0)| = |\lim_{z\to 0} g(z)/z| = |g'(0)| = \alpha$. Let $\Gamma := [0, h(z)] \subset h(\mathbb{D})$, $z \in \mathbb{D} \setminus \{0\}$; then

$$|g(z)| = \left| \int_\Gamma d\left(g \circ h^{-1}(w)\right) \right| \le \int_\Gamma \left| \frac{d\left(g \circ h^{-1}(w)\right)}{dw} \right| |dw| < \int_\Gamma |dw| = |h(z)|. \tag{11}$$

Hence,

$$\left|G'(z)\right| = \lim_{t\to z} \left| \frac{g(t)}{t} \right| < \lim_{t\to z} \left| \frac{h(t)}{t} \right| = \left|H'(z)\right|, \qquad z \in \mathbb{D} \setminus \{0\}, \tag{12}$$

which implies that $F(z)$ is a sense-preserving and locally univalent harmonic mapping in $\mathbb{D}$. By [11, Corollary 2.3], we obtain that $F \in S_H^\alpha(C_\beta)$. □

Applying Lemma 1, we can prove the following theorem.

Theorem 5. *If $f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n} \in S_H^\alpha(C_\beta)$, then*

$$|a_n| \le \frac{\prod_{k=2}^{n}(k - 2\beta)}{n!}, \quad n = 2, 3, 4, \ldots, \tag{13}$$

$$|b_n| \le \frac{\alpha \prod_{k=2}^{n}(k - 2\beta)}{n!} + \frac{1 - \alpha^2}{n} \left(1 + \sum_{k=2}^{n-1} \left(\frac{\prod_{t=2}^{k}(t - 2\beta)}{(k-1)!} \right) \right), \quad n = 3, 4, 5, \ldots. \tag{14}$$

Specially,

$$|b_2| \le \frac{1 + 2\alpha(1 - \beta) - \alpha^2}{2}, \quad \text{where } |b_1| = \alpha,\ 0 \le \beta \le 1. \tag{15}$$

The estimate for $|b_2|$ is sharp; the extremal functions are

$$\Omega(z) := H_0(z) + \overline{G_0(z)} = \begin{cases} \dfrac{1 - (1-z)^{2\beta-1}}{2\beta - 1} + \overline{\displaystyle\int_0^z \frac{\xi + \alpha}{(1 + \alpha\xi)(1-\xi)^{2-2\beta}} d\xi}, & \beta \ne \dfrac{1}{2}, \\ \log \dfrac{1}{1-z} + \overline{\displaystyle\int_0^z \frac{\xi + \alpha}{1 - (1-\alpha)\xi - \alpha\xi^2} d\xi}, & \beta = \dfrac{1}{2}. \end{cases} \tag{16}$$

Proof. Assuming $f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n} \in S_H^\alpha(C_\beta)$, $z \in \mathbb{D}$, then by [7] we have (13). Let $g'(z) = \omega(z)h'(z)$, where $\omega(z)$ is the dilatation of f. Since $\omega(z)$ is analytic in $\mathbb{D}$, it has a power series expansion

$$\omega(z) = \sum_{n=0}^{\infty} c_n z^n, \quad z \in \mathbb{D}, \tag{17}$$

where $c_n \in \mathbb{C}$, $n = 0, 1, 2, \ldots$, and $|c_0| = |\omega(0)| = |g'(0)| = |b_1| = \alpha$. Recall that $|\omega(z)| < 1$ for all $z \in \mathbb{D}$; then, by Lemma 1, we have

$$|c_n| \le 1 - |c_0|^2, \quad n = 1, 2, 3, \ldots. \tag{18}$$

Together with formulas (5), (6), and (17) we give

$$\sum_{n=1}^{\infty} n b_n z^{n-1} = \sum_{n=0}^{\infty} c_n z^n \sum_{n=1}^{\infty} n a_n z^{n-1}, \quad z \in \mathbb{D}, \tag{19}$$

which leads to

$$\sum_{n=0}^{\infty} (n+1) b_{n+1} z^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} (k+1) a_{k+1} c_{n-k} \right) z^n, \quad z \in \mathbb{D}. \tag{20}$$

Comparing coefficients, we obtain

$$(n+1) b_{n+1} = \sum_{k=0}^{n} (k+1) a_{k+1} c_{n-k}, \quad n = 0, 1, 2, \ldots. \tag{21}$$

Applying formulas (13) and (18) and by simple calculation, we have

$$\begin{aligned} n|b_n| &\le |c_{n-1}| + 2|a_2||c_{n-2}| + \cdots + (n-1)|a_{n-1}||c_1| + n|a_n||c_0| \\ &\le \left(1 + \sum_{k=2}^{n-1} k|a_k|\right)\left(1 - |c_0|^2\right) + n|a_n||c_0| \\ &\le \left(1 + \sum_{k=2}^{n-1}\left(\frac{\prod_{t=2}^{k}(t-2\beta)}{(k-1)!}\right)\right)\left(1-\alpha^2\right) + \frac{\prod_{k=2}^{n}(k-2\beta)}{(n-1)!}\alpha, \quad n = 3, 4, 5, \ldots. \end{aligned} \tag{22}$$

In particular,

$$\begin{aligned} 2|b_2| &\le |a_1||c_1| + 2|a_2||c_0| \le 1 - |c_0|^2 + 2|a_2||c_0| \\ &\le 1 + 2(1-\beta)\alpha - \alpha^2. \end{aligned} \tag{23}$$

Next, we will prove the estimate is sharp. For $\alpha \in [0, 1) \subset \mathbb{D}$, consider a function $\Omega(z) := H_0(z) + \overline{G_0(z)}$, such that

$$C_\beta \ni H_0(z) := \begin{cases} \dfrac{1 - (1-z)^{2\beta-1}}{2\beta - 1}, & \beta \ne \dfrac{1}{2}, \\ \log \dfrac{1}{1-z}, & \beta = \dfrac{1}{2}, \end{cases} \quad z \in \mathbb{D}. \tag{24}$$

and suppose that the dilatation of $\Omega(z)$ satisfies

$$\omega_0(z) := \frac{z + \alpha}{1 + \alpha z}, \quad z \in \mathbb{D}. \tag{25}$$

Applying formula (5), we obtain

$$G_0'(z) = \begin{cases} \dfrac{z+\alpha}{(1+\alpha z)(1-z)^{2-2\beta}} = \alpha + \left(1 + 2\alpha(1-\beta) - \alpha^2\right) z + \cdots, & \beta \ne \dfrac{1}{2}, \\ \dfrac{z+\alpha}{(1+\alpha z)(1-z)} = \alpha + \left(1 + \alpha - \alpha^2\right) z + \cdots, & \beta = \dfrac{1}{2}. \end{cases} \tag{26}$$

which implies the estimate of (15) is sharp. Obviously, $|\omega_0(z)| < 1$, $z \in \mathbb{D}$, which means $\Omega(z) := H_0(z) + \overline{G_0(z)} \in S_H^\alpha(C_\beta)$. Hence, the proof is completed. □

Corollary 6. *If* $f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n} \in L_H^\alpha(S_\beta^*)$, *then* $|a_n| \le \prod_{k=2}^{n}(k-2\beta)/(n-1)!$, $n = 2, 3, 4, \ldots$,

$$\begin{aligned} |b_2| &\le 2(1-\beta)\alpha + \frac{1-\alpha^2}{2}, \\ |b_n| &\le \frac{\alpha \prod_{k=2}^{n}(k-2\beta)}{(n-1)!} + \left(1-\alpha^2\right)\left(1 + \sum_{k=2}^{n-1}\left(\frac{\prod_{t=2}^{k}(t-2\beta)}{(k-1)!}\right)\right), \\ &\qquad n = 3, 4, 5, \ldots. \end{aligned} \tag{27}$$

Proof. If $f(z) \in L_H^\alpha(S_\beta^*)$, then, by Lemma 4, the function $F(z) := H(z) + \overline{G(z)} \in S_H^\alpha(C_\beta)$, where $zH'(z) = h(z)$, $zG'(z) = g(z)$, $z \in \mathbb{D}$. Let $G(z)$ be expanded in the power series

$$G(z) = \sum_{n=1}^{\infty} B_n z^n, \quad z \in \mathbb{D},\ B_n \in \mathbb{C}. \tag{28}$$

Together with expansion (6) of $g(z)$ and formula $g(z) = zG'(z)$, we have $b_n = nB_n$; then by Theorem 5 we can easily obtain the coefficient estimates of $f(z) \in L_H^\alpha(S_\beta^*)$.

Specially, by comparing coefficients, we have $2b_2 = 2a_2c_0 + c_1$, which easily leads to the estimate $|b_2| \le 2(1-\beta)\alpha + (1-\alpha^2)/2$ by the condition of Corollary 6. □

Since the analytic part h of $f \in S_H^\alpha(C_\beta)$ belongs to C_β, then, by [7], we have the following distortion estimate of h:

$$\frac{1}{(1+r)^{2(1-\beta)}} \le |h'(z)| \le \frac{1}{(1-r)^{2(1-\beta)}}, \quad z = re^{i\theta} \in \mathbb{D}. \tag{29}$$

Our next aim is to give the distortion estimate of the coanalytic part g of $f \in S_H^\alpha(C_\beta)$.

Theorem 7. *If $f(z) = h(z) + \overline{g(z)} \in S_H^\alpha(C_\beta)$, then*

$$\left|g'(z)\right| \geq \frac{|\alpha - r|}{(1-\alpha r)(1+r)^{2(1-\beta)}}, \quad z = re^{i\theta} \in \mathbb{D}, \tag{30}$$

$$\left|g'(z)\right| \leq \frac{\alpha + r}{(1+\alpha r)(1-r)^{2(1-\beta)}}, \quad z = re^{i\theta} \in \mathbb{D}. \tag{31}$$

These inequalities are sharp. The equalities hold for the harmonic function $\Omega(z)$ which is defined in (16).

Proof. Let $g'(0) = \alpha e^{i\mu}$. Consider the function

$$f_0(z) := \frac{e^{-i\mu}\omega(z) - \alpha}{1 - \alpha e^{-i\mu}\omega(z)}, \quad z = re^{i\theta} \in \mathbb{D}, \tag{32}$$

which satisfies assumptions of the Schwarz lemma; then we have

$$\left|e^{-i\mu}\omega(z) - \alpha\right| \leq r\left|1 - \alpha e^{-i\mu}\omega(z)\right|, \quad z = re^{i\theta} \in \mathbb{D}. \tag{33}$$

It is equivalent to

$$\left|e^{-i\mu}\omega(z) - \frac{\alpha\left(1-r^2\right)}{1-\alpha^2 r^2}\right| \leq \frac{r\left(1-\alpha^2\right)}{1-\alpha^2 r^2}, \quad z = re^{i\theta} \in \mathbb{D}. \tag{34}$$

Hence, applying the triangle inequalities to formula (34) we have

$$\frac{|\alpha - r|}{1-\alpha r} \leq |\omega(z)| \leq \frac{\alpha + r}{1+\alpha r}, \quad z = re^{i\theta} \in \mathbb{D}. \tag{35}$$

Finally, applying formula (29) together with (35) to the identity $g' = \omega h'$, we obtain (30) and (31). The function $\Omega(z)$ defined in (16) shows that inequalities (30) and (31) are sharp. The proof is completed. □

Corollary 8. *If $f(z) = h(z) + \overline{g(z)} \in L_H(S_\beta^*)$, then*

$$\begin{aligned} \left|g'(z)\right| &\geq \frac{|\alpha - r|(1-r+2\beta r)}{(1+r)^{3-2\beta}}, \quad z = re^{i\theta} \in \mathbb{D}, \\ \left|g'(z)\right| &\leq \frac{(\alpha + r)(1+r-2\beta r)}{(1-r)^{3-2\beta}}, \quad z = re^{i\theta} \in \mathbb{D}. \end{aligned} \tag{36}$$

Proof. In [7], we know that if $f(z) = h(z) + \overline{g(z)} \in L_H(S_\beta^*)$, then

$$\frac{1-(1-2\beta)r}{(1+r)^{3-2\beta}} \leq \left|h'(z)\right| \leq \frac{1+(1-2\beta)r}{(1-r)^{3-2\beta}}, \quad z = re^{i\theta} \in \mathbb{D}. \tag{37}$$

Using inequality (35) to identity $g'(z) = \omega(z)h'(z)$, then the corollary can be obtained immediately. □

By [7], we have the following growth estimate of $h \in C_\beta$, where $f(z) = h(z) + \overline{g(z)} \in S_H^\alpha(C_\beta)$.

In the case $\beta \neq 1/2$,

$$\frac{(1+r)^{2\beta-1} - 1}{2\beta - 1} \leq |h(z)| \leq \frac{1-(1-r)^{2\beta-1}}{2\beta-1}, \quad z = re^{i\theta} \in \mathbb{D}. \tag{38}$$

In the case $\beta = 1/2$,

$$\log(1+r) \leq |h(z)| \leq -\log(1-r), \quad z = re^{i\theta} \in \mathbb{D}. \tag{39}$$

In the next results, we give the growth estimate of coanalytic part g of $f \in S_H^\alpha(C_\beta)$.

Theorem 9. *If $f(z) = h(z) + \overline{g(z)} \in S_H^\alpha(C_\beta)$, then*

$$|g(z)| \leq \int_0^r \frac{\alpha + \rho}{(1+\alpha\rho)(1-\rho)^{2(1-\beta)}} d\rho, \quad z = re^{i\theta} \in \mathbb{D}. \tag{40}$$

The inequality is sharp. The equality holds for the harmonic function $\Omega(z)$ which is defined in (16).

Proof. Let $\gamma := [0, z]$; applying estimate (31) we have

$$\begin{aligned} |g(z)| &= \left|\int_\gamma g'(\xi)\, d\xi\right| \leq \int_\gamma \left|g'(\xi)\right| |d\xi| \\ &\leq \int_0^r \frac{\alpha + \rho}{(1+\alpha\rho)(1-\rho)^{2(1-\beta)}} d\rho, \end{aligned} \tag{41}$$

where $z = re^{i\theta} \in \mathbb{D}$. The function $\Omega(z)$ defined (16) shows that inequality (40) is sharp. □

For $f(z) = h(z) + \overline{g(z)} \in L_H(S_\beta^*)$, by [7], we have

$$\frac{r}{(1+r)^{2(1-\beta)}} \leq |h(z)| \leq \frac{r}{(1-r)^{2(1-\beta)}}, \quad z = re^{i\theta} \in \mathbb{D}. \tag{42}$$

Now, we give the growth estimates of coanalytic part $g(z)$ of $f(z) \in L_H(S_\beta^*)$.

Corollary 10. *If $f(z) = h(z) + \overline{g(z)} \in L_H(S_\beta^*)$, then*

$$|g(z)| \leq \int_0^r \frac{(\alpha + \rho)(1+\rho-2\beta\rho)}{(1-\rho)^{3-2\beta}} d\rho, \quad z = re^{i\theta} \in \mathbb{D}. \tag{43}$$

Using the distortion estimates in (29) and (35), we can easily deduce the following area estimates of $f(z) \in S_H^\alpha(C_\beta)$.

Theorem 11. *Let $\beta \in (1/2, 1]$ and $A := \iint_{\mathbb{D}} J_f(z)\,dx\,dy$; if $f(z) = h(z) + \overline{g(z)} \in S_H^\alpha(C_\beta)$, then*

$$\begin{aligned} 2\pi \int_0^1 \frac{r\left(1-r^2\right)\left(1-\alpha^2\right)}{(1+r)^{4(1-\beta)}(1+\alpha r)^2} dr &\leq A \\ &\leq 2\pi \int_0^1 \frac{r\left(1-r^2\right)\left(1-\alpha^2\right)}{(1-r)^{4(1-\beta)}(1-\alpha r)^2} dr, \end{aligned} \tag{44}$$

where $z = re^{i\theta} \in \mathbb{D}$.

Proof. Observe that if $f(z) \in S_H^{\alpha}(C_\beta)$, then $h'(z)$ does not vanish in $\mathbb{D}$. We can give the Jacobian of $f(z) = h(z) + \overline{g(z)}$ in the form

$$J_f(z) = \left|h'(z)\right|^2 \left(1 - |\omega(z)|^2\right), \quad z \in \mathbb{D}, \tag{45}$$

where $\omega(z)$ is the dilatation of $f(z)$. Applying (29) and (35) to (45) we obtain

$$\begin{aligned}
A &:= \iint_{\mathbb{D}} J_f(z)\,dx\,dy = \int_0^{2\pi} d\theta \int_0^1 J_f\left(re^{i\theta}\right) r\,dr \\
&= 2\pi \int_0^1 r J_f\left(re^{i\theta}\right) dr \\
&= 2\pi \int_0^1 r \left|h'\left(re^{i\theta}\right)\right|^2 \left(1 - \left|\omega\left(re^{i\theta}\right)\right|^2\right) dr \\
&\geq 2\pi \int_0^1 r \left(\frac{1}{(1+r)^{2(1-\beta)}}\right)^2 \left(1 - \left(\frac{\alpha + r}{1 + \alpha r}\right)^2\right) dr \\
&= 2\pi \int_0^1 r \frac{\left(1-\alpha^2\right)\left(1-r^2\right)}{(1+r)^{4(1-\beta)}(1+\alpha r)^2} dr, \\
A &:= 2\pi \int_0^1 r \left|h'\left(re^{i\theta}\right)\right|^2 \left(1 - \left|\omega\left(re^{i\theta}\right)\right|^2\right) dr \\
&\leq 2\pi \int_0^1 r \left(\frac{1}{(1-r)^{2(1-\beta)}}\right)^2 \left(1 - \left(\frac{\alpha - r}{1 - \alpha r}\right)^2\right) dr \\
&= 2\pi \int_0^1 r \frac{\left(1-\alpha^2\right)\left(1-r^2\right)}{(1-r)^{4(1-\beta)}(1-\alpha r)^2} dr,
\end{aligned} \tag{46}$$

where $z = re^{i\theta} \in \mathbb{D}$; this completes the proof. □

Remark 12. To avoid the maximum of A having no sense, we give the limiting condition $\beta \in (1/2, 1]$ in Theorem 11.

Corollary 13. *Let $\beta \in [0,1)$ and $A := \iint_{\mathbb{D}} J_f(z)dx\,dy$; if $f(z) = h(z) + \overline{g(z)} \in L_H(S_\beta^*)$, then*

$$\begin{aligned}
A &\geq 2\pi \int_0^1 \frac{\left(1-\alpha^2\right) r \left(1-r^2\right)(1 - r + 2\beta r)}{(1+r)^{6-4\beta}(1+\alpha r)^2} dr, \\
A &\leq 2\pi \int_0^1 \frac{\left(1-\alpha^2\right) r \left(1-r^2\right)(1 + r - 2\beta r)^2}{(1-r)^{6-4\beta}(1-\alpha r)^2} dr.
\end{aligned} \tag{47}$$

Theorem 14. *If $f(z) \in S_H^{\alpha}(C_\beta)$, then*

$$|f(z)| \geq \int_0^r \frac{(1-\alpha)(1-\rho)}{(1+\alpha\rho)(1+\rho)^{2(1-\beta)}} d\rho, \quad z = re^{i\theta} \in \mathbb{D}, \tag{48}$$

$$|f(z)| \leq \begin{cases} \dfrac{1-(1-r)^{2\beta-1}}{2\beta - 1} + \displaystyle\int_0^r \frac{\alpha + \rho}{(1+\alpha\rho)(1-\rho)^{2(1-\beta)}} d\rho, & \beta \neq \dfrac{1}{2}, \\ \log \dfrac{1+\alpha r}{1-r}, & \beta = \dfrac{1}{2}, \end{cases} \quad z = re^{i\theta} \in \mathbb{D}. \tag{49}$$

Proof. For any point $z = re^{i\theta} \in \mathbb{D}$, let $\mathbb{D}_r := \mathbb{D}(0,r) = \{z \in \mathbb{D} : |z| < r\}$ and denote

$$d := \min_{z \in \mathbb{D}_r} \left|f\left(\mathbb{D}_r\right)\right| \tag{50}$$

and then $\mathbb{D}(0,d) \subseteq f(\mathbb{D}_r) \subseteq f(\mathbb{D})$. Hence, there exists $z_r \in \partial\mathbb{D}_r$ such that $d = |f(z_r)|$. Let $L(t) := tf(z_r)$, $t \in [0,1]$; then $l(t) := f^{-1}(L(t))$ and $t \in [0,1]$ is a well-defined Jordan arc. Since $f = h + \overline{g}$, then we can obtain

$$\begin{aligned}
d &= \left|f\left(z_r\right)\right| = \int_L |dw| = \int_l |df| \\
&= \int_l \left|h'(\xi)\,d\xi + \overline{g'(\xi)} d\overline{\xi}\right| \\
&\geq \int_l \left(\left|h'(\xi)\right| - \left|g'(\xi)\right|\right) |d\xi|.
\end{aligned} \tag{51}$$

By $\omega = g'/h'$ with formulas (29) and (35), we have

$$\begin{aligned}
\left|h'(\xi)\right| - \left|g'(\xi)\right| &= \left|h'(\xi)\right| (1 - |\omega(\xi)|) \\
&\geq \frac{1}{(1+|\xi|)^{2(1-\beta)}} \left(1 - \frac{\alpha + |\xi|}{1 + \alpha|\xi|}\right) \\
&= \frac{(1-\alpha)(1-|\xi|)}{(1+\alpha|\xi|)(1+|\xi|)^{2(1-\beta)}}.
\end{aligned} \tag{52}$$

Hence, we obtain

$$\begin{aligned}
d &\geq \int_l \frac{(1-\alpha)(1-|\xi|)}{(1+\alpha|\xi|)(1+|\xi|)^{2(1-\beta)}} |d\xi| \\
&= \int_0^1 \frac{(1-\alpha)(1-|l(t)|)}{(1+\alpha|l(t)|)(1+|l(t)|)^{2(1-\beta)}} dt \\
&\geq \int_0^r \frac{(1-\alpha)(1-\rho)}{(1+\alpha\rho)(1+\rho)^{2(1-\beta)}} d\rho.
\end{aligned} \tag{53}$$

To prove (49) we simply use the inequality

$$|f(z)| = \left|h(z) + \overline{g(z)}\right| \leq |h(z)| + |g(z)|. \tag{54}$$

By formulas (38), (39), and (40) with simple calculation we have (49); this completes the proof. □

Corollary 15. *If $f(z) \in S_H(S_\beta^*)$, then*

$$|f(z)| \geq \int_0^r \frac{(1-\alpha)(1-\rho)(1-\rho+2\beta\rho)}{(1+\alpha\rho)(1+\rho)^{3-2\beta}} d\rho, \quad z = re^{i\theta} \in \mathbb{D}, \tag{55}$$

$$|f(z)| \leq \frac{r}{(1-r)^{2(1-\beta)}} + \int_0^r \frac{(\alpha+\rho)(1+\rho-2\beta\rho)}{(1-\rho)^{3-2\beta}} d\rho, \quad z = re^{i\theta} \in \mathbb{D}. \tag{56}$$

Finally, the growth estimate of $f \in S_H^\alpha(C_\beta)$ yields the following covering estimate.

Theorem 16. *If $f(z) \in S_H^\alpha(C_\beta)$, then*

$$D(0, R) \subset f(D), \tag{57}$$

where

$$R := \int_0^1 \frac{(1-\alpha)(1-\rho)}{(1+\alpha\rho)(1+\rho)^{2(1-\beta)}} d\rho. \tag{58}$$

Proof. Let r tend to 1 in estimate (48); then Theorem 16 follows immediately from the argument principle for harmonic mappings. □

Corollary 17. *If $f(z) \in S_H(S_\beta^*)$, then*

$$D(0, R) \subset f(D), \tag{59}$$

where

$$R := \int_0^1 \frac{(1-\alpha)(1-\rho)(1-\rho+2\beta\rho)}{(1+\alpha\rho)(1+\rho)^{3-2\beta}} d\rho. \tag{60}$$

Proof. Let r tend to 1 in estimate (55); then Corollary 17 follows immediately from the argument principle for harmonic mappings. □

Remark 18. The univalence problem of a locally univalent harmonic mapping with starlike analytic parts is an open problem. Though S_β^* have stronger properties than S_0^*, we cannot obtain the sharp value of β such that $f(z) = h(z) + \overline{g(z)} \in L_H(S_\beta^*)$ is univalent. It has important sense to study. Moreover, case of $\beta = 0, 1$ was given a systematic study in [4, 10, 11].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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