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Research Article

Killing Vector Fields in Generalized Conformal β-Change of Finsler Spaces

Mallikarjun Yallappa Kumbar,[1] Narasimhamurthy Senajji Kampalappa,[2] Thippeswamy Komalobiah Rajanna,[2] and Kavyashree Ambale Rajegowda[2]

[1] *Department of P.G. Mathematics, Vijaya College, Rashtriya Vidyalaya Road, Basavanagudi, Bangalore, Karnataka 560004, India*
[2] *Department of P.G. Studies and Research in Mathematics, Kuvempu University, Shankaraghatta, Shimoga, Karnataka 577451, India*

Correspondence should be addressed to Mallikarjun Yallappa Kumbar; mallikarjunykumbar@gmail.com

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We consider a Finsler space equipped with a Generalized Conformal β-change of metric and study the Killing vector fields that correspond between the original Finsler space and the Finsler space equipped with Generalized Conformal β-change of metric. We obtain necessary and sufficient condition for a vector field Killing in the original Finsler space to be Killing in the Finsler space equipped with Generalized Conformal β-change of metric.

1. Introduction

In 1976, Hashiguchi [1] studied the conformal change of Finsler metrics; namely, $\overline{L} = e^{\sigma(x)}L$. In particular, he also dealt with the special conformal transformation named C-conformal transformation. This change has been studied by Izumi [2] and Kropina [3]. In 2008, Abed [4, 5] introduced the transformation $\overline{L} = e^{\sigma(x)}L + \beta$, thus generalizing the conformal, Randers, and generalized Randers changes. Moreover, he established the relationships between some important tensors associated with (M, L) and the corresponding tensors associated with $(M, \overline{L})$. He also studied some invariant and σ-invariant properties and obtained a relationship between the Cartan connection associated with (M, L) and the transformed Cartan connection associated with $(M, \overline{L})$.

In this paper, we deal with a general change of Finsler metrics defined by

$$L(x, y) \longrightarrow \overline{L}(x, y) = f\left(e^{\sigma(x)}L(x, y), \beta(x, y)\right), \quad (1)$$

where f is a positively homogeneous function of degree one in $\overline{L} := e^{\sigma}L$ and β. This change will be referred to as a generalized β-conformal change. It is clear that this change is a generalization of the abovementioned changes and deals simultaneously with β-change and conformal change. It combines also the special case of Shibata ($\overline{L} = f(L, \beta)$) and that of Abed ($\overline{L} = e^{\sigma}L, \beta$).

In 1984, Shibata [6] studied β-change of Finsler metrics and discussed certain invariant tensors under such a change. Killing equations play important role in the study of a Finsler space which undergoes a change in the metric. In fact, they give an equivalent characterization for the transformations to preserve distances. In 1979, Singh et al. [7] studied a Randers space $F^n(M, L(x, y) = (g_{ij}(x)y^i y^j)^{1/2} + b_i(x)y^i)$, $n \geq 2$, which undergoes a change $L(x, y) \mapsto L^*(x, y) = L^2(x, y) + (\alpha_i(x)y^i)^2$. They discussed Killing correspondence of spaces $F^n(M, L)$ and $F^{*n}(M, L^*)$.

In the present paper, we consider a general Finsler space $F^n(M, L)$ which undergoes conformal and β-change; that is, $L(x, y) \rightarrow \overline{L}(x, y) = f(e^{\sigma(x)}L(x, y), \beta(x, y))$, where $\beta(x, y) = b_i(x)y^i$ is a 1-form. We study Killing correspondence of Finsler spaces $F^n(M, L)$ and $\overline{F}^n(M, \overline{L})$. For the notations and terminology, we refer the reader to the books [8, 9] and the papers [6] by Shibata and [10] by Youssef et al.

2. Preliminaries

Let $F^n = (M, L)$, $n \geq 2$, be an n-dimensional C^∞ Finsler manifold with fundamental function $L = L(x, y)$. Consider the following change of Finsler structures which will be referred to as a generalized β-conformal change:

$$L(x,y) \longrightarrow \overline{L}(x,y) = f\left(e^{\sigma(x)}L(x,y), \beta(x,y)\right), \quad (2)$$

where f is a positively homogeneous function of degree one in $e^{\sigma}L$ and 1-form β, where $\beta = b_i(x)dx^i$.

We define

$$f_1 := \frac{\partial f}{\partial \widetilde{L}}, \quad f_2 := \frac{\partial f}{\partial \beta}, \quad f_{12} := \frac{\partial^2 f}{\partial \widetilde{L}\partial \beta}, \ldots, \quad (3)$$

where $\widetilde{L} = e^{\sigma}L$.

The angular metric tensor $\overline{h}_{ij}$ of the space $\overline{F}^n$ is given by [10]

$$\overline{h}_{ij} = e^{\sigma}ph_{ij} + q_0 m_i m_j, \quad (4)$$

where

$$\begin{aligned} p &= \frac{ff_1}{L}, \\ q &= ff_2, \\ q_0 &= ff_{22}, \\ p_0 &= f_2^2 + q_0, \\ q_{-1} &= \frac{ff_{12}}{L}, \\ p_{-1} &= q_{-1} + \frac{pf_2}{f}, \\ q_{-2} &= \frac{f\left(e^{\sigma}f_{11} - f_1/L\right)}{L^2}, \\ p_{-2} &= q_{-2} + \frac{e^{\sigma}p^2}{f^2}, \\ m_i &= b_i - \frac{\beta y^i}{L^2} \neq 0, \\ \sigma_i &= \partial_i \sigma. \end{aligned} \quad (5)$$

h_{ij} is the angular metric tensor of F^n. The fundamental metric tensor $\overline{g}_{ij}$ and its inverse $\overline{g}^{ij}$ of $\overline{F}^n$ are expressed as [10]

$$\begin{aligned} \overline{g}_{ij} &= e^{\sigma}pg_{ij} + p_0 b_i b_j + e^{\sigma}p_{-1}\left(b_i y_j + b_j y_i\right) + e^{\sigma}p_{-2}y_i y_j, \\ \overline{g}^{ij} &= \left(\frac{e^{-\sigma}}{p}\right)g^{ij} - s_0 b^i b^j - s_{-1}\left(b^i y^j + b^j y^i\right) - s_{-2}y^i y^j, \end{aligned} \quad (6)$$

where

$$\begin{aligned} s_0 &= \frac{e^{-\sigma}f^2 q_0}{(\varepsilon pL^2)}, \\ s_{-1} &= \frac{p_{-1}f^2}{(\varepsilon pL^2)}, \\ s_{-2} &= \frac{p_{-1}\left(e^{\sigma}m^2 pL^2 - b^2 f^2\right)}{(\varepsilon p\beta L^2)}, \\ \varepsilon &= \frac{f^2\left(e^{\sigma}p + m^2 q_0\right)}{L^2} \neq 0, \\ m^2 &= g^{ij}m_i m_j. \end{aligned} \quad (7)$$

g_{ij} and g^{ij}, respectively, are the metric tensor and inverse metric tensor of F^n. The Cartan tensor $\overline{C}_{ijk}$ and the associate Cartan tensor $\overline{C}_{ij}^l$ of $\overline{F}^n$ are given by the following expressions:

$$\begin{aligned} \overline{C}_{ijk} &= e^{\sigma}pC_{ijk} + \frac{1}{2}e^{\sigma}p_{-1}\left(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j\right) \\ &\quad + \frac{1}{2}p_{02}m_i m_j m_k. \end{aligned} \quad (8)$$

The $(h)hv$-torsion tensor $\overline{C}_{ij}^l$ is expressed in terms of C_{ij}^l as [10]

$$\overline{C}_{ij}^l = C_{ij}^l + M_{ij}^l, \quad (9)$$

where

$$\begin{aligned} M_{ij}^l &= \frac{1}{2p}\left[e^{-\sigma}m^l - pm^2\left(s_0 b^l + s_{-1}y^l\right)\right] \\ &\quad \cdot \left(e^{\sigma}p_{-1}h_{ij} + p_{02}m_i m_j\right) - e^{\sigma}\left(s_0 b^l + s_{-1}y^l\right) \\ &\quad \cdot \left(pC_{isj}b^s + p_{-1}m_i m_j\right) + \frac{p_{-1}}{2p}\left(h_i^l m_j + h_j^l m_i\right); \\ h_j^i &= g^{il}h_{lj}, \\ p_{02} &= \frac{\partial p_0}{\partial \beta}. \end{aligned} \quad (10)$$

C_{ijk} and C_{ij}^l are, respectively, the Cartan tensor and the associate Cartan tensor of F^n. The spray coefficients $\overline{G}^i$ of $\overline{F}^n$ in terms of the spray coefficients G^i of F^n are expressed as [10]

$$\overline{G}^i = G^i + D^i, \quad (11)$$

where

$$D^i = \frac{\sigma_0}{2p}\left\{\left[2p - \beta p_{-1} - e^\sigma p^2 L^2 s_{-2} - p s_{-1}\left(2e^\sigma p\beta + e^\sigma p_{-1} L^2 m^2\right)\right] y^i - 2e^\sigma p^2 \beta s_0 b^i\right\} + \frac{q}{p} e^{-\sigma} F_0^i - \frac{1}{2} L^2 \sigma^i + \frac{1}{2}\left(e^\sigma p E_{00} - 2q F_{\beta 0} + e^\sigma p L^2 \sigma_\beta\right)\left(s_0 b^i + s_{-1} y^i\right); \quad (12)$$

$$E_{jk} = \left(\frac{1}{2}\right)\left(b_{j|k} + b_{k|j}\right),$$

$$F_{jk} = \left(\frac{1}{2}\right)\left(b_{j|k} - b_{k|j}\right),$$

$$F^i_j = g^{ik} F_{kj}.$$

The symbol "|" denotes h-covariant derivative with respect to Cartan connection $C\Gamma$ and lower index "0" (except in s_0) denotes the contraction by y^i.

The relation between the coefficients $\overline{N}^i_j$ of Cartan nonlinear connection in $\overline{F}^n$ and the coefficients N^i_j of the corresponding Cartan nonlinear connection in F^n is given by [10]

$$\overline{N}^i_j = N^i_j + D^i_j, \quad (13)$$

where

$$D^i_j = \frac{e^{-\sigma}}{p} A^i_j - \left(s_0 b^i + s_{-1} y^i\right) A_{tj} b^t - \left(q b_{0|j} + e^\sigma p L^2 \sigma_j\right)\left(s_{-1} b^i + s_{-2} y^i\right);$$

$$A_{ij} = E_{00} B_{ij} + F_{i0} Q_j + q F_{ij} + E_{j0} Q_i - 2\left(e^\sigma p C_{sij} + V_{sij}\right) D^s + \frac{1}{2}\sigma_0\left[2e^\sigma p g_{ij} + 2e^\sigma p_{-1} m_j y_i - 2\beta B_{ij} + e^\sigma p_{-1}\left(b_i y_j - b_j y_i\right)\right] - \frac{1}{2}\sigma_i\left(e^\sigma L^2 p_{-1} m_j + 2e^\sigma p y_j\right) + \frac{1}{2}\sigma_j\left(2e^\sigma p y_i + e^\sigma L^2 p_{-1} m_i\right); \quad (14)$$

$$A^i_j = g^{li} A_{lj},$$

$$2B_{ij} = e^\sigma p_{-1} h_{ij} + p_{02} m_i m_j,$$

$$Q_i = e^\sigma p_{-1} y_i + p_0 b_i.$$

The coefficients $\overline{F}^i_{jk}$ of Cartan connection $C\overline{\Gamma}$ in $\overline{F}^n$ and the coefficients F^i_{jk} of the corresponding Cartan connection $C\Gamma$ in F^n are related as follows [10]:

$$\overline{F}^i_{jk} = F^i_{jk} + D^i_{jk}, \quad (15)$$

where

$$D^i_{jk} = \left\{\left(\frac{e^{-\sigma}}{p}\right) g^{it} - \left(s_0 b^i + s_{-1} y^i\right) b^t - \left(s_{-1} b^i + s_{-2} y^i\right) y^t\right\}\left\{F_{tk} Q_j + F_{tj} Q_k + E_{jk} Q_t + \frac{1}{2} \cdot \Theta_{(j,k,t)}\left(2e^\sigma p C_{jkm} D^m_t + 2V_{jkm} D^m_t - K_{jk}\sigma_t - 2B_{jk} b_{0|t}\right)\right\},$$

$$V_{ijk} = \frac{1}{2} e^\sigma p_{-1}\left(h_{ij} m_k + h_{jk} m_i + h_{ki} m_j\right) + \frac{1}{2} \cdot p_{02} m_i m_j m_k, \quad (16)$$

$$K_{ij} = A_1 g_{ij} + A_2 b_i b_j + A_3\left(b_i y_j + b_j y_i\right) + A_4 y_i y_j,$$

$$A_1 = e^\sigma\left(2p - \beta p_{-1}\right),$$

$$A_2 = -\beta p_{02},$$

$$A_3 = e^\sigma p_{-1} + \left(\frac{\beta^2}{L^2}\right) p_{02},$$

$$A_4 = e^\sigma p_{-2} - \left(\frac{\beta^3}{L^4}\right) p_{02},$$

$$\Theta_{(j,k,t)}\left\{A_{jkt}\right\} = A_{jkt} - A_{ktj} - A_{tjk}.$$

The tensor D^i_{jk} has the properties

$$D^i_{j0} = B^i_{j0} = D^i_j;$$

$$D^i_{00} = 2D^i, \quad (17)$$

$$\text{where } B^i_{jk} = \partial_k D^i_j.$$

3. Killing Vector Fields in Correspondence of F^n and $\overline{F}^n$

Let us consider an infinitesimal transformation

$$'x^i = x^i + \epsilon v^i(x), \quad (18)$$

where ϵ is an infinitesimal constant and $v^i(x)$ is a contravariant vector field.

The vector field $v^i(x)$ is said to be a Killing vector field in F^n if the metric tensor of the Finsler space with respect to the infinitesimal transformation (18) is Lie invariant; that is,

$$\pounds_v g_{ij} = 0, \quad (19)$$

with $\pounds_v$ being the operator of Lie differentiation. Equivalently, the vector field $v^i(x)$ is Killing in F^n if

$$v_{i|j} + v_{j|i} + 2C^l_{ij} v_{l|0} = 0, \quad (20)$$

where $v_i = g_{il} v^l$.

Now, we prove the following result which gives a necessary and sufficient condition for a Killing vector field in F^n to be Killing in $\overline{F}^n$.

Theorem 1. *A Killing vector field $v^i(x)$ in F^n is Killing in $\overline{F}^n$ if and only if*

$$M^l_{ij} v_{l|0} + C_{rjt} v^t D^r_i + C_{rit} v^t D^r_j + v_r D^r_{ij} + \overline{C}^l_{ij} \left(2C_{rlt} v^t D^r + v_r D^r_l \right) = 0, \tag{21}$$

where $\overline{C}^l_{ij}$ is the associate Cartan tensor of $\overline{F}^n$.

Proof. Assume that $v^i(x)$ is Killing in F^n. Then (20) is satisfied. By definition, the h-covariant derivatives of v_i with respect to $C\overline{\Gamma}$ and $C\Gamma$ are, respectively, given as

$$\begin{aligned} &\text{(a)}\quad v_{i\|j} = \partial_j v_i - \left(\dot{\partial}_r v_i\right) \overline{G}^r_j - v_r \overline{F}^r_{ij}, \\ &\text{(b)}\quad v_{i|j} = \partial_j v_i - \left(\dot{\partial}_r v_i\right) G^r_j - v_r F^r_{ij}, \end{aligned} \tag{22}$$

where $\partial_j = \partial/\partial x^j$ and "$\|$" denote the h-covariant differentiation with respect to $C\overline{\Gamma}$. Equation (22)(a), by virtue of (11), (15), and (22)(b), takes the form

$$v_{i\|j} = v_{i|j} - 2C_{rit} v^t D^r_j - v_r D^r_{ij}. \tag{23}$$

Now, from (23), we have

$$\begin{aligned} v_{i\|j} + v_{j\|i} + 2\overline{C}^l_{ij} v_{l\|0} = {} & v_{i|j} + v_{j|i} + 2C^l_{ij} v_{l|0} - 2C_{rit} v^t D^r_j \\ & - 2C_{rjt} v^t D^r_i - 2v_r D^r_{ij} \\ & - 2\overline{C}^l_{ij} \left(2C_{rlt} v^t D^r + v_r D^r_l \right). \end{aligned} \tag{24}$$

Using (9) in (24) and applying (20), we get

$$\begin{aligned} v_{i\|j} + v_{j\|i} + 2\overline{C}^l_{ij} v_{l\|0} = {} & 2M^l_{ij} v_{l|0} - 2C_{rit} v^t D^r_j \\ & - 2C_{rjt} v^t D^r_i - 2v_r D^r_{ij} \\ & - 2\overline{C}^l_{ij} \left(2C_{rlt} v^t D^r + v_r D^r_l \right). \end{aligned} \tag{25}$$

Proof is complete with the observation that $v^i(x)$ is Killing in $\overline{F}^n$ if and only if $v_{i\|j} + v_{j\|i} + 2\overline{C}^l_{ij} v_{l\|0} = 0$, that is, if and only if (21) holds. □

If a vector field $v^i(x)$ is Killing in F^n and $\overline{F}^n$, then, from Theorem 1, (21) holds, which on transvection by y^i yields

$$2C_{rlt} v^t D^r + v_r D^r_l = 0. \tag{26}$$

Equation (21), in view of (26), enables us to state the following.

Corollary 2. *If a vector field $v^i(x)$ is Killing in F^n and $\overline{F}^n$, then*

$$C_{rit} v^t D^r_j + C_{rjt} v^t D^r_i + v_r D^r_{ij} - M^l_{ij} v_{l|0} = 0. \tag{27}$$

As another important consequence of Theorem 1, we have the following.

Corollary 3. *If a vector field $v^i(x)$ is Killing in F^n and $\overline{F}^n$, then the vector $v_i(x, y)$ is orthogonal to the vector $D^i(x, y)$.*

Proof. As $v^i(x)$ is Killing in F^n and $\overline{F}^n$, (21) holds, which on transvection by y^i gives (26). Again Transvection (26) by y^j, it follows that $v_r D^r = 0$. This proves the result. □

4. Conclusion

The main purpose of the present paper is to examine the classical approach to the problem of existence of Killing vector fields and study how they vary from point to point and how they are related to Killing vector fields defined on the whole manifold. In this respect, our purpose is similar to that of Shukla and Gupta on the study of projective motion. Actually, there is a more substantial relation of our work to theirs where we proved Theorem 1 as the main result and as its consequences we obtained Corollaries 2 and 3. Since the Killing equation (19) is a necessary and sufficient condition for the transformation (18) to be a motion in F^n, condition (21) obtained in Theorem 1 may be taken as the necessary and sufficient condition for the vector field $v^i(x)$, generating a motion in F^n, to generate a motion in $\overline{F}^n$ as well. It is clear that vector field $v^i(x)$, generating an affine motion (resp., projective motion) in F^n, generates an affine motion (resp., projective motion) in $\overline{F}^n$ if condition (21) holds. Our study has applications to link various transformations in F^n with the corresponding transformations in $\overline{F}^n$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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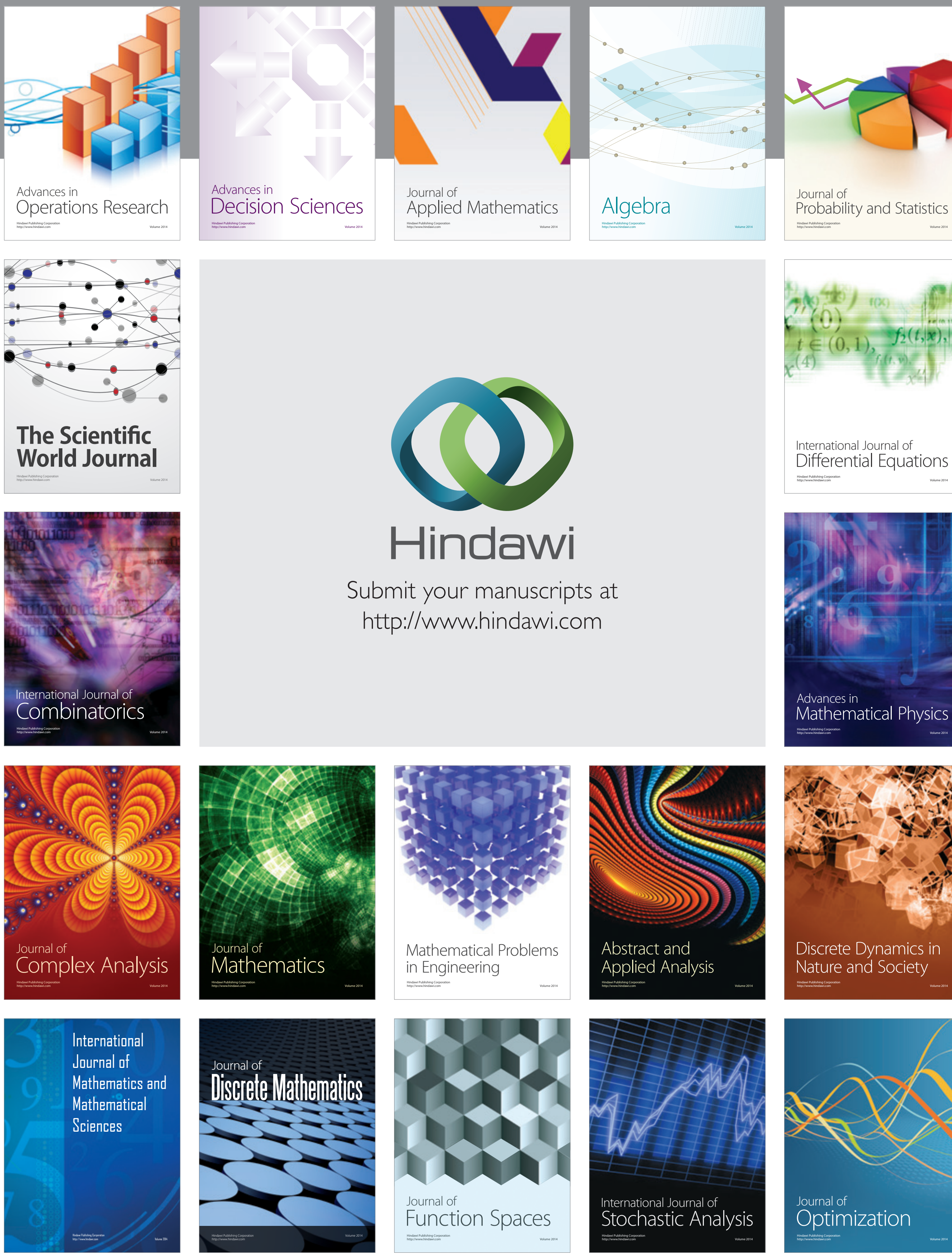
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