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Research Article

Coefficient Bounds for Certain Subclasses of m-Fold Symmetric Biunivalent Functions

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We consider two new subclasses $S_{\Sigma_m}(\alpha,\lambda)$ and $S_{\Sigma_m}(\beta,\lambda)$ of Σ_m consisting of analytic and m-fold symmetric biunivalent functions in the open unit disk U. Furthermore, we establish bounds for the coefficients for these subclasses and several related classes are also considered and connections to earlier known results are made.

1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$, and let S be the subclass of A consisting of form (1) which is also univalent in U.

The Koebe one-quarter theorem [1] states that the image of U under every function f from S contains a disk of radius 1/4. Thus, every such univalent function has inverse f^{-1} which satisfies

$$\begin{gathered} f^{-1}(f(z)) = z \quad (z \in U), \\ f(f^{-1}(w)) = w \quad \left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right), \end{gathered} \quad (2)$$

where

$$\begin{aligned} f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 \\ - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots. \end{aligned} \quad (3)$$

Function $f \in A$ is said to be biunivalent in U if both f and f^{-1} are univalent in U. Let Σ denote the class of biunivalent functions defined in unit disk U.

For a brief history and interesting examples in class Σ, see [2]. Examples of functions in class Σ are

$$\begin{gathered} \frac{z}{1-z}, \\ -\log(1-z), \\ \frac{1}{2}\log\left(\frac{1+z}{1-z}\right), \end{gathered} \quad (4)$$

and so on. However, the familiar Koebe function is not a member of Σ. Other common examples of functions in S such as

$$\begin{gathered} z - \frac{z^2}{2}, \\ \frac{z}{1-z^2} \end{gathered} \quad (5)$$

are also not members of Σ (see [2]).

For each function $f \in S$, function

$$h(z) = \sqrt[m]{f(z^m)} \quad (z \in U,\ m \in \mathbb{N}) \quad (6)$$

is univalent and maps unit disk U into a region with m-fold symmetry. A function is said to be m-fold symmetric (see [3, 4]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \quad (z \in U,\ m \in \mathbb{N}). \quad (7)$$

We denote by S_m the class of m-fold symmetric univalent functions in U, which are normalized by the series expansion (7). In fact, the functions in class S are *one*-fold symmetric.

Analogous to the concept of m-fold symmetric univalent functions, we here introduced the concept of m-fold symmetric biunivalent functions. Each function $f \in \Sigma$ generates an m-fold symmetric biunivalent function for each integer $m \in \mathbb{N}$. The normalized form of f is given as in (7) and the series expansion for f^{-1}, which has been recently proven by Srivastava et al. [5], is given as follows:

$$\begin{aligned} g(w) = {} & w - a_{m+1} w^{m+1} + \left[(m+1) a_{m+1}^2 - a_{2m+1}\right] \\ & \cdot w^{2m+1} - \left[\frac{1}{2}(m+1)(3m+2) a_{m+1}^3 \right. \\ & \left. - (3m+2) a_{m+1} a_{2m+1} + a_{3m+1}\right] w^{3m+1} + \cdots, \end{aligned} \tag{8}$$

where $f^{-1} = g$. We denote by Σ_m the class of m-fold symmetric biunivalent functions in U. For $m = 1$, formula (8) coincides with formula (3) of class Σ. Some examples of m-fold symmetric biunivalent functions are given as follows:

$$\begin{gathered} \left(\frac{z^m}{1-z^m}\right)^{1/m}, \\ \left[-\log\left(1-z^m\right)\right]^{1/m}, \\ \left[\frac{1}{2}\log\left(\frac{1+z^m}{1-z^m}\right)^{1/m}\right]. \end{gathered} \tag{9}$$

Lewin [6] studied the class of biunivalent functions, obtaining the bound 1.51 for modulus of the second coefficient $|a_2|$. Subsequently, Brannan and Clunie [7] conjectured that $|a_2| \le \sqrt{2}$ for $f \in \Sigma$. Later, Netanyahu [8] showed that $\max |a_2| = 4/3$ if $f(z) \in \Sigma$. Brannan and Taha [9] introduced certain subclasses of biunivalent function class Σ similar to the familiar subclasses. $S^\star(\beta)$ and $K(\beta)$ are of starlike and convex function of order β $(0 \le \beta < 1)$, respectively (see [8]). Classes $S_\Sigma^\star(\alpha)$ and $K_\Sigma(\alpha)$ of bistarlike functions of order α and biconvex functions of order α, corresponding to function classes $S^\star(\alpha)$ and $K(\alpha)$, were also introduced analogously. For each of function classes $S_\Sigma^\star(\alpha)$ and $K_\Sigma(\alpha)$, they found nonsharp estimates on the initial coefficients. In fact, the aforecited work of Srivastava et al. [2] essentially revived the investigation of various subclasses of biunivalent function class Σ in recent years. Recently, many authors investigated bounds for various subclasses of biunivalent functions (see [2, 10–15]). Not much is known about the bounds on general coefficient $|a_n|$ for $n \ge 4$. In the literature, only few works determine general coefficient bounds $|a_n|$ for the analytic biunivalent functions (see [16–18]). The coefficient estimate problem for each of $|a_n|$ $(n \in \mathbb{N} \setminus \{1,2\};\ \mathbb{N} = \{1,2,3,\ldots\})$ is still an open problem.

The aim of the this paper is to introduce two new subclasses of function class Σ_m and derive estimates on initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in these new subclasses. We have to remember the following lemma here so as to derive our basic results.

Lemma 1 (see [4]). *If $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$ is an analytic function in U with positive real part, then*

$$\begin{gathered} |p_n| \le 2 \quad (n \in \mathbb{N} = \{1,2,\ldots\}), \\ \left|p_2 - \frac{p_1^2}{2}\right| \le 2 - \frac{|p_1|^2}{2}. \end{gathered} \tag{10}$$

2. Coefficient Bounds for Function Class $S_{\Sigma_m}(\alpha,\lambda)$

Definition 2. A function $f \in \Sigma_m$ is said to be in class $S_{\Sigma_m}(\alpha,\lambda)$ if the following conditions are satisfied:

$$\begin{gathered} \left|\arg\left(\frac{zf'(z)}{(1-\lambda) f(z) + \lambda z f'(z)}\right)\right| < \frac{\alpha\pi}{2} \\ f \in \Sigma,\ (0 < \alpha \le 1,\ 0 \le \lambda < 1,\ z \in U), \\ \left|\arg\left(\frac{\lambda g'(w)}{(1-\lambda) g(w) + \lambda w g'(w)}\right)\right| < \frac{\alpha\pi}{2} \\ (0 < \alpha \le 1,\ 0 \le \lambda < 1,\ w \in U), \end{gathered} \tag{11}$$

where function $g = f^{-1}$.

Theorem 3. *Let f given by (7) be in class $S_{\Sigma_m}(\alpha,\lambda)$, $0 < \alpha \le 1$. Then,*

$$\begin{gathered} |a_{m+1}| \le \frac{2\alpha}{m(1-\lambda)\sqrt{\alpha+1}}, \\ |a_{2m+1}| \le \frac{\alpha}{m(1-\lambda)} + \frac{2(m+1)\alpha^2}{m^2(1-\lambda)^2}. \end{gathered} \tag{12}$$

Proof. Let $f \in S_{\Sigma_m}(\alpha,\lambda)$. Then,

$$\begin{gathered} \frac{zf'(z)}{(1-\lambda) f(z) + \lambda z f'(z)} = [p(z)]^\alpha, \\ \frac{\lambda g'(w)}{(1-\lambda) g(w) + \lambda w g'(w)} = [q(w)]^\alpha, \end{gathered} \tag{13}$$

where $g = f^{-1}$ and p, q in P have the following forms:

$$\begin{gathered} p(z) = 1 + p_m z^m + p_{2m} z^{2m} + \cdots, \\ q(w) = 1 + q_m w^m + q_{2m} w^{2m} + \cdots. \end{gathered} \tag{14}$$

Now, equating the coefficients in (13), we get

$$m(1-\lambda) a_{m+1} = \alpha p_m, \tag{15}$$

$$\begin{aligned} & m(1-\lambda)\left[2a_{2m+1} - (\lambda m + 1) a_{m+1}^2\right] \\ & \quad = \alpha p_{2m} + \frac{\alpha(\alpha-1)}{2} p_m^2, \end{aligned} \tag{16}$$

$$-m(1-\lambda) a_{m+1} = \alpha q_m, \tag{17}$$

$$\begin{aligned} & m(1-\lambda)\left[(1 + m(2-\lambda)) a_{m+1}^2 - 2a_{2m+1}\right] \\ & \quad = \alpha q_{2m} + \frac{\alpha(\alpha-1)}{2} q_m^2. \end{aligned} \tag{18}$$

From (15) and (17), we obtain

$$p_m = -q_m, \tag{19}$$

$$2m^2(1-\lambda)^2 a_{m+1}^2 = \alpha^2\left(p_m^2 + q_m^2\right). \tag{20}$$

Also from (16), (18), and (20), we have

$$\begin{aligned} 2m^2(1-\lambda)^2 a_{m+1}^2 &= \alpha\left(p_{2m}+q_{2m}\right) + \frac{\alpha(\alpha-1)}{2}\left(p_m^2+q_m^2\right) \\ &= \alpha\left(p_{2m}+q_{2m}\right) + \frac{\alpha(\alpha-1)}{2}\frac{2m^2(1-\lambda)^2}{\alpha^2}a_{m+1}^2. \end{aligned} \tag{21}$$

Therefore, we have

$$a_{m+1}^2 = \frac{\alpha^2\left(p_{2m}+q_{2m}\right)}{m^2(1-\lambda)^2(\alpha+1)}. \tag{22}$$

Applying Lemma 1 for coefficients p_{2m} and q_{2m}, we obtain

$$\left|a_{m+1}\right| \le \frac{2\alpha}{m(1-\lambda)\sqrt{\alpha+1}}. \tag{23}$$

Next, in order to find the bound on $|a_{2m+1}|$, by subtracting (18) from (16), we obtain

$$\begin{aligned} &4m(1-\lambda)a_{2m+1} - 2m(m+1)(1-\lambda)a_{m+1}^2 \\ &\quad = \alpha\left(p_{2m}-q_{2m}\right) + \frac{\alpha(\alpha-1)}{2}\left(p_m^2-q_m^2\right). \end{aligned} \tag{24}$$

Then, in view of (19) and (20) and applying Lemma 1 for coefficients p_{2m}, p_m and q_{2m}, q_m, we have

$$\left|a_{2m+1}\right| \le \frac{\alpha}{m(1-\lambda)} + \frac{2(m+1)\alpha^2}{m^2(1-\lambda)^2}. \tag{25}$$

This completes the proof of Theorem 3. □

3. Coefficient Bounds for Function Class $S_{\Sigma_m}(\beta,\lambda)$

Definition 4. Function $f \in \Sigma_m$ given by (7) is said to be in class $S_{\Sigma_m}(\beta,\lambda)$ if the following conditions are satisfied:

$$\begin{aligned} &\operatorname{Re}\left(\frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)}\right) > \beta \\ &\qquad f \in \Sigma,\ (0 \le \beta < 1,\ 0 \le \lambda < 1,\ z \in U), \\ &\operatorname{Re}\left(\frac{\lambda g'(w)}{(1-\lambda)g(w)+\lambda wg'(w)}\right) > \beta \\ &\qquad (0 \le \beta < 1,\ 0 \le \lambda < 1,\ w \in U), \end{aligned} \tag{26}$$

where function $g = f^{-1}$.

Theorem 5. *Let f given by (7) be in class $S_{\Sigma_m}(\beta,\lambda)$, $0 \le \beta < 1$. Then,*

$$\begin{aligned} \left|a_{m+1}\right| &\le \frac{\sqrt{2(1-\beta)}}{m(1-\lambda)}, \\ \left|a_{2m+1}\right| &\le \frac{2(m+1)(1-\beta)^2}{m^2(1-\lambda)^2} + \frac{1-\beta}{m(1-\lambda)}. \end{aligned} \tag{27}$$

Proof. Let $f \in S_{\Sigma_m}(\beta,\lambda)$. Then,

$$\begin{aligned} \frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)} &= \beta + (1-\beta)p(z), \\ \frac{\lambda g'(w)}{(1-\lambda)g(w)+\lambda wg'(w)} &= \beta + (1-\beta)q(w), \end{aligned} \tag{28}$$

where $p, q \in P$ and $g = f^{-1}$.

It follows from (28) that

$$m(1-\lambda)a_{m+1} = (1-\beta)p_m, \tag{29}$$

$$m(1-\lambda)\left[2a_{2m+1} - (\lambda m+1)a_{m+1}^2\right] = (1-\beta)p_{2m}, \tag{30}$$

$$-m(1-\lambda)a_{m+1} = (1-\beta)q_m, \tag{31}$$

$$\begin{aligned} &m(1-\lambda)\left[(1+m(2-\lambda))a_{m+1}^2 - 2a_{2m+1}\right] \\ &\quad = (1-\beta)q_{2m}. \end{aligned} \tag{32}$$

From (29) and (31), we obtain

$$\begin{aligned} p_m &= -q_m, \\ 2m^2(1-\lambda)^2 a_{m+1}^2 &= (1-\beta)^2\left(p_m^2+q_m^2\right). \end{aligned} \tag{33}$$

Adding (30) and (32), we have

$$2m^2(1-\lambda)^2 a_{m+1}^2 = (1-\beta)\left(p_{2m}+q_{2m}\right). \tag{34}$$

Therefore, we obtain

$$a_{m+1}^2 = \frac{(1-\beta)\left(p_{2m}+q_{2m}\right)}{2m^2(1-\lambda)^2}. \tag{35}$$

Applying Lemma 1 for coefficients p_{2m} and q_{2m}, we obtain

$$\left|a_{m+1}\right| \le \frac{\sqrt{2(1-\beta)}}{m(1-\lambda)}. \tag{36}$$

Next, in order to find the bound on $|a_{2m+1}|$, by subtracting (32) from (30), we obtain

$$\begin{aligned} &4m(1-\lambda)a_{2m+1} - 2m(m+1)(1-\lambda)a_{m+1}^2 \\ &\quad = (1-\beta)\left(p_{2m}-q_{2m}\right). \end{aligned} \tag{37}$$

Then, in view of (33), applying Lemma 1 for coefficients p_{2m}, p_m and q_{2m}, q_m, we have

$$\left|a_{2m+1}\right| \le \frac{2(m+1)(1-\beta)^2}{m^2(1-\lambda)^2} + \frac{1-\beta}{m(1-\lambda)}. \tag{38}$$

This completes the proof of Theorem 5. □

If we set $\lambda = 0$ in Theorems 3 and 5, then classes $S_{\Sigma_m}(\alpha, \lambda)$ and $S_{\Sigma_m}(\beta, \lambda)$ reduce to classes $S_{\Sigma_m}^{\alpha}$ and $S_{\Sigma_m}^{\beta}$ and thus we obtain the following corollaries.

Corollary 6. *Let f given by (7) be in class $S_{\Sigma_m}^{\alpha}$ $(0 < \alpha \leq 1)$. Then,*

$$|a_{m+1}| \leq \frac{2\alpha}{m\sqrt{\alpha+1}}, \qquad |a_{2m+1}| \leq \frac{\alpha}{m} + \frac{2(m+1)\alpha^2}{m^2}. \tag{39}$$

Corollary 7. *Let f given by (7) be in class $S_{\Sigma_m}^{\beta}$ $(0 \leq \beta < 1)$. Then,*

$$|a_{m+1}| \leq \frac{\sqrt{2(1-\beta)}}{m}, \qquad |a_{2m+1}| \leq \frac{2(m+1)(1-\beta)^2}{m^2} + \frac{1-\beta}{m}. \tag{40}$$

Classes $S_{\Sigma_m}^{\alpha}$ and $S_{\Sigma_m}^{\beta}$ are, respectively, defined as follows.

Definition 8. Function $f \in \Sigma_m$ given by (7) is said to be in class $S_{\Sigma_m}^{\alpha}$ if the following conditions are satisfied:

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \quad f \in \Sigma,\ (0 < \alpha \leq 1,\ z \in U), \qquad \left|\arg\left(\frac{\lambda g'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1,\ w \in U), \tag{41}$$

where function $g = f^{-1}$.

Definition 9. Function $f \in \Sigma_m$ given by (7) is said to be in class $S_{\Sigma_m}^{\beta}$ if the following conditions are satisfied:

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta \quad f \in \Sigma,\ (0 \leq \beta < 1,\ z \in U), \qquad \operatorname{Re}\left(\frac{\lambda g'(w)}{g(w)}\right) > \beta \quad (0 \leq \beta < 1,\ w \in U), \tag{42}$$

where function $g = f^{-1}$.

For *one*-fold symmetric biunivalent functions and $\lambda = 0$, Theorems 3 and 5 reduce to Corollaries 10 and 11, respectively, which were proven earlier by Murugusundaramoorty et al. [19].

Corollary 10. *Let f given by (7) be in class $S_{\Sigma}^{\star}(\alpha)$ $(0 < \alpha \leq 1)$. Then,*

$$|a_2| \leq \frac{2\alpha}{\sqrt{\alpha+1}}, \qquad |a_3| \leq 4\alpha^2 + \alpha. \tag{43}$$

Corollary 11. *Let f given by (7) be in class $S_{\Sigma}^{\star}(\beta)$ $(0 \leq \beta < 1)$. Then,*

$$|a_2| \leq \sqrt{2(1-\beta)}, \qquad |a_3| \leq 4(1-\beta)^2 + (1-\beta). \tag{44}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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