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Research Article

Weak Comparison Principle for Weighted Fractional p-Laplacian Equation

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The aim of this paper is to establish a weak comparison principle for a class fractional p-Laplacian equation with weight. The nonlinear term $f(x,s)>0$ is a Carathéodory function which is possibly unbounded both at the origin and at infinity and such that $f(x,s)s^{1-p}$ decreases with respect to s for a.e. $x\in\Omega$.

1. Introduction and Main Results

In this paper, we study a weak comparison principle for the following fractional p-Laplacian problem

$$\begin{cases} (-\Delta)^{\alpha}_{p,\beta}u(x)=f(x,u), & x\in\Omega,\\ u(x)>0, & x\in\Omega,\\ u(x)=0, & x\in\mathbb{R}^N\setminus\Omega, \end{cases}\tag{1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$ containing the origin, $0\le\beta<((N-p\alpha)/2)$, $1<p$, $p\alpha<N$, and $f:\Omega\times(0,+\infty)\longrightarrow(0,+\infty)$ is a general Carathéodory function, which is possibly unbounded both at the origin and at infinity and such that $f(x,s)s^{1-p}$ decreases with respect to s for a.e. $x\in\Omega$. The weighted fractional p-Laplacian $(-\Delta)^{\alpha}_{p,\beta}$ is the pseudodifferential operator defined as

$$\begin{aligned}(-\Delta)^{\alpha}_{p,\beta}u(x)&=P.V.\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+p\alpha}}\frac{dy}{|x|^{\beta}|y|^{\beta}}\\ &=\lim_{\varepsilon\to0}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+p\alpha}}\frac{dy}{|x|^{\beta}|y|^{\beta}},\end{aligned}\tag{2}$$

here P.V. denotes the principal value of the integral.

The simplest model is

$$\begin{cases} (-\Delta)^{\alpha}_{p,\beta}u(x)=\dfrac{h(x)}{u^{\gamma}(x)}+g(x)u^{q}(x), & x\in\Omega,\\ u(x)>0, & x\in\Omega,\\ u(x)=0, & x\in\mathbb{R}^N\setminus\Omega, \end{cases}\tag{3}$$

where $\gamma\ge0$ and $0\le q\le p$, and h and g are nonnegative functions.

The interest on the nonlocal operators continues to grow in recent years since such problems arise in various fields. The fractional p-Laplacian $(-\Delta)^{\alpha}_{p,\beta}$, on one hand, is an extension of the local operator $-\mathrm{div}\,(|x|^{-\beta}|\nabla u|^{p-2}\nabla u)$. Note that, for this type of operator, the Caffarelli-Kohn-Nirenberg inequality plays an important role, see [1–10]. On the other hand,$p=2$, which appears in a natural way when dealing with the fractional Laplace problem with Hardy potential. More precisely, let u be a solution to the following problem

$$\begin{cases} (-\Delta)^{\alpha}u-\lambda\dfrac{u}{|x|^{2\alpha}}=f(x,u), & x\in\Omega,\\ u(x)=0, & x\in\mathbb{R}^N\setminus\Omega, \end{cases}\tag{4}$$

where $\lambda:=\Lambda_{N,\alpha}+\Phi_{N,\alpha}(\beta)$, $\Lambda_{N,\alpha}$ is the Hardy constant, and

$$\Phi_{N,\alpha}(\beta) = 2^{2\alpha}\left(\frac{\Gamma((\beta+2\alpha)/2)\Gamma((N-\beta)/2)}{\Gamma((N-\beta-2\alpha)/2)\Gamma(\beta/2)} - \frac{\Gamma^2((N+2\alpha)/4)}{\Gamma^2((N-2\alpha)/4)}\right). \tag{5}$$

Then, according to ground state representation [11, 12], $v(x) = u(x)|x|^{\beta}$ satisfies

$$\begin{cases} L_\beta v = |x|^{-\beta} f\left(x, |x|^{-\beta} v\right), & x \in \Omega, \\ v = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{6}$$

where

$$L_\beta v = C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{v(x) - v(y)}{|x-y|^{N+2\alpha}} \frac{dy}{|x|^\beta |y|^\beta}. \tag{7}$$

Fractional Laplace operator $(-\Delta)^\alpha$ can be defined using Fourier analysis, functional calculus, singular integrals, or Lévy processes. Thus, rich mathematical concepts allow in general rich properties. For some abstract definitions and tools of fractional Laplace operator, see [13]. For more recent results of fractional Laplace elliptic problem, see [14–16] and the reference therein.

There are many works on the study of fractional p-Laplacian equations. Canino et al. [17] investigated the existence and uniqueness of solutions to

$$\begin{cases} (-\Delta)_p^\alpha u(x) = \dfrac{h(x)}{u^\gamma(x)}, & x \in \Omega, \\ u(x) > 0, & x \in \Omega, \\ u(x) = 0, & x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{8}$$

When $\beta = 0$, Mukherjee and Sreenadh [18] used variational methods to show the existence and multiplicity of positive solution problem (1) with critical growth and singular nonlinearity. Abdellaoui et al. [19] prove the existence of a weak solution to (1) under some hypotheses on $f(x, u)$.

The main idea of this paper comes from the seminal papers [20, 21]. In [21], Brezis and Oswald have shown the existence and uniqueness of a solution to a Laplace elliptic problem. Recently, Durastanti and Oliva [20] obtained the existence and uniqueness of positive solutions of an elliptic boundary value problem modeled by

$$\begin{cases} -\Delta_p u(x) = \dfrac{h(x)}{u^\gamma(x)} + g(x) u^q(x), & x \in \Omega, \\ u(x) > 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{9}$$

The main result of this paper is the following weak comparison principle.

Theorem 1. *Assume that f_1, f_2 are nonnegative functions such that either $f_1(x,s)s^{1-p}$ or $f_2(x,s)s^{1-p}$ is decreasing with respect to s, and for almost every $x \in \Omega$,*

$$f_1(x,s) \le f_2(x,s), \tag{10}$$

for almost every $x \in \Omega$ and for all $s \in (0, +\infty)$. Suppose that u_1 and u_2 are weak solutions to the problem

$$\begin{cases} (-\Delta)_{p,\beta}^\alpha u_i = f_i(x, u_i), & x \in \Omega, \\ u_i(x) > 0, & x \in \Omega, \\ u_i(x) = 0, & x \in \mathbb{R}^N \setminus \Omega, i = 1, 2. \end{cases} \tag{11}$$

Then, $u_1(x) \le u_2(x)$ almost everywhere in Ω.

Consequently, we obtain the uniqueness of the solution to problem (1).

Corollary 2. *Assume that f is a nonnegative function such that $f(x,s)s^{1-p}$ is decreasing with respect to s for almost every $x \in \Omega$. Then, there exists at most one weak solution to problem (1).*

The paper is organized as follows. In Section 2, we present some definitions and preliminary tools, which will be used in the proof of Theorem 1. The proof of Theorem 1 and Corollary 2 be given in Section 3.

2. Preparations

First of all, we give the definition of truncation function.

Definition 3. For every $k \ge 0$ and $\sigma \in \mathbb{R}^N$, define

$$T_k(\sigma) = \max\{-k, \min\{k, \sigma\}\}, G_k(\sigma) = \sigma - T_k(\sigma). \tag{12}$$

Let $\Omega \subset \mathbb{R}^N$, the weighted fractional Sobolev space $W_\beta^{\alpha,p}(\Omega)$ is defined by

$$W_\beta^{\alpha,p}(\Omega) = \left\{\varphi \in L^p(\Omega, d\mu) \colon \int_\Omega \int_\Omega |\varphi(x) - \varphi(y)|^p d\nu < \infty\right\}, \tag{13}$$

endowed with the norm

$$\|\varphi\|_{W_\beta^{\alpha,p}(\Omega)} = \left(\int_\Omega |\varphi(x)|^p d\mu\right)^{1/p} + \left(\int_\Omega \int_\Omega |\varphi(x) - \varphi(y)|^p d\nu\right)^{1/p}, \tag{14}$$

where

$$d\mu \coloneqq \frac{dx}{|x|^{2\beta}}, d\nu \coloneqq \frac{dxdy}{|x-y|^{n+p\alpha}|x|^\beta |y|^\beta}. \tag{15}$$

The space $W_{0,\beta}^{\alpha,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the above norm.

Definition 4. A positive function $u \in W_{0,\beta}^{\alpha,p}(\Omega) \cap L^{p-1}(\Omega)$ is a weak solution to problem (1) if $f(x,u) \in L_{loc}^{p-1}(\Omega)$ and for any $\phi \in C_0^1(\Omega)$,

$$\left\langle (-\Delta)^{\alpha}_{p,\beta} u, \phi \right\rangle = \int_{\Omega} f(x,u)\phi(x)dx, \tag{16}$$

where

$$\left\langle (-\Delta)^{\alpha}_{p,\beta} u, \phi \right\rangle = \iint_{D_\Omega} |u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))d\nu,$$

$$D_\Omega = \mathbb{R}^N \times \mathbb{R}^N \setminus (\Omega^c \times \Omega^c). \tag{17}$$

It is worth pointing out that the formulation (16) can be extended for $W^{\alpha,p}_0(\Omega)$-test functions by standard argument.

The following fractional Picone inequality appears in Proposition 2.15 of [22] with $\beta = 0$. A slight change in this proof actually shows that the fractional Picone inequality also holds for fractional p-Laplacian $(-\Delta)^{\alpha}_{p,\beta}$.

Lemma 5. *Consider $u, v \in W^{\alpha,p}_{0,\beta}(\Omega)$ with $u \geq 0$. Assume that $(-\Delta)^{\alpha}_{p,\beta} u$ is a positive bounded Radon measure in Ω. Then*

$$2\int_{\Omega} (-\Delta)^{\alpha}_{p,\beta} u \frac{v^p}{u^{p-1}} \leq \|v\|^p_{W^{\alpha,p}_{0,\beta}(\Omega)}. \tag{18}$$

Proof. Take $v_k = T_k(v)$ and $\tilde{u} = u + \eta$, where $\eta > 0$ is a constant. It is easy to show that $(v_k^p/\tilde{u}^{p-1}) \in W^{\alpha,p}_{0,\beta}(\Omega)$. By similar arguments as in the proof of Proposition 2.10 of [22], we can easily prove the following equation

$$\begin{aligned} 2\int_{\Omega} (-\Delta)^{\alpha}_{p,\beta} u \frac{v_k^p}{\tilde{u}^{p-1}} &= \left\langle (-\Delta)^{\alpha}_{p,\beta} u, \frac{v_k^p}{\tilde{u}^{p-1}} \right\rangle \\ &= \iint_{D_\Omega} |u(x)-u(y)|^{p-2}(u(x)-u(y)) \\ &\quad \cdot \left(\frac{v_k^p(x)}{\tilde{u}^{p-1}(x)} - \frac{v_k^p(y)}{\tilde{u}^{p-1}(y)} \right) d\nu \\ &\leq \iint_{D_\Omega} |v_k(x)-v_k(y)|^p d\nu \leq \|v\|^p_{W^{\alpha,p}_{0,\beta}(\Omega)}, \end{aligned} \tag{19}$$

where we use discrete Picone inequality. For more details, see Proposition 4.2 of [23]. Letting $k \longrightarrow +\infty$ and $\eta \longrightarrow 0$ in (19) leads to (18).

3. Proof of Main Theorem

Proof. Suppose that $f_1(x,s)s^{1-p}$ is decreasing with respect to s for almost every $x \in \Omega$. A slight change will be needed provided $f_2(x,s)s^{1-p}$ is decreasing with respect to s for almost every $x \in \Omega$ but no essential difference.

For fixed $\varepsilon > 0$ and $k \in \mathbb{N}$, define

$$\begin{aligned} &u_{1,\varepsilon}(x) = u_1(x) + \varepsilon, u_{2,\varepsilon}(x) = u_2(x) + \varepsilon, w_\varepsilon(x) = u^p_{1,\varepsilon}(x) - u^p_{2,\varepsilon}(x), \\ &\phi_1(x) = \frac{T_k(w^+_\varepsilon(x))}{u^{p-1}_{1,\varepsilon}(x)}, \phi_2(x) = \frac{T_k(w^+_\varepsilon(x))}{u^{p-1}_{2,\varepsilon}(x)}. \end{aligned} \tag{20}$$

where T_k is defined by (12) and $w^+_\varepsilon := \max\{w_\varepsilon, 0\}$ is the positive part of the function w_ε.

In the following proof, I show that $w^+_\varepsilon = 0$, which leads to $u_1(x) \leq u_2(x)$ almost everywhere in Ω.

Choosing ϕ_1 and ϕ_2 as test functions in equations of u_1 and u_2, respectively, we find

$$\left\langle (-\Delta)^{\alpha}_{p,\beta} u_1, \phi_1 \right\rangle = \int_{\Omega} f_1(x,u_1)\phi_1(x)dx, \tag{21}$$

$$\left\langle (-\Delta)^{\alpha}_{p,\beta} u_2, \phi_2 \right\rangle = \int_{\Omega} f_2(x,u_2)\phi_2(x)dx. \tag{22}$$

Subtracting the two equations (21) and (22), we obtain

$$\begin{aligned} &\left\langle (-\Delta)^{\alpha}_{p,\beta} u_1, \phi_1 \right\rangle - \left\langle (-\Delta)^{\alpha}_{p,\beta} u_2, \phi_2 \right\rangle \\ &\quad = \int_{\Omega} (f_1(x,u_1)\phi_1(x) - f_2(x,u_2)\phi_2(x))dx \\ &\quad = \int_{\Omega} \left(\frac{f_1(x,u_1)}{u^{p-1}_{1,\varepsilon}(x)} - \frac{f_2(x,u_2)}{u^{p-1}_{2,\varepsilon}(x)} \right) T_k(w^+_\varepsilon)dx. \end{aligned} \tag{23}$$

Decompose $\mathbb{R}^N$ as $\mathbb{R}^N = D_1 \cup D_2 \cup D_3$, where

$$\begin{aligned} D_1 &= \{x \in \mathbb{R}^N : w^+_\varepsilon(x) = 0\}, \\ D_2 &= \{x \in \mathbb{R}^N : 0 < w^+_\varepsilon(x) < k\}, \\ D_3 &= \{x \in \mathbb{R}^N : w^+_\varepsilon(x) \geq k\}. \end{aligned} \tag{24}$$

Therefore,

$$\begin{aligned} T_k(w^+_\varepsilon(x)) &= \begin{cases} 0, & x \in D_1, \\ w^+_\varepsilon, & x \in D_2, \\ k, & x \in D_3, \end{cases} \\ \mathbb{R}^N \times \mathbb{R}^N &= \bigcup_{i,j=1}^{i,j=3} D_{ij}, \end{aligned} \tag{25}$$

here $D_{ij} = \{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : x \in D_i, y \in D_j\}$.

By the definition of ϕ_i and D_{ij}, we get

$$\phi_i(x)-\phi_i(y)=\begin{cases} 0, \ x\in D_{11}, \\ -\dfrac{w_\varepsilon^+(y)}{u_{i,\varepsilon}^{p-1}(y)}, & x\in D_{12}, \\ -\dfrac{k}{u_{i,\varepsilon}^{p-1}(y)}, & x\in D_{13}, \\ \dfrac{w_\varepsilon^+(x)}{u_{i,\varepsilon}^{p-1}(x)}, & x\in D_{21}, \\ \dfrac{w_\varepsilon^+(x)}{u_{i,\varepsilon}^{p-1}(x)}-\dfrac{w_\varepsilon^+(y)}{u_{i,\varepsilon}^{p-1}(y)}, & x\in D_{22}, \\ \dfrac{w_\varepsilon^+(x)}{u_{i,\varepsilon}^{p-1}(x)}-\dfrac{k}{u_{i,\varepsilon}^{p-1}(y)}, & x\in D_{23}, \\ \dfrac{k}{u_{i,\varepsilon}^{p-1}(x)}, & x\in D_{31}, \\ \dfrac{k}{u_{i,\varepsilon}^{p-1}(x)}-\dfrac{w_\varepsilon^+(y)}{u_{i,\varepsilon}^{p-1}(y)}, & x\in D_{32}, \\ \dfrac{k}{u_{i,\varepsilon}^{p-1}(x)}-\dfrac{k}{u_{i,\varepsilon}^{p-1}(y)}, & x\in D_{33}. \end{cases} \tag{26}$$

Now, rewrite $\langle(-\Delta)_{p,\beta}^{\alpha}u_1,\phi_1\rangle-\langle(-\Delta)_{p,\beta}^{\alpha}u_2,\phi_2\rangle$ as

$$\begin{aligned}&\left\langle(-\Delta)_{p,\beta}^{\alpha}u_1,\phi_1\right\rangle-\left\langle(-\Delta)_{p,\beta}^{\alpha}u_2,\phi_2\right\rangle \\ &\quad=\iint_{\bigcup_{i,j=1}^{i,j=3}D_{ij}}|u_1(x)-u_1(y)|^{p-2}(u_1(x)-u_1(y))(\phi_1(x) \\ &\quad-\phi_1(y))d\nu-\iint_{\bigcup_{i,j=1}^{i,j=3}D_{ij}}|u_2(x)-u_2(y)|^{p-2}(u_2(x) \\ &\quad-u_2(y))(\phi_2(x)-\phi_2(y))d\nu=\sum_{i,j=1}^{i,j=3}\left(I_{ij}-J_{ij}\right),\end{aligned} \tag{27}$$

where

$$\begin{aligned}I_{ij}&=\iint_{D_{ij}}|u_1(x)-u_1(y)|^{p-2}(u_1(x)-u_1(y))(\phi_1(x)-\phi_1(y))d\nu,\\ J_{ij}&=\iint_{D_{ij}}|u_2(x)-u_2(y)|^{p-2}(u_2(x)-u_2(y))(\phi_2(x)-\phi_2(y))d\nu.\end{aligned} \tag{28}$$

For simplicity of notation, denote

$$A_i(u)=|u_i(x)-u_i(y)|^{p-2}(u_i(x)-u_i(y)),\ i=1,2. \tag{29}$$

Obviously,

$$I_{11}=J_{11}=0. \tag{30}$$

Now consider $I_{12}-J_{12}$. By (26), we get

$$\begin{aligned}I_{12}-J_{12}&=-\iint_{D_{12}}A_1(u)\frac{u_{1,\varepsilon}^p(y)-u_{2,\varepsilon}^p(y)}{u_{1,\varepsilon}^{p-1}(y)}d\nu \\ &\quad+\iint_{D_{12}}A_2(u)\frac{u_{1,\varepsilon}^p(y)-u_{2,\varepsilon}^p(y)}{u_{2,\varepsilon}^{p-1}(y)}d\nu \\ &=\iint_{D_{12}}\left[u_{1,\varepsilon}^p(y)-u_{2,\varepsilon}^p(y)\right]M_{12}d\nu,\end{aligned} \tag{31}$$

where

$$M_{12}=\frac{A_2(u)}{u_{2,\varepsilon}^{p-1}(y)}-\frac{A_1(u)}{u_{1,\varepsilon}^{p-1}(y)}. \tag{32}$$

Note that $u_1(x)\le u_2(x)$ and $u_1(y)\ge u_2(y)$ for $(x,y)\in D_{12}$. Thus, $u_1(x)-u_1(y)\le u_2(x)-u_2(y)$. By the monotonicity of the function $h(t)=|t|^{p-2}t$, we find $M_{12}\ge 0$, which implies that

$$I_{12}-J_{12}\ge 0. \tag{33}$$

For $I_{13}-J_{13}$, we have

$$I_{13}-J_{13}=k\iint_{D_{13}}M_{12}d\nu\ge 0, \tag{34}$$

since $M_{12}\ge 0$ for $(x,y)\in D_{13}$.
For $I_{21}-J_{21}$, we find

$$\begin{aligned}I_{21}-J_{21}&=\iint_{D_{21}}A_1(u)\frac{u_{1,\varepsilon}^p(x)-u_{2,\varepsilon}^p(x)}{u_{1,\varepsilon}^{p-1}(x)}d\nu \\ &\quad-\iint_{D_{21}}A_2(u)\frac{u_{1,\varepsilon}^p(x)-u_{2,\varepsilon}^p(x)}{u_{2,\varepsilon}^{p-1}(x)}d\nu \\ &=\iint_{D_{21}}\left[u_{1,\varepsilon}^p(x)-u_{2,\varepsilon}^p(x)\right]M_{21}d\nu,\end{aligned} \tag{35}$$

where

$$M_{21}=\frac{A_1(u)}{u_{1,\varepsilon}^{p-1}(x)}-\frac{A_2(u)}{u_{2,\varepsilon}^{p-1}(x)}. \tag{36}$$

Recalling that $u_1(x)-u_1(y)\ge u_2(x)-u_2(y)$ since $u_1(x)\ge u_2(x)$ and $u_1(y)\le u_2(y)$ for $(x,y)\in D_{21}$. This fact, together with $A_i(u)(x)/u_{i,\varepsilon}^{p-1}(x)<1$ for $i=1,2$, yields $M_{21}\ge 0$. Consequently

$$I_{21}-J_{21}\ge 0. \tag{37}$$

Now, we consider $I_{22} - J_{22}$.

$$\begin{aligned}
I_{22} - J_{22} &= \iint_{D_{22}} A_1(u)\left[\frac{w_\varepsilon(x)}{u_{1,\varepsilon}^{p-1}(x)} - \frac{w_\varepsilon(y)}{u_{1,\varepsilon}^{p-1}(y)}\right]dv \\
&\quad - \iint_{D_{22}} A_2(u)\left[\frac{w_\varepsilon(x)}{u_{1,\varepsilon}^{p-1}(x)} - \frac{w_\varepsilon(y)}{u_{1,\varepsilon}^{p-1}(y)}\right]dv \\
&= \iint_{D_{22}} A_1(u)\left[u_{1,\varepsilon}(x) - \frac{u_{2,\varepsilon}^p(x)}{u_{1,\varepsilon}^{p-1}(x)} - u_{1,\varepsilon}(y) + \frac{u_{2,\varepsilon}^p(y)}{u_{1,\varepsilon}^{p-1}(y)}\right]dv \\
&\quad + \iint_{D_{22}} A_2(u)\left[u_{2,\varepsilon}(x) - \frac{u_{1,\varepsilon}^p(x)}{u_{2,\varepsilon}^{p-1}(x)} - u_{2,\varepsilon}(y) + \frac{u_{1,\varepsilon}^p(y)}{u_{2,\varepsilon}^{p-1}(y)}\right]dv \\
&= \|u_{1,\varepsilon}\|_{W_\beta^{s,p}}^p - \left\langle \frac{u_{2,\varepsilon}^p(x)}{u_{1,\varepsilon}^{p-1}(x)}, (-\Delta)_{p,\beta}^\alpha u_{1,\varepsilon}\right\rangle + \|u_{2,\varepsilon}\|_{W_\beta^{s,p}}^p \\
&\quad - \left\langle \frac{u_{1,\varepsilon}^p(x)}{u_{2,\varepsilon}^{p-1}(x)}, (-\Delta)_{p,\beta}^\alpha u_{2,\varepsilon}\right\rangle \ge 0,
\end{aligned} \tag{38}$$

here we use the fractional Picone inequality; see Lemma 5.

For $I_{23} - J_{23}$, we have

$$\begin{aligned}
I_{23} - J_{23} &= \iint_{D_{23}} A_1(u)\left[\frac{u_{1,\varepsilon}^p(x) - u_{2,\varepsilon}^p(x)}{u_{1,\varepsilon}^{p-1}(x)} - \frac{k}{u_{1,\varepsilon}^{p-1}(y)}\right]dv \\
&\quad - \iint_{D_{23}} A_2(u)\left[\frac{u_{1,\varepsilon}^p(x) - u_{2,\varepsilon}^p(x)}{u_{2,\varepsilon}^{p-1}(x)} - \frac{k}{u_{2,\varepsilon}^{p-1}(y)}\right]dv \\
&= \iint_{D_{23}} A_1(u)\left[\frac{u_{1,\varepsilon}^p(x) - u_{2,\varepsilon}^p(x)}{u_{1,\varepsilon}^{p-1}(x)} - \frac{k}{u_{1,\varepsilon}^{p-1}(y)}\right]dv \\
&\quad + \iint_{D_{23}} A_2(u)\left[\frac{u_{2,\varepsilon}^p(x) - u_{1,\varepsilon}^p(x)}{u_{2,\varepsilon}^{p-1}(x)} + \frac{k}{u_{2,\varepsilon}^{p-1}(y)}\right]dv.
\end{aligned} \tag{39}$$

Now, we consider the first term of the right-hand side of (39). Suppose that $u_1(x) - u_1(y) \ge 0$,

$$\begin{aligned}
&\iint_{D_{23}} A_1(u)\left[\frac{u_{1,\varepsilon}^p(x) - u_{2,\varepsilon}^p(x)}{u_{1,\varepsilon}^{p-1}(x)} - \frac{k}{u_{1,\varepsilon}^{p-1}(y)}\right]dv \\
&\quad = \iint_{D_{23}} A_1(u)\left[\frac{u_{1,\varepsilon}^p(x) - u_{1,\varepsilon}^p(y)}{u_{1,\varepsilon}^{p-1}(x)} + \frac{u_{1,\varepsilon}^p(y) - u_{2,\varepsilon}^p(x)}{u_{1,\varepsilon}^{p-1}(x)} - \frac{k}{u_{1,\varepsilon}^{p-1}(y)}\right]dv \\
&\quad \ge \iint_{D_{23}} A_1(u)\left[\frac{u_{1,\varepsilon}^p(y) - u_{2,\varepsilon}^p(x)}{u_{1,\varepsilon}^{p-1}(x)} - \frac{k}{u_{1,\varepsilon}^{p-1}(y)}\right]dv \ge \|u_{1,\varepsilon}\|_{W_\beta^{s,p}(D_{23})} \ge 0.
\end{aligned} \tag{40}$$

Suppose that $u_1(x) - u_1(y) \le 0$, the above inequation holds also since $k \le u_{1,\varepsilon}^{p-1}(y)$ for $(x, y) \in D_{23}$.

For the second term of the right-hand side of (39), by a similar argument, we get

$$\begin{aligned}
&\iint_{D_{23}} A_2(u)\left[\frac{u_{2,\varepsilon}^p(x) - u_{1,\varepsilon}^p(x)}{u_{2,\varepsilon}^{p-1}(x)} + \frac{k}{u_{2,\varepsilon}^{p-1}(y)}\right]dv \\
&\quad = \iint_{D_{23}} A_2(u)\left[\frac{u_{2,\varepsilon}^p(x) - u_{2,\varepsilon}^p(y)}{u_{2,\varepsilon}^{p-1}(x)} + \frac{u_{2,\varepsilon}^p(y) - u_{1,\varepsilon}^p(x)}{u_{2,\varepsilon}^{p-1}(x)} + \frac{k}{u_{2,\varepsilon}^{p-1}(y)}\right]dv \\
&\quad \ge \iint_{D_{23}} A_2(u)\left[\frac{u_{2,\varepsilon}^p(y) - u_{1,\varepsilon}^p(x)}{u_{2,\varepsilon}^{p-1}(x)} + \frac{k}{u_{2,\varepsilon}^{p-1}(y)}\right]dv \ge \|u_{2,\varepsilon}\|_{W_\beta^{s,p}(D_{23})} \ge 0.
\end{aligned} \tag{41}$$

Thus, taking into account (40) and (41), we derive that

$$I_{23} - J_{23} \ge 0. \tag{42}$$

For $I_{31} - J_{31}$, it is easily seen that

$$\begin{aligned}
I_{31} - J_{31} &= k\iint_{D_{21}} \frac{|u_1(x) - u_1(y)|^{p-2}(u_1(x) - u_1(y))}{u_{1,\varepsilon}^{p-1}(x)}dv \\
&\quad - k\iint_{D_{21}} \frac{|u_2(x) - u_2(y)|^{p-2}(u_2(x) - u_2(y))}{u_{1,\varepsilon}^{p-1}(x)}dv \ge 0.
\end{aligned} \tag{43}$$

For $I_{32} - J_{32}$, repeating the previous argument of $I_{23} - J_{23}$ leads to

$$\begin{aligned}
I_{32} - J_{32} &= \iint_{D_{32}} A_1(u)\left[\frac{k}{u_{1,\varepsilon}^{p-1}(x)} - \frac{u_{1,\varepsilon}^p(y) - u_{2,\varepsilon}^p(y)}{u_{1,\varepsilon}^{p-1}(y)}\right]dv \\
&\quad - \iint_{D_{32}} A_2(u)\left[\frac{k}{u_{2,\varepsilon}^{p-1}(x)} - \frac{u_{1,\varepsilon}^p(y) - u_{2,\varepsilon}^p(y)}{u_{2,\varepsilon}^{p-1}(y)}\right]dv \ge 0.
\end{aligned} \tag{44}$$

For $I_{33} - J_{33}$, we have

$$\begin{aligned}
I_{33} - J_{33} &= \iint_{D_{33}} A_1(u)\left[\frac{k}{u_{1,\varepsilon}^{p-1}(x)} - \frac{k}{u_{1,\varepsilon}^{p-1}(y)}\right]dv \\
&\quad - \iint_{D_{33}} A_2(u)\left[\frac{k}{u_{2,\varepsilon}^{p-1}(x)} - \frac{k}{u_{2,\varepsilon}^{p-1}(y)}\right]dv \\
&= \left\langle (-\Delta)_{p,\beta}^\alpha u_{1,\varepsilon}, \frac{k}{u_{1,\varepsilon}^{p-1}}\right\rangle_{D_{33}} - \left\langle (-\Delta)_{p,\beta}^\alpha u_{2,\varepsilon}, \frac{k}{u_{2,\varepsilon}^{p-1}}\right\rangle_{D_{33}} \\
&= \left(f_1(x, u_1), \frac{k}{u_{1,\varepsilon}^{p-1}}\right)_{\mathscr{D}} - \left(f_2(x, u_2), \frac{k}{u_{2,\varepsilon}^{p-1}}\right)_{\mathscr{D}} \\
&= k\int_{\mathscr{D}} \left(\frac{f_1(x, u_1)}{u_{1,\varepsilon}^{p-1}} - \frac{f_2(x, u_2)}{u_{2,\varepsilon}^{p-1}}\right),
\end{aligned} \tag{45}$$

where $\mathscr{D}$ satisfies $D_{33} = \mathbb{R}^N \setminus \mathscr{D}^c \times \mathscr{D}^c$.

For the right-hand side of (23), according to (10), we have

$$\begin{aligned} &\int_\Omega \left(\frac{f_1(x,u_1)}{u_{1,\varepsilon}^{p-1}} - \frac{f_2(x,u_2)}{u_{2,\varepsilon}^{p-1}} \right) T_k(w_\varepsilon^+) dx \\ &\quad \le \int_\Omega \left(\frac{f_1(x,u_1)}{u_{1,\varepsilon}^{p-1}} - \frac{f_1(x,u_2)}{u_{2,\varepsilon}^{p-1}} \right) T_k(w_\varepsilon^+) dx. \end{aligned} \tag{46}$$

Using the monotonicity of $f_1(x,s)s^{1-p}$, we know that

$$\frac{f_1(x,u_1)}{u_1^{p-1}} \le \frac{f_1(x,u_2)}{u_2^{p-1}}, \tag{47}$$

since $u_1(x) \ge u_2(x)$ for $x \in \mathscr{D}$, where $\mathscr{D}$ appears in (45). Thus, taking into account (45)–(47), we obtain, for small enough $\varepsilon > 0$,

$$\begin{aligned} &\int_\Omega \left(\frac{f_1(x,u_1)}{u_{1,\varepsilon}^{p-1}} - \frac{f_2(x,u_2)}{u_{2,\varepsilon}^{p-1}} \right) T_k(w_\varepsilon^+) dx - (I_{33} - J_{33}) \\ &\quad = \int_{\Omega\setminus\mathscr{D}} \left(\frac{f_1(x,u_1)}{u_{1,\varepsilon}^{p-1}} - \frac{f_2(x,u_2)}{u_{2,\varepsilon}^{p-1}} \right) w_\varepsilon^+ dx \le 0. \end{aligned} \tag{48}$$

This fact, together with (23), (30), (33), (34), (37), (38), (42)–(44), leads to

$$0 \le \int_{\Omega\setminus\mathscr{D}} \left(\frac{f_1(x,u_1)}{u_{1,\varepsilon}^{p-1}} - \frac{f_2(x,u_2)}{u_{2,\varepsilon}^{p-1}} \right) w_\varepsilon^+ dx \le 0, \tag{49}$$

which implies that $w_\varepsilon^+ = 0$ for $x \in \Omega \setminus \mathscr{D}$, that is $u_1(x) \ge u_2(x)$ for any $x \in \Omega$. This completes the proof of Theorem 1.

The proof of Corollary 2 is immediate, which is omitted.

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare no conflict of interest.

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