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*Research Article*

# Optimal Linear Estimators for Time-Delay Systems with Fading Measurements and Correlated Noises

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The optimal linear estimation problems are investigated in this paper for a class of discrete linear systems with fading measurements and correlated noises. Firstly, the fading measurements occur in a random way where the fading probabilities are regulated by probability mass functions in a given interval. Furthermore, time-delay exists in the system state and observation simultaneously. Additionally, the multiplicative noises are considered to describe the uncertainty of the state. Based on the projection theory, the linear minimum variance optimal linear estimators, including filter, predictor, and smoother are presented in the paper. Compared with conventional state augmentation, the new algorithm is finite-dimensionally computable and does not increase computational and storage load when the delay is large. A numerical example is provided to illustrate the effectiveness of the proposed algorithms.

## 1. Introduction

NCSs have received significant attention for their successful applications in space exploration, target tracking, remote surgery, unmanned aerial vehicles, industrial monitoring, and other areas in recent years [1–12]. As is well known, network-induced phenomena, such as communication delays, fading measurements or packet dropouts, quantization effects, and sensor saturations, are unavoidable in data transmission of practical networked systems due mainly to the sudden environment changes, intermittent transmission congestions, random failures, and repairs of components [13]. Hence, the data received by the estimator may not be real-time ones, which leads to the traditional estimation algorithms being no longer applicable.

Fading measurements are important issues in NCSs. The phenomenon of packet dropouts in the network can be seen as a special case of fading measurement. Fortunately, many efficient approaches have been developed for the systems with fading measurements [14–17]. It is considered in [14] that a sensor network where single or multiple sensors amplify and forward their measurements of a common linear dynamic system to a remote fusion center via noisy fading wireless channels and shows that the expected error covariance (with respect to the fading process) of the time-varying Kalman filter is bounded and converges to a steady state value. Yan et al. [15] concentrated on the $H_\infty$ state estimator's design problems for a kind of discrete-time artificial neural networks (ANNs) with multiple fading measurements. The phenomenon of multiple fading measurements is represented by a set of individual stochastic variables obeying a predetermined distribution on interval [0,1]. In [16], a modified stochastic fading model with disturbance-dependent Gaussian noise is put forward to better reflect the fading phenomena in complex wireless communication networks. By introducing a novel concept of finite-time stochastic exponential dissipative, a state-feedback controller is designed. For a class of nonlinear systems with stochastic nonlinearities and multiple fading measurements, the stochastic nonlinearities are represented by statistical means which indicates multiplicative stochastic disturbances, and sufficient conditions are obtained to ensure stochastic stability of the modified unscented Kalman filter [17].

However, in practical applications, considering that the dynamic system is a discretized version of a continuous dynamic system with noise and the state of the dynamic system is observed by some sensors in a time-correlated noisy environment, such as during noise jamming generated by some target, the noises may be correlated and even finite-step correlated [18]. At present, many experts and scholars have adopted different algorithms to estimate the state of the systems with correlated noises. Sun et al. proposed some filtering algorithms for systems with fading measurements and correlated noises [7, 19]. Sun et al. [7] consider that different sensor channels have different fading measurement rates, and the process and measurement noises are finite-step autocorrelated and/or cross correlated with each other. In such complex systems, the optimal linear state estimators in the linear minimum variance (LMV) sense are presented by using the innovation analysis approach. Liu concentrated on the problems of state estimation for discrete-time linear systems with fading measurements and time-correlated channel noise [20–22]. The fading measurement appears in a random way, and the fading phenomenon for each sensor is described by an individual random variable taking a value in a given interval [20]. Furthermore, some results have been reported on the Kalman filtering problems of systems with uncertain correlated noise [23–26]. By introducing the fictitious noises to compensate the stochastic uncertainties, the system under consideration can be converted into one with only uncertain noise variances [23, 24]. However, in all these papers, the results focus on finding the optimal estimators, under which the state delay and observation delay are not considered simultaneously. Moreover, a few phenomena of imperfect transmission including the fading measurement and the time delay could be easily incorporated, and the optimal estimation problems for linear uncertain systems with single delayed measurement have not taken fading measurements into account [25, 26].

Based on the discussions above, we aim to solve the optimal linear estimation problems for a class of state delay and observation delay systems with fading measurements and correlated noises. In this paper, the aforementioned problems are considered fully. The probability mass functions in a giving interval are used to describe a discrete random variable, and the mean and covariance of the variable depend on the distribution law of each probability mass function. Based on the minimum mean square error (MMSE) estimation principle, we present the optimal linear state estimators, including filter, predictor, and smoother by using the projection theory [27]. Compared with conventional state augmentation, the new algorithm is finite-dimensionally computable and does not increase computational and storage load with time. Hence, the proposed algorithm is suitable for real-time applications.

The rest of this work is organized as follows. Section 2 formulates the problems for a class of time-delay systems with fading measurements and correlated noises and states the assumptions under which we prove the results. The preliminary lemmas of this work are derived in Section 3. In Section 4, the optimal linear estimators including filter, predictor, and smoother are designed. A numerical example is given in Section 5, which is followed by some conclusions in Section 6.

## 2. Problem Formulation

Consider the state delay and observation delay systems with fading measurements and correlated noises as follows:

$$x(t+1)=\left[A+\sum_{i=1}^{h}A_i\zeta_i(t)\right]x(t-d)+\Gamma w(t), \tag{1}$$

$$y(t)=\gamma(t)Hx(t-d)+v(t), \tag{2}$$

where $t$ is the discrete time, $x(t)\in R^n$ is the state, $y(t)\in R^{\mathrm{m}}$ is the measurement received by the sensors, $w(t)\in R^{\mathrm{p}}$ is the process noise, $v(t)\in R^{\mathrm{m}}$ is the measurement noise, $d$ is constant time delay, $\zeta_i(t), i=1,2,\ldots,h$ is the multiplicative noise, and $A$, $A_i$, $H$, and $\Gamma$ are known constant matrices with appropriate dimensions.

We now have four assumptions upon the initial values, statistical characteristic of system noise and random fading variable $\gamma(t)$.

*Assumption 1.* The process and measurement noise $w(t)$ and $v(t)$ are cross-correlated with zero mean and

$$\mathrm{E}\left\{\begin{bmatrix} w(t)\\ v(t)\end{bmatrix}\begin{bmatrix} w^T(j) & v^T(j)\end{bmatrix}\right\}=\begin{bmatrix} Q_w & S\\ S^T & Q_v\end{bmatrix}\delta_{tj}. \tag{3}$$

*Assumption 2.* $\zeta_i(t)$ is white noise with zero mean and variance $Q_{\zeta_i}(t)$ independent of $\zeta_j(t)$, $j\neq i$. $\zeta_i(t)$ is also uncorrelated with other noise signals.

*Assumption 3.* $\gamma(t)\in R$ is the random fading variable with mean $\overline{\gamma}(t)$ and variance $\sigma^2(t)$, and the fading probabilities are regulated by the probability mass functions in a given interval $[\alpha,\beta]$, $0\le\alpha\le\beta\le 1$. $\gamma(t)$ is uncorrelated with other noise signals.

*Assumption 4.* The initial state $x(0)$ is uncorrelated with $w(t)$, $v(t)$, $\zeta_i(t)$, $i=1,2,\ldots,h$ and $\mathrm{E}[x(0)]=\overline{x}(0)$, $\mathrm{E}[(x(0)-\overline{x}(0))(x(0)-\overline{x}(0))^T]=P(0)$.

Our aim is to find the optimal linear state estimators $\widehat{x}(j\mid t)$ based on the measurements $L(y(t),y(t-1),\ldots,y(0))$ for $j=t$, $j>t$, and $j<t$, which is called state filter, predictor, and smoother, respectively. Here, we will design the estimators that depend on the attenuation rate $\gamma(t)$ based on the received measurements.

## 3. Preliminary Lemmas

Firstly, system (1) can be converted to

$$x(t+1)=\left[A+\sum_{i=1}^{h}A_i\zeta_i(t)\right]x(t-d)+\Gamma w(t)+J[y(t)-(\gamma(t)-\overline{\gamma}(t))Hx(t-d)-\overline{\gamma}(t)Hx(t-d)-v(t)], \tag{4}$$

where $J$ is the pending matrix.

From (4), system (1) can be rewritten as follows:

$$x(t+1) = \overline{A}x(t-d) + \Omega(t) + Jy(t), \quad (5)$$

where

$$\Omega(t) = \sum_{i=1}^{h} A_i\zeta_i(t)x(t-d) - J(\gamma(t)-\overline{\gamma}(t))Hx(t-d),$$
$$\overline{A} = A - J\overline{\gamma}(t)H,$$
$$\overline{w}(t) = \Gamma w(t) - Jv(t). \quad (6)$$

Then, it holds that $\mathrm{E}[\Omega(t)] = 0$ in view of Assumptions 1 and 2, where $\overline{w}(t)$ is zero-mean white noise and satisfies

$$\mathrm{E}\left[\overline{w}(t)v^T(j)\right] = [\Gamma S - JQ_v]\delta_{tj}. \quad (7)$$

From (7), we can see $\overline{w}(t)$ is uncorrelated with $v(t)$ when $J = \Gamma S Q_v^{-1}$, then we have

$$\mathrm{E}\left[\overline{w}(t)v^T(j)\right] = 0. \quad (8)$$

The variance of $\overline{w}(t)$ can be computed by

$$Q_{\overline{w}} = \Gamma\left[Q_w - SQ_v^{-1}S^T\right]\Gamma^T. \quad (9)$$

Defining the state expectation $q(t) = \mathrm{E}[x(t)]$ and the state second moment $\Lambda(t) = \mathrm{E}[x(t)x^T(t)]$, we have

$$\Lambda(t+1) = \overline{A}\Lambda(t-d)\overline{A}^T + Jy(t)y^T(t)J^T + \sum_{i=1}^{h} Q_{\zeta_i}A_i\Lambda(t-d)A_i^T + Q_{\overline{w}} + J\sigma^2(t)H\Lambda(t-d)H^TJ^T, \quad (10)$$

where the initial value $\Lambda(0) = x(0)\overline{x}^T(0) + P(0)$.

The variance of $\Omega(t)$ can be computed by

$$\sum(t) = \sum_{i=1}^{h} Q_{\zeta_i}A_i\Lambda(t-d)A_i^T + Q_{\overline{w}} + J\sigma^2(t)H\Lambda(t-d)H^TJ^T. \quad (11)$$

The estimation error covariance matrix at different times is given by

$$\Phi_t(i,j) = \mathrm{E}\left[\tilde{x}(i \mid t)\tilde{x}^T(j \mid t)\right]. \quad (12)$$

Before giving the main results of optimal linear estimators, some lemmas are presented firstly.

**Lemma 1.** *For systems (2) and (5), the estimate $\hat{x}(t-d \mid t-i), i = 0, 1, \ldots, d-1$ can be computed according to the following equations:*

$$\hat{x}(t-d \mid t-i) = \hat{x}(t-d \mid t-i-1) + K(t-d \mid t-i)\varepsilon(t-i), \quad (13)$$

$$\varepsilon(t-i) = y(t-i) - \overline{\gamma}(t)H\hat{x}(t-d-i \mid t-i-1), \quad (14)$$

$$Q_\varepsilon(t-i) = \sigma^2(t)H\Lambda(t-d-i)H^T + Q_v + \overline{\gamma}^2(t)HP(t-d-i \mid t-i-1)H^T, \quad (15)$$

$$K(t-d \mid t-i) = \overline{\gamma}(t)\Phi_{t-i-1}(t-d, t-d-i)H^TQ_\varepsilon^{-1}(t-i), \quad (16)$$

$$P(t-d \mid t-i) = P(t-d \mid t-i-1) - \overline{\gamma}(t)K(t-d \mid t-i) \times H\Phi_{t-i-1}(t-d-i, t-d), \quad (17)$$

*where the innovation $\varepsilon(t) = y(t) - \hat{y}(t \mid t-1)$ and its covariance matrix $Q_\varepsilon(t) = \mathrm{E}[\varepsilon(t)\varepsilon^T(t)]$. $\Lambda(t-d-i)$ can be obtained from (10).*

*Proof.* According to the projection theory, we can easily get (13).

The gain matrix $K(t-d \mid t-i)$ is defined by

$$K(t-d \mid t-i) = \mathrm{E}\left[x(t-d)\varepsilon^T(t-i)\right]Q_\varepsilon^{-1}(t-i). \quad (18)$$

From (2), we have

$$y(t-i) = \gamma(t-i)Hx(t-d-i) + v(t-i). \quad (19)$$

Taking projection on both sides of (19) yields

$$\hat{y}(t-i \mid t-i-1) = \overline{\gamma}(t)H\hat{x}(t-d-i \mid t-i-1) + \hat{v}(t-i \mid t-i-1), \quad (20)$$

where $\hat{v}(t-i \mid t-i-1) = 0$.

Substituting (20) in the definition of innovation, we obtain (14).

From (2) and (14), we have

$$\varepsilon(t-i) = (\gamma(t-i) - \overline{\gamma}(t))Hx(t-d-i) + v(t-i) + \overline{\gamma}(t)H\tilde{x}(t-d-i \mid t-i-1). \quad (21)$$

Noting $\mathrm{E}[(\gamma(t)-\overline{\gamma}(t))^2] = \sigma^2(t)$, $\mathrm{E}[\gamma(t)-\overline{\gamma}(t)] = 0$, and $v(t-i) \perp \tilde{x}(t-d-i \mid t-i-1)$, we can easily obtain (13).

From Assumptions 1 and 2 and noting $\hat{x}(t-d \mid t-i-1) \perp \tilde{x}(t-d-i \mid t-i-1)$, we have

$$\mathrm{E}\left[x(t-d)\varepsilon^T(t-i)\right] = \overline{\gamma}(t)\Phi_{t-i-1}(t-d, t-d-i)H^T. \quad (22)$$

Substituting (22) in (18), we obtain (16).

According to the definition of the covariance matrix and noting $v(t-i) \perp \tilde{x}(t-d-i \mid t-i-1)$, we have

$$\begin{aligned} P(t-d \mid t-i) &= \mathrm{E}\left[\tilde{x}(t-d \mid t-i)\tilde{x}^T(t-d \mid t-i)\right] \\ &= P(t-d \mid t-i-1) - \overline{\gamma}(t)\Phi_{t-i-1} \\ &\quad \times (t-d, t-d-i)H^TK^T(t-d \mid t-i) \\ &\quad - \overline{\gamma}(t)K(t-d \mid t-i)H\Phi_{t-i-1}(t-d-i, t-d) \\ &\quad + \overline{\gamma}^2(t)K(t-d \mid t-i)HP(t-d-i \mid t-i-1) \\ &\quad \times H^TK^T(t-d \mid t-i) + \sigma^2(t)K(t-d \mid t-i) \\ &\quad \times HP(t-d-i \mid t-i-1)H^TK^T(t-d \mid t-i) \\ &\quad + K(t-d \mid t-i)Q_vK^T(t-d \mid t-i). \end{aligned} \quad (23)$$

From (16), we have

$$\overline{\gamma}(t)\Phi_{t-i-1}(t-d,t-d-i)H^T = K(t-d \mid t-i)Q_\varepsilon(t-i). \tag{24}$$

Substituting (24) in (23), we have (17).

**Lemma 2.** *For systems (2) and (5) under the precondition of Lemma 1, the estimation error covariance matrix* $\Phi_{t-i-j}(t-d-i+1,t-d+1)$ *of the state is calculated by*

$$\begin{aligned}\Phi_{t-i-j}&(t-d-i+1,t-d+1)\\ &= \Phi_{t-i-j-1}(t-d-i+1,t-d+1) - \overline{\gamma}(t)K(t-d-i+1 \mid t-i-j)\\ &\quad\times H\Phi_{t-j-i-1}(t-d-i-j,t-d+1),\end{aligned} \tag{25}$$

*where* $i = 1,2,\ldots,d-1$, $j = 1,2,\ldots,d-1-i$.

*Proof.* From Lemma 1, we can easily get

$$\begin{aligned}\Phi_{t-i-j}&(t-d-i+1,t-d+1)\\ &= \mathrm{E}\{[\tilde{x}(t-d-i+1 \mid t-i-j-1) - K(t-d-i+1 \mid t-i-j)\\ &\quad\times \varepsilon(t-i-j)][\tilde{x}(t-d+1 \mid t-i-j-1) - K(t-d+1 \mid t-i-j)\\ &\quad\times \varepsilon(t-i-j)]^T\},\end{aligned} \tag{26}$$

where the innovation $\varepsilon(t-i-j)$ can be calculated by

$$\begin{aligned}\varepsilon(t-i-j) &= (\gamma(t-i-j) - \overline{\gamma}(t))Hx(t-d-i-j)\\ &\quad + v(t-i-j) + \overline{\gamma}(t)H\tilde{x}(t-d-i-j \mid t-i-j-1).\end{aligned} \tag{27}$$

Substituting (27) in (26), we get

$$\begin{aligned}\Phi_{t-i-j}&(t-d-i+1,t-d+1)\\ &= \Phi_{t-i-j-1}(t-d-i+1,t-d+1) - \overline{\gamma}(t)\\ &\quad\times \Phi_{t-i-j-1}(t-d-i+1,t-d-i-j)H^T\\ &\quad\times K^T(t-d+1 \mid t-i-j)\\ &\quad- \overline{\gamma}(t)K(t-d-i+1 \mid t-i-j)H\\ &\quad\times \Phi_{t-i-j-1}(t-d-i-j,t-d+1)\\ &\quad+ K(t-d-i+1 \mid t-i-j)\\ &\quad\times Q_\varepsilon(t-i-j)K^T(t-d+1 \mid t-i-j),\end{aligned} \tag{28}$$

where the gain matrix $K(t-d \mid t-i)$ is calculated by

$$\begin{aligned}K&(t-d-i+1 \mid t-i-j)\\ &= \mathrm{E}\left[x(t-d-i+1)\varepsilon^T(t-i-j)\right]Q_\varepsilon^{-1}(t-i-j)\\ &= \overline{\gamma}(t)\Phi_{t-i-j-1}(t-d-i+1,t-d-i-j)H^TQ_\varepsilon^{-1}(t-i-j).\end{aligned} \tag{29}$$

Substituting (29) in (28), we obtain (25).

**Lemma 3.** *For the systems (2) and (5), the estimation error covariance matrix* $\Phi_{t-d+1}(t-d-i+1,t-d+1)$ *of the state is calculated by*

$$\begin{aligned}\Phi_{t-d+1}&(t-d-i+1,t-d+1)\\ &= \{[\Psi_{d-i+1}(t-i)]^T - \overline{\gamma}(t)K(t-d-i \mid t-d)H\\ &\quad\times P(t-2d|t-d-1)\}\overline{A}^T\\ &\quad- \overline{\gamma}(t)K(t-d-i+1 \mid t-d+1)H[\Psi_d(t)]^T,\end{aligned} \tag{30}$$

*where* $i = 1,2,\ldots,d-1$.

*Proof.* Similar to Lemma 2, we can easily derive

$$\begin{aligned}\Phi_{t-d+1}&(t-d-i,t-d+1)\\ &= \Phi_{t-d}(t-d-i+1,t-d+1) - \overline{\gamma}(t)\\ &\quad\times K(t-d-i+1 \mid t-d+1)H\\ &\quad\times \Phi_{t-d}(t-2d+1,t-d+1).\end{aligned} \tag{31}$$

From (5), we have $\tilde{x}(t-d+1 \mid t-d) = \overline{A}\tilde{x}(t-2d|t-d) + \Omega(t-d)$, and noting $\Omega(t-d) \perp \tilde{x}(t-d-i+1 \mid t-d)$, we can easily get

$$\Phi_{t-d}(t-d+1,t-d-i+1) = \overline{A}\Phi_{t-d}(t-2d,t-d-i+1). \tag{32}$$

Noting $\Psi_i(t) = \Phi_{t-i}(t-d-i+1,t-d+1)$, we have

$$\Phi_{t-d-1}(t-2d,t-d-i+1) = \Psi_{d-i+1}(t-i). \tag{33}$$

Let $j = d-i$ and similar to (25), (32) can be rewitten by

$$\begin{aligned}\Phi_{t-d}&(t-d-i+1,t-d+1)\\ &= [\Phi_{t-d-1}(t-d-i+1,t-2d) - \overline{\gamma}(t)\\ &\quad\times K(t-d-i+1 \mid t-d)HP(t-2d|t-d-1)]\overline{A}^T.\end{aligned} \tag{34}$$

Substituting (34) and (33) in (31), we have (30).

## 4. Optimal Linear Estimators

In this section, we obtain the main results on optimal filter, predictor, smoother, and corresponding estimation error covariance matrices for the system under consideration in the sense of linear MMSE. At the end of this section, the realization steps of the proposed algorithm are explained.

### *4.1. Optimal Linear Filter*

**Theorem 1.** *For systems (2) and (5) under Assumptions 1–4, the optimal linear filter is given by*

$$\begin{aligned}\hat{x}(t+1|t+1) &= \hat{x}(t+1|t) + K(t+1)\\ &\quad\times(y(t+1) - \overline{\gamma}(t)H\hat{x}(t-d+1 \mid t)),\end{aligned} \tag{35}$$

*where the gain matrix* $K(t+1)$ *and the covariance matrix* $Q_\varepsilon(t+1)$ *are calculated, respectively, by*

$$K(t+1) = \overline{\gamma}(t)\Psi_d^T(t+d)H^T Q_\varepsilon^{-1}(t+1), \quad (36)$$

$$\begin{aligned} Q_\varepsilon(t+1) &= \sigma^2(t)H\Lambda(t-d+1)H^T + Q_v \\ &\quad + \overline{\gamma}^2(t)HP(t-d+1 \mid t)H^T. \end{aligned} \quad (37)$$

The covariance matrix $P(t+1|t+1)$ of the state filter and $\Psi_d(t+d)$ are calculated, respectively, by

$$P(t+1|t+1) = P(t+1|t) - \overline{\gamma}(t)K(t+1)H\Psi_d(t+d), \quad (38)$$

$$\Psi_d^T(t+d) = \overline{A}[\Psi_1(t) - \overline{\gamma}(t)K(t-d \mid t)H\Psi_1(t)]. \quad (39)$$

*Proof.* According to the projection theory and (2), we can easily get

$$\widehat{x}(t+1|t+1) = \widehat{x}(t+1|t) + K(t+1)\varepsilon(t+1), \quad (40)$$

$$y(t+1) = \gamma(t+1)Hx(t-d+1) + v(t+1), \quad (41)$$

where $K(t+1) = K(t+1|t+1)$.

The gain matrix $K(t+1)$ of the state filter is calculated by

$$K(t+1) = \mathrm{E}\left[x(t+1)\varepsilon^T(t+1)\right]Q_\varepsilon^{-1}(t+1), \quad (42)$$

where the innovation $\varepsilon(t+1)$ can be calculated by

$$\begin{aligned} \varepsilon(t+1) &= [\gamma(t+1) - \overline{\gamma}(t)]Hx(t-d+1) + v(t+1) \\ &\quad + \overline{\gamma}(t)H\widetilde{x}(t-d+1 \mid t). \end{aligned} \quad (43)$$

Substituting (42) and (43) in (40), we obtain (35).

From $\Psi_i(t) = \Phi_{t-i}(t-d-i+1, t-d+1)$, we obtain (36). According to the definition of the covariance matrix, we can easily derive (37).

Noting

$$\widetilde{x}(t+1|t+1) = \widetilde{x}(t+1|t) - K(t+1)\varepsilon(t+1). \quad (44)$$

The covariance matrix $P(t+1|t+1)$ of the state filter is calculated by

$$\begin{aligned} P(t+1|t+1) &= P(t+1|t) - \overline{\gamma}^2(t)\Phi_t(t+1, t-d+1)H^T \\ &\quad \times Q_\varepsilon^{-1}(t+1)H\Phi_t^T(t+1, t-d+1). \end{aligned} \quad (45)$$

Let $i = 0$ and noting $\Phi_{t-1}(t-d, t-d) = P(t-d \mid t-1)$. Then, (16) can be rewritten by

$$K(t-d \mid t) = \overline{\gamma}(t)P(t-d \mid t-1)H^T Q_\varepsilon^{-1}(t). \quad (46)$$

Noting $\Psi_d(t+d) = \Phi_t(t-d+1, t+1)$, from (36) and transforming (45), we have (38).

From (5), we can easily derive

$$\widetilde{x}(t+1|t) = \overline{A}\widetilde{x}(t-d \mid t) + \Omega(t). \quad (47)$$

From (47) and noting $\Omega(t) \perp \widetilde{x}(t-d+1 \mid t)$, we obtain

$$\Phi_t(t+1, t-d+1) = \overline{A}\Phi_t(t-d, t-d+1). \quad (48)$$

Let $i = 1$, $j = -1$. Then, (25) can be rewritten by

$$\begin{aligned} \Phi_t(t-d, t-d+1) &= \Phi_{t-1}(t-d, t-d+1) \\ &\quad - \overline{\gamma}(t)K(t-d \mid t)H\Phi_{t-1}(t-d, t-d+1). \end{aligned} \quad (49)$$

Noting $\Psi_i(t) = \Phi_{t-i}(t-d-i+1, t-d+1)$ and substituing (49) in (48), we get (39).

*4.2. Optimal Linear Predictor*

**Theorem 2.** *For systems (2) and (5) under Assumptions 1–4, the optimal linear predictor is given by*

$$\begin{aligned} \widehat{x}(t+1|t) &= \overline{A}[\widehat{x}(t-d \mid t-1) + K(t-d \mid t) \\ &\quad \times (y(t) - \overline{\gamma}(t)H\widehat{x}(t-d \mid t-1)] + Jy(t), \end{aligned} \quad (50)$$

*where the gain matrix $K(t-d \mid t)$ of the state predictor is calculated by*

$$\begin{aligned} K(t-d \mid t) &= \overline{\gamma}(t)P(t-d \mid t-1)H^T(\sigma^2(t)H\Lambda(t-d)H^T \\ &\quad + Q_v + \overline{\gamma}^2(t)HP(t-d \mid t-1)H^T)^{-1}. \end{aligned} \quad (51)$$

*The estimation error covariance matrix $P(t+1|t)$ of the state predictor is calculated by*

$$\begin{aligned} P(t+1|t) &= \overline{A}[P(t-d \mid t-1) - \overline{\gamma}(t)K(t-d \mid t)H \\ &\quad \times P(t-d \mid t-1)]\overline{A}^T + \Sigma(t). \end{aligned} \quad (52)$$

*Proof.* From (5) and (17), we can easily derive

$$\begin{aligned} \widehat{x}(t+1|t) &= \overline{A}\widehat{x}(t-d \mid t) \\ &= \overline{A}[\widehat{x}(t-d \mid t-1) + K(t-d \mid t)(y(t) \\ &\quad - \overline{\gamma}(t)H\widehat{x}(t-d \mid t-1)] + Jy(t), \end{aligned} \quad (53)$$

$$\begin{aligned} P(t+1|t) &= \overline{A}P(t-d \mid t)\overline{A}^T + \Sigma(t) \\ &= \overline{A}[P(t-d \mid t-1) - \overline{\gamma}(t)K(t-d \mid t) \\ &\quad \times HP(t-d \mid t-1)]\overline{A}^T + \Sigma(t). \end{aligned} \quad (54)$$

(51) has been obtained in Theorem 1.

*4.3. Optimal Linear Smoother*

**Theorem 3.** *For the systems (2) and (5) under Assumptions 1–4, the optimal linear smoother is given by*

$$\begin{aligned} \widehat{x}(t-d+1 \mid t) &= \widehat{x}(t-d+1 \mid t-d+1) \\ &\quad + \sum_{i=1}^{d-1} K(t-d+1 \mid t-i+1)(y(t-i+1) \\ &\quad - \overline{\gamma}(t)H\widehat{x}(t-d-i+1 \mid t-i)), \end{aligned} \quad (55)$$

*where the gain matrix $K(t-d+1 \mid t-i+1)$ of the state predictor is calculated by*

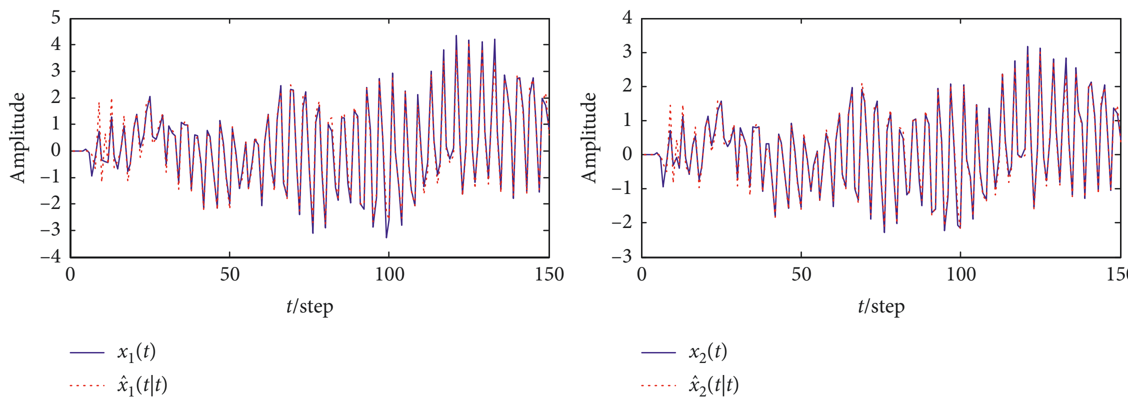


Figure 1: Tracking performance of the state filter.

$$K(t-d+1\mid t-i+1) = \overline{\gamma}(t)\Psi_i^T(t)H^T\left[\sigma^2(t)H\Lambda(t-d-i+1)H^T + Q_v + \overline{\gamma}^2(t)HP(t-d-i+1\mid t-i)H^T\right]^{-1}. \tag{56}$$

*The estimation error covariance matrix $P(t-d+1\mid t)$ of the state predictor is calculated by*

$$P(t-d+1\mid t) = P(t-d+1\mid t-d+1) - \sum_{i=1}^{d-1}\overline{\gamma}(t)K(t-d+1\mid t-i+1)H\Psi_i(t), \tag{57}$$

$$\begin{aligned}\Psi_i(t) = {} & \left[\Psi_{d-i+1}^T(t-i) - \overline{\gamma}(t)K(t-d-i\mid t-d)H \times P(t-2d|t-d-1)\right]\overline{A}^T - \overline{\gamma}(t)K \\ & \times (t-d-i+1\mid t-d+1)H \times \Psi_d^T(t) \\ & - \sum_{j=0}^{d-1-i}\overline{\gamma}(t)K(t-d-i+1\mid t-i-j)H\Psi_{i+j+1}(t),\end{aligned} \tag{58}$$

*where* $\Psi_i(t) = \Phi_{t-i}(t-d-i+1, t-d+1)$, $i = 1,2,\ldots,d$.

*Proof.* According to the projection theorem, we obtain

$$\hat{x}(t-d\mid t-i) = \hat{x}(t-d\mid t-i-1) + K(t-d\mid t-1)\varepsilon(t-i), \quad i = 0,1,\ldots,d-1. \tag{59}$$

Let $i = 1,2,\ldots,d-1$, and according to the iteration, we have (55).

From (55), we can easily get

$$\begin{aligned}\hat{x}(t-d+1\mid t) = {} & \hat{x}(t-d+1\mid t-d+1) - \sum_{i=1}^{d-1}K(t-d+1\mid t-i+1)\big(y(t-i+1) \\ & - \overline{\gamma}(t)H\hat{x}(t-d-i+1\mid t-i)\big).\end{aligned} \tag{60}$$

From the definition of the gain matrix and the covariance matrix, we can obtain (56) and (57), respectively.

From Lemma 2, we have the covariance matrixes as follows:

$$\begin{aligned}&\Phi_{t-i}(t-d-i+1, t-d+1) = \Phi_{t-i-1}(t-d-i+1, t-d+1) - \overline{\gamma}(t) \times K(t-d-i+1\mid t-i)H \times \Phi_{t-i-1}(t-d-i, t-d+1), \quad j = 0, \\ &\Phi_{t-i-1}(t-d-i+1, t-d+1) = \Phi_{t-i-2}(t-d-i+1, t-d+1) - \overline{\gamma}(t) \times K(t-d-i+1\mid t-i-1)H \times \Phi_{t-i-2}(t-d-i-1, t-d+1), \quad j = 1, \\ &\vdots \\ &\Phi_{t-d+2}(t-d-i+1, t-d+1) = \Phi_{t-d+1}(t-d-i+1, t-d+1) - \overline{\gamma}(t)K(t-d-i+1\mid t-d+2)H \times \Phi_{t-d+1}(t-2d+2, t-d+1), \quad j = d-2-i.\end{aligned} \tag{61}$$

According to the iteration, we have

$$\begin{aligned}\Phi_{t-i}(t-d-i+1, t-d+1) = {} & \Phi_{t-d+1}(t-d-i+1, t-d+1) - \sum_{j=0}^{d-1-i}\overline{\gamma}(t) \\ & \times K(t-d-i+1\mid t-i-j)H \\ & \times \Phi_{t-i-j-1}(t-d-i-j, t-d+1).\end{aligned} \tag{62}$$

Substituting (30) in (62), we have (58).

Based on the above discussion, we propose a new algorithm for the system under consideration in a recursive form. Starting with the initial estimates $\Psi_i(1), \Psi_i(2), \ldots, \Psi_i(d)$, $\hat{x}(d|d)$, $P(d|d)$, $\varepsilon(d)$, $Q_\varepsilon(d)$, $i = 1,2,\ldots,2d+1$, the proposed algorithm is given by the following steps:

*Step 1.* Computing $\Psi_{d-1}(t), \Psi_{d-2}(t), \ldots,$ and $\Psi_1(t)$ in sequence using (58).

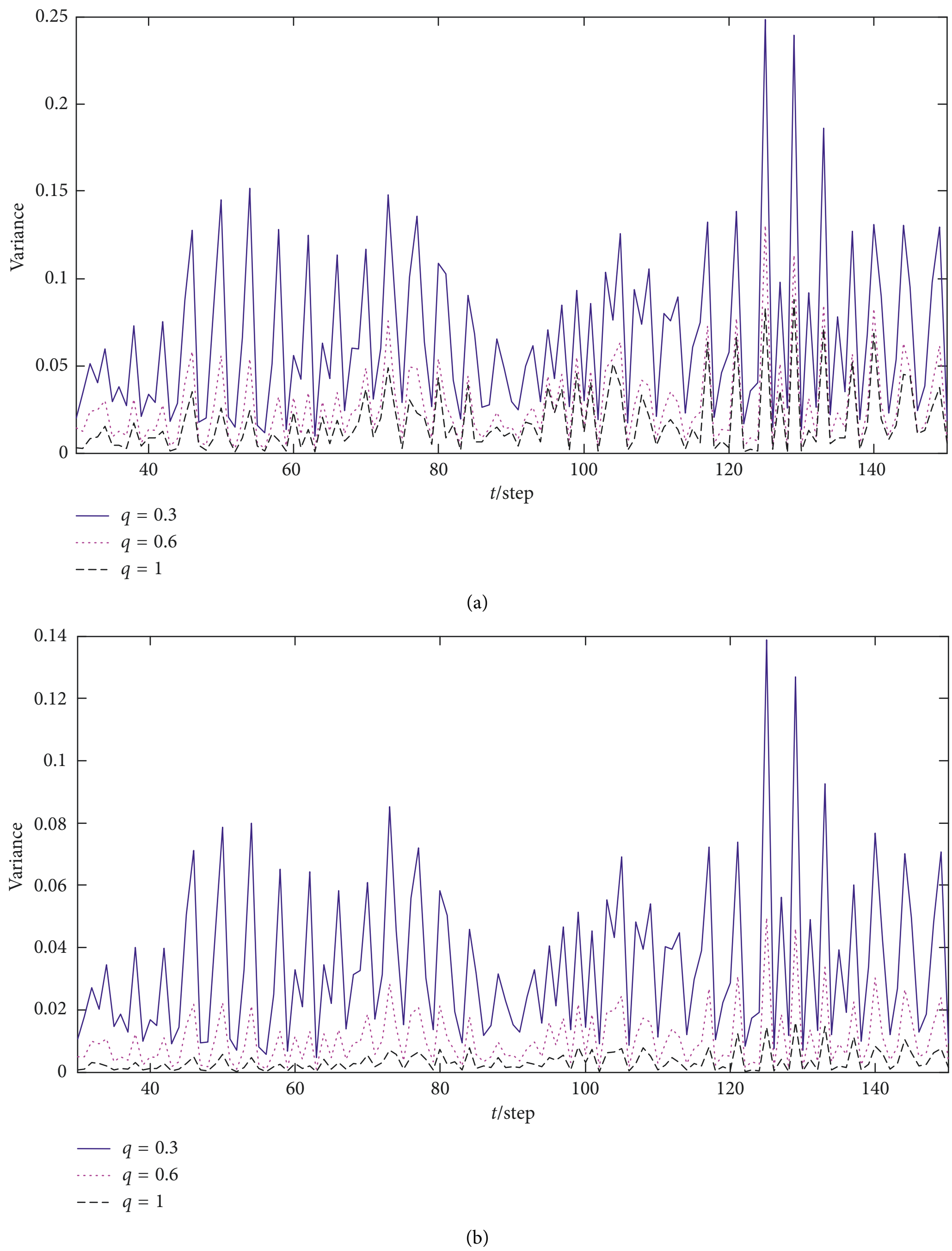


Figure 2: Comparison of error variances of the state filter ($P(\gamma(t) = q) = 0.8, q = 0.3, 0.6, 1$). (a) Estimation error variances for $x_1(t)$. (b) Estimation error variances for $x_2(t)$.

*Step 2*. Substituting the results of Step 1 in (56), computing $K(t-d+1 \mid t-1), \ldots, K(t-d+1 \mid t-i+1), \ldots$, and $K(t-d+1 \mid t-d+2)$ in sequence.

*Step 3*. Substituting the results of Step 1 and Step 2 in (55) and (57), computing $\hat{x}(t-d+1 \mid t)$ and $P(t-d+1 \mid t)$, respectively.

*Step 4*. Substituting the results of Step 3 in (50)–(52), computing $K(t-d \mid t), \hat{x}(t+1|t), P(t+1|t)$ in sequence. $\hat{x}(t-d \mid t-1)$ and $P(t-d \mid t-1)$ can be immediately obtained from $\hat{x}(t-d+1 \mid t)$ and $P(t-d+1 \mid t)$.

*Step 5*. Substituting the results of Step 1 and Step 4 in (35)–(39), computing $\Psi_d(t+d), \hat{x}(t+1|t+1)$ and $P(t+1|t+1)$ in sequence.

*Step 6*. Storing the results of Step 1–5 for computing the optimal estimators at time $t+1$.

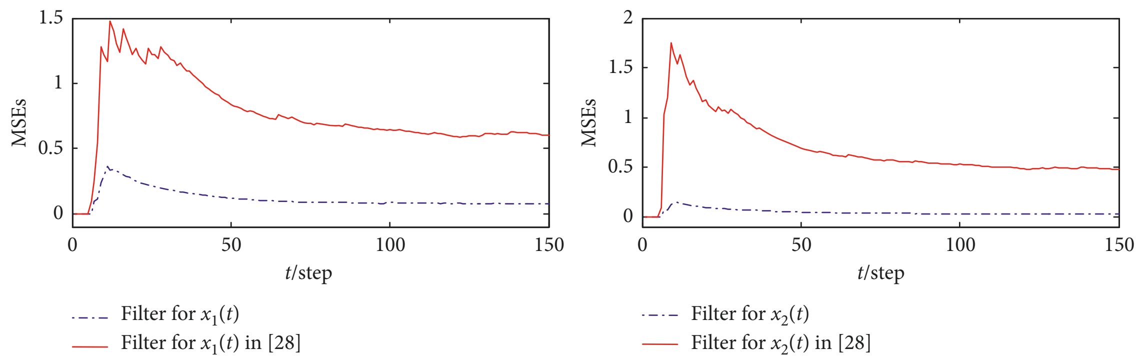


FIGURE 3: Comparison between the MSE curves of the filter and that of the predictor in [28].

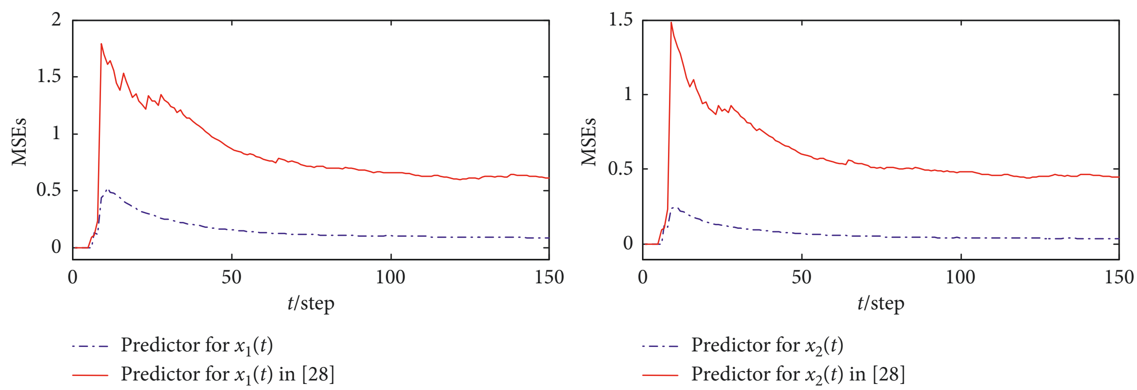


FIGURE 4: Comparison between the MSE curves of the one-step predictor and that of the predictor in [28].

## 5. Numerical Example

Consider the systems (1) and (2), let $h = 1$, the time delay $d = 3$, the state $x(t) = [x_1(t), x_2(t)]^T$, and $w(t)$ and $v(t)$ are zero-mean white noise sequences with variance $Q_w = 0.3$ and $Q_v = 0.3$. The multiplicative noise $\zeta_1(t)$ is white noise with zero mean and variance $Q_{\zeta_1}(t) = 0.16$, $\gamma(t)$ is a discrete random variable with the probability mass function $P(\gamma(t) = 0.95) = 0.05$, $P(\gamma(t) = 0.98) = 0.15$, and $P(\gamma(t) = 1) = 0.8$. $\Gamma = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$, $H = \begin{bmatrix} 0.5 & 1 \end{bmatrix}$, $A = \begin{bmatrix} 0.4 & 0.7 \\ 0.2 & 0.6 \end{bmatrix}$, $A_1 = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & -0.1 \end{bmatrix}$, $\hat{x}(0|0) = [0, 0]^T$, $P(0|0) = 0.01 I_2$, and $\varepsilon(i) = 0$, $i = 1, 2, \ldots, 2d + 1$. The optimal estimated value $\hat{x}(t|t)$ and $\hat{x}(t+1|t)$ can be calculated using Theorems 1, 2, and 3, respectively. To demonstrate the effectiveness of our proposed estimation algorithm, the tracking curves of the filter are shown in Figure 1. We can see the filter has the good tracking performance. From the probability mass function of $\gamma(t)$, it is obvious that the fading value $\gamma(t) = 1$ plays a bigger role than others because it has the greatest probability. In order to study the effect of the state filter further, we present the comparison of filter error variances when $P(\gamma(t) = q) = 0.8$, $q = 0.3, 0.6, 1$ in Figure 2. From Figure 2, we can see the filter has the highest estimation accuracy when $q = 1$. Figures 3 and 4 show the simulation results under the 150 Monte Carlo experiments. From the comparisons of mean square error (MSE) curves of filter and predictor with the filter and predictor in [28] in Figures 3 and 4, we can see the MSE curves of estimators in [28] sit on the top because Chen et al. [28] have not considered fading measurements and correlated noises. It indicates the estimation accuracy in this paper is better than that in [28]. Tables 1 and 2 give some specific values for the MSEs of filter and predictor for the filter and predictor in [28]. These simulation results show that the estimation algorithm proposed in this paper can provide satisfactory performance.

TABLE 1: Comparison between the MSEs of the filter and that of the filter in [28] ($t = 25, 50, 75, 100, 125, 150$).

| $t$ | 25 | 50 | 75 | 100 | 125 | 150 |
|---|---|---|---|---|---|---|
| $\hat{x}_1(t\|t)$ in this paper | 0.2156 | 0.1196 | 0.0887 | 0.0868 | 0.0784 | 0.0750 |
| $\hat{x}_1(t\|t)$ in [28] | 1.2208 | 0.8383 | 0.6919 | 0.6456 | 0.5965 | 0.6040 |
| $\hat{x}_2(t\|t)$ in this paper | 0.0836 | 0.0486 | 0.0358 | 0.0332 | 0.0297 | 0.0287 |
| $\hat{x}_2(t\|t)$ in [28] | 1.0700 | 0.6941 | 0.5705 | 0.5296 | 0.4876 | 0.4783 |

## 6. Conclusion

In this paper, we have investigated the recursive estimation problems for time-delay systems with fading measurements

Table 2: Comparison between the MSEs of the predictor and that of the predictor in [28] ($t = 25, 50, 75, 100, 125, 150$).

| $t$ | 25 | 50 | 75 | 100 | 125 | 150 |
|---|---|---|---|---|---|---|
| $\hat{x}_1(t+1\|t)$ in this paper | 0.2876 | 0.1555 | 0.1127 | 0.1048 | 0.0928 | 0.0870 |
| $\hat{x}_1(t+1\|t)$ in [28] | 1.2881 | 0.8718 | 0.7143 | 0.6624 | 0.6099 | 0.6152 |
| $\hat{x}_2(t+1\|t)$ in this paper | 0.1258 | 0.0697 | 0.0499 | 0.0437 | 0.0382 | 0.0357 |
| $\hat{x}_1(t+1\|t)$ in [28] | 0.8933 | 0.6056 | 0.5115 | 0.4854 | 0.4522 | 0.4488 |

and correlated noises. The time-delay exists in the system state and observation simultaneously. The fading measurements have been addressed by the mean and covariance of the variable which depend on the distribution law of each probability mass function. Based on the projection theory, the linear minimum variance optimal linear estimators, including filter, predictor, and smoother have been proposed. Finally, the effectiveness of the proposed estimators has been illustrated by a numerical example. In addition, it should be noted that the time-delay discrete system discussed in this paper defines that state and observation delay are the same constant, but in practical applications, the time-delay may be stochastic. Therefore, one of the future research topics is to develop more efficient algorithms for the complex discrete or continuous systems subject to network-induced phenomena [29, 30].

## Notations

| | |
|---|---|
| $R^n$: | $n$-dimensional Euclidean space |
| Superscript $T$: | Transpose |
| $\mathrm{E}(x)$: | Mathematical expectation of random variable $x$ |
| $\mathrm{tr}(\circ)$: | Trace of matrix $\circ$ |
| $\delta_{tj}$: | Kronecker delta function |
| $I_n$: | $n$ by $n$ identity matrix |
| $\perp$: | Orthogonality |
| $\mathrm{Prob}(*)$: | Probability of the occurrence of the event $*$ |
| $\hat{x}(\circ\|\cdot)$: | Estimate of the stochastic variable $x(\circ)$ based on measurements before time $\cdot$, i.e., the projection of $x(\circ)$ on the linear space generated by the measurements before time $\cdot$ |
| $\tilde{x}(\circ\|\cdot) = x(\circ) - \hat{x}(\circ\|\cdot)$: | Estimation error. |

## Data Availability

The data used to support the findings of this study are included within the article.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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