Research Letter

Performance of Equal Power Allocation in Multiuser OFDM-Based Cognitive Radio Systems

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The performance difference between a simple equal subchannel power allocation and the optimal water-filling subchannel power allocation schemes is studied for a multiuser OFDM-based cognitive radio (CR) system. It is shown that this difference depends on the average subchannel gain variations among the CR users as well as the activity levels of the primary users.

1. Introduction

Cognitive radio (CR) [1–3] is a concept which can potentially alleviate the pending spectrum shortage crisis. As discussed in [2], orthogonal frequency division multiplexing (OFDM) is an attractive modulation candidate for CR systems. It is well known that the optimal solution to the problem of determining the capacity of a set of M parallel additive white Gaussian noise (AWGN) subchannels, each of which may have a different noise power level, subject to a total input signal power constraint, has a nice water-filling interpretation [4]. We will refer to this as optimal water-filling (OWF). In OWF, the signal powers allocated to different subchannels are in general different and no power is allocated to "silent" subchannels on which the noise power exceeds a certain threshold (water level). The scheme in which the total available signal power is shared equally among *all* (silent and nonsilent) subchannels is referred to as plain equal power allocation (PEPA).

Simulation results in [5] indicate that the difference between OWF and PEPA is quite small in a multiple user system with Rayleigh fading when each subchannel is assigned to the user with the best channel quality for that subchannel. Analytical results in [6] show that the performance difference between OWF and PEPA decreases with the number of users and average signal-to-noise ratio (SNR). In this paper, we study the performance difference between PEPA and OWF in a multiuser OFDM-based CR system. It is found that PEPA performs almost as well as OWF when there is little variation in CR user (CRU) average subchannel gains or PU activity level is high.

2. System Model

We consider a CR system with a total bandwidth of W Hz and L PUs; PU l, $l = 1, 2, \ldots, L$ has a bandwidth allocation of W_l Hz. Frequency bands carrying PU signals are referred to as *active*; nonactive bands are also termed *spectrum holes*. In order to reduce the mutual interference between secondary CRUs and PUs to acceptable levels, some subchannels adjacent to active PU bands are not used by the CRUs.

We are interested in downlink transmissions from one CR base station (CRBS) to K CRUs. It is assumed that the CRBS and the CRUs are able to accurately locate the spectrum holes. The system bandwidth of W Hz can accommodate M OFDM subbands (or subchannels), each with noise power σ_0^2. Interference among the subchannels is assumed to be negligible.

The system is time-slotted with a slot duration equal to an OFDM symbol duration (T_s). The subchannels are modelled in discrete time, with the gain for subchannel m and time slot t from the CRBS to CRU k denoted by $\sqrt{g_{k,m}^t}$. For simplicity, it is assumed that for any given value of k, $\{g_{k,m}, m = 1, 2, \ldots, M\}$ are identically distributed random variables (RVs) with a common probability function

(pdf) and cumulated distribution function (cdf) denoted as $f_{G_k}(g_k)$ and $F_{G_k}(g_k)$, respectively.

At each time slot t, each subchannel within $\mathcal{M}_t$, the set of available subchannels at time slot t, can be used by the CRBS to transmit to at most one CRU. We use $f_G(g)$ and $F_G(g)$ to denote the pdf and the cdf of the selected CRUs, respectively. The number (r_m^t) of bits per OFDM symbol which can be supported by subchannel m in time slot t is given by [7]

$$r_m^t = \log_2\left(1 + \frac{g_m^t s_m^t}{\Gamma \sigma_0^2}\right), \qquad (1)$$

where g_m^t is the subchannel gain of the selected CRU, s_m^t is the power allocated to subchannel m at time slot t, and Γ is an SNR gap parameter which indicates how far the system is operating from capacity. The available power constraint implies that

$$\sum_{m\in\mathcal{M}_t} s_m^t \leq S \quad \forall t, \qquad (2)$$

where S is the total power per time slot.

The availability of a PU band is modelled by a two-state Markov chain. During a time slot t, a PU band can be in one of two modes: *active* or *inactive* [8]. A PU band can change mode once every T_{state} slots. At a transition time, the probability of a PU band changing from active to inactive mode is $1 - p_a$, and the probability of changing from inactive to active mode is $1 - p_n$. The number ($l_{\text{CR},t}$) of available PU bands at time slots $\{t, t = 1,2,\ldots\}$ then forms a Markov chain, with a transition probability matrix $\mathbf{Q} = \{q_{ij}\}$, $i, j = 0, 1, 2, \ldots, L$, where state i corresponds to the event that the number of available PU bands is equal to i and the probability (q_{ij}) of moving from state i to state j is given by

$$q_{ij} = \sum_{n=0}^{L} \binom{i}{n}(1-p_n)^n p_n^{i-n}\binom{L-i}{n-i+j}(1-p_a)^{n-i+j}p_a^{L-j-n}. \qquad (3)$$

The steady-state probability column vector $\mathbf{\Pi} = (\pi_0, \pi_1, \ldots, \pi_L)^T$ is given by [9]

$$\mathbf{\Pi} = \mathbf{U}^{-1}\mathbf{V}, \qquad (4)$$

where

$$\mathbf{U} = \begin{bmatrix} 1, & 1, & 1, & \cdots & 1 \\ 0, & 1-q_{00}+q_{10}, & 1-q_{00}+q_{20}, & \cdots & 1-q_{00}+q_{L0} \\ 0, & q_{01}+1-q_{11}, & q_{01}-q_{21}, & \cdots & q_{01}-q_{L1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0, & q_{0L}-q_{1L}, & q_{0L}-q_{2L}, & \cdots & q_{0L}+1-q_{LL} \end{bmatrix}, \qquad (5)$$

and $\mathbf{V} = (1, 1-q_{00}, q_{01}, \ldots, q_{0L})^T$. If each PU band can accommodate an equal number (M/L) of subchannels, the probability of having m available subchannels is

$$b_m = \begin{cases} \pi_l, & \text{if } m = \dfrac{lM}{L}, \ l = 0,1,2,\ldots,L \\ 0, & \text{otherwise.} \end{cases} \qquad (6)$$

3. Achievable Bit Rates for OWF and PEPA

Although OWF provides the optimal solution for subchannel power allocation, it is more complex-to-implement than PEPA. We now derive expressions to compare the bit rates achievable by the two schemes.

3.1. Upper Bounds on Achievable Bit Rate for OWF

Assuming that OWF is applied to the $m_{\text{CR},t}$ subchannel gains of the selected CRUs at time slot t, the average bit rate (ABR) ($B_{\text{OWF}} \overset{\Delta}{=} \lim_{T\to\infty} B_{\text{OWF}}^{(T)}$) is given by [4]

$$B_{\text{OWF}}^{(T)} = \frac{1}{T_s T}\sum_{t=1}^{T}\sum_{m\in\mathcal{M}_t:\,\Gamma\sigma_0^2/g_m^t\leq L_{\text{OWF}}^t}\log_2\left(\frac{L_{\text{OWF}}^t g_m^t}{\Gamma\sigma_0^2}\right). \qquad (7)$$

In (7), $\Gamma\sigma_0^2/g_m^t$ can be viewed as the *equivalent noise power* on subchannel m and L_{OWF}^t is the water level at time t.

Let $\mathcal{T}_i$ be the set of time slots with $m_{\text{CR},t} = i$, and T_i be the number of elements in set $\mathcal{T}_i$. Grouping the time slots with $m_{\text{CR},t} = i$, we can rewrite (7) as

$$B_{\text{OWF}}^{(T)} = \frac{1}{T_s T}\sum_{i=1}^{M}\sum_{t_i\in\mathcal{T}_i}\sum_{m\in\mathcal{M}_{t_i}:\,\Gamma\sigma_0^2/g_m^{t_i}\leq L_{\text{OWF},i}^{t_i}}\log_2\left(\frac{L_{\text{OWF},i}^{t_i} g_m^{t_i}}{\Gamma\sigma_0^2}\right), \qquad (8)$$

where $L_{\text{OWF},i}^{t_i}$ is the water level at time slot t_i.

When $m_{\text{CR},t} = i$, the ABR if OWF is applied at each time slot t_i is smaller than that if OWF is applied *in one shot* to all the iT_i subchannel gains of the selected CRUs over the T_i time slots, that is,

$$B_{\text{OWF}}^{(T)} \leq \frac{1}{T_s T}\sum_{i=1}^{M}\sum_{t_i\in\mathcal{T}_i}\sum_{m\in\mathcal{M}_{t_i}:\,\Gamma\sigma_0^2/g_m^{t_i}\leq L_{\text{OWF},i}^{(T_i)}}\log_2\left(\frac{L_{\text{OWF},i}^{(T_i)} g_m^{t_i}}{\Gamma\sigma_0^2}\right), \qquad (9)$$

where $L_{\text{OWF},i}^{(T_i)}$ is the global water level for set $\mathcal{T}_i$.

Letting $T_i = b_i T$, and taking the limit as $T\to\infty$, we have

$$B_{\text{OWF}} \leq \frac{1}{T_s}\sum_{i=1}^{M} i b_i \int_{\Gamma\sigma_0^2/L_{\text{OWF},i}^{(\infty)}}^{\infty}\log_2\left(\frac{L_{\text{OWF},i}^{(\infty)} g}{\Gamma\sigma_0^2}\right)f_G(g)dg. \qquad (10)$$

In (9), $L_{\text{OWF},i}^{(T_i)}$, is lower than the level calculated by including all subchannels at time slots $t \in \mathcal{T}_i$ because there may exist some subchannels with above water level noises. Therefore,

$$L_{\text{OWF},i}^{(T_i)} \leq \frac{\sum_{t\in\mathcal{T}_i}\sum_{m\in\mathcal{M}_t}(\Gamma\sigma_0^2/g_m^t)}{iT_i} + \frac{S}{i}. \qquad (11)$$

Taking the limit as $T_i\to\infty$ in (11) yields

$$L_{\text{OWF},i}^{(\infty)} \leq E\left\{\frac{1}{G}\right\}\Gamma\sigma_0^2 + \frac{S}{i}. \qquad (12)$$

Substituting for the two occurrences of $L_{\mathrm{OWF},i}^{(\infty)}$ in (10) by the RHS of (12), we obtain

$$
\begin{aligned}
&B_{\mathrm{OWF}} \\
&\leq \frac{1}{T_s} \sum_{i=1}^{M} i b_i \left[\left(1 - F_G\left(\frac{1}{L_i}\right) \right) \log_2(L_i) + \int_{1/L_i}^{\infty} \log_2(g) f_G(g) dg \right],
\end{aligned}
\tag{13}
$$

where $L_i = \mathbf{L}_i$ and $\mathbf{L}_i \triangleq E\{1/G\} + S/(i\Gamma\sigma_0^2)$.

The upper bound for $L_{\mathrm{OWF},i}^{(\infty)}$ in (12) can be quite loose since it includes every subchannel, regardless of its equivalent noise power $\Gamma\sigma_0^2/g_m^t$. As $g_m^t \to 0$, $\Gamma\sigma_0^2/g_m^t$ increases without bound. Consequently, the bound in (13) is also loose. We obtain a tighter bound by turning off any subchannel $m \in \mathcal{M}_t$ for which $\Gamma\sigma_0^2/g_m^t$ is greater than the RHS of (12), that is, we consider only the subchannels for which $g_m^t > 1/\mathbf{L}_i$ in calculating the water levels. The resulting water level is still higher than $L_{\mathrm{OWF},i}^{(\infty)}$ so that

$$
L_{\mathrm{OWF},i}^{(\infty)} \leq \frac{\Gamma\sigma_0^2 \int_{1/\mathbf{L}_i}^{\infty} (1/g) f_G(g) dg + S/i}{1 - F_G(1/\mathbf{L}_i)}.
\tag{14}
$$

Substituting $L_{\mathrm{OWF},i}^{(\infty)}$ in (10) by the RHS of (14), we obtain a tighter bound, namely, (13) with

$$
L_i = \frac{\int_{1/\mathbf{L}_i}^{\infty} (1/g) f_G(g) dg + S/(i\Gamma\sigma_0^2)}{1 - F_G(1/\mathbf{L}_i)}.
\tag{15}
$$

3.2. Achievable Bit Rate for PEPA

The ABR for PEPA is $B_{\mathrm{PEPA}} \triangleq \lim_{T \to \infty} B_{\mathrm{PEPA}}^{(T)}$, where

$$
B_{\mathrm{PEPA}}^{(T)} = \frac{1}{T_s T} \sum_{t=1}^{T} \sum_{m \in \mathcal{M}_t} \log_2\left(1 + \frac{S g_m^t}{m_{\mathrm{CR},t}\Gamma\sigma_0^2} \right).
\tag{16}
$$

Grouping the time slots with $m_{\mathrm{CR},t} = i$, we have

$$
B_{\mathrm{PEPA}}^{(T)} = \frac{1}{T_s T} \sum_{i=1}^{M} \sum_{t \in \mathcal{T}_i} \sum_{m \in \mathcal{M}_t} \log_2\left(1 + \frac{S g_m^t}{i\Gamma\sigma_0^2} \right).
\tag{17}
$$

Letting $T_i \to \infty$, we obtain

$$
B_{\mathrm{PEPA}} = \frac{1}{T_s} \sum_{i=1}^{M} i b_i E\left\{ \log_2\left(1 + \frac{S}{i\Gamma\sigma_0^2} G \right) \right\}.
\tag{18}
$$

4. Rayleigh Fading Channel

In Section 3, we studied the ABR for OWF and PEPA for arbitrary pdf's and cdf's of the subchannel gains of the selected CRUs. In this section, we obtain the pdf and cdf of the subchannel gains of the *selected* CRUs for two different subchannel allocation strategies. The subchannel gains of the CRUs are assumed to be Rayleigh-distributed, that is, the power gains are exponentially distributed.

4.1. Opportunistic Subchannel Assignment

Suppose that at each time t, each of the M subchannels is assigned to the CRU with the highest gain for that subchannel. If the average subchannel power gains for all CRUs are equal, the pdf of the power gain for the CRU assigned to any subchannel is readily obtained using a standard result in order statistics [10], that is,

$$
f_G(g) = K \left[1 - e^{-g/E\{G\}} \right]^{K-1} \frac{e^{-g/E\{G\}}}{E\{G\}}
\tag{19}
$$

with corresponding cdf

$$
F_G(g) = \left[1 - e^{-g/E\{G\}} \right]^K.
\tag{20}
$$

4.2. A Fairer Subchannel Assignment Scheme

If the average subchannel gains for CRUs are quite different, assigning a subchannel to the CRU with the highest gain may be too unfair to CRUs with poor average subchannel gains. A fairer scheme [11] is to select, for each subchannel, the CRU with the best channel gain relative to its own mean gain, that is,

$$
k^*(t) = \underset{k}{\mathrm{argmax}}\ \frac{g_{k,m}(t)}{E\{G_k\}}.
\tag{21}
$$

The distribution of a CRU's subchannel gain relative to its own mean is exponential with a mean of 1. Thus, the probability of selecting CRU i is $1/K$, that is, $P(k^* = i) = 1/K$, $i = 1, 2, \ldots, K$. The cdf of the power gain of the selected CRU for a subchannel is

$$
\begin{aligned}
F_G(g) &= P(G \leq g) \\
&= \sum_{i=1}^{K} P(G \leq g \mid k^* = i) P(k^* = i) \\
&= \frac{1}{K} \sum_{i=1}^{K} \prod_{j=1}^{K} P\left(G_j \leq \frac{g E\{G_j\}}{E\{G_i\}} \right) \\
&= \frac{1}{K} \sum_{i=1}^{K} \left(1 - e^{-g/E\{G_i\}} \right)^K.
\end{aligned}
\tag{22}
$$

The corresponding pdf is

$$
f_G(g) = \sum_{i=1}^{K} \left(1 - e^{-g/E\{G_i\}} \right)^{(K-1)} \frac{e^{-g/E\{G_i\}}}{E\{G_i\}}.
\tag{23}
$$

5. Numerical Results

To compare the ABR for OWF and PEPA in a multiuser OFDM-based CR system, the expressions in (7), (18), and (13) with L_i equal to the RHS of (15) are evaluated. The two subchannel allocation strategies in Sections 4.1 and 4.2, hereafter referred to as Case A and Case B, respectively, are considered. In Case A, the average subchannel power gain for each CRU is chosen as 2×10^{-13}. In Case B, we increase the
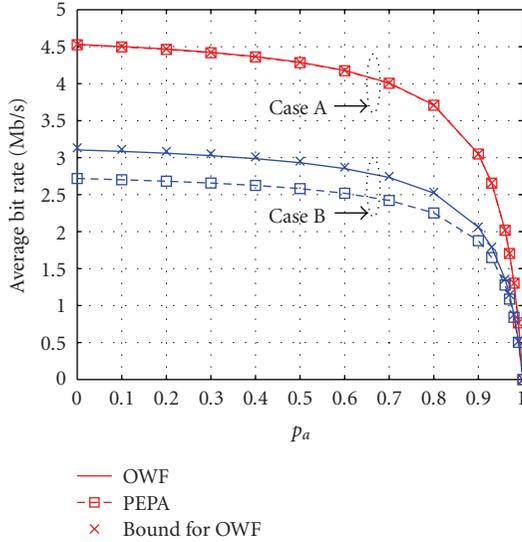
FIGURE 1: ABR as a function of p_a for OWF and PEPA with $K = 6$.

the CRU average gains are similar, the simpler PEPA scheme suffers little loss relative to OWF.

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number of CRUs by six at a time. The average subchannel power gains of the six CRUs are chosen as follows: one with value 10^{-12}, two with value 10^{-13} and three with value 10^{-14}. The resulting overall average subchannel gain for the six CRUs is 2×10^{-13}. In our calculations, we also use the following parameter values: $\Gamma = 1$, $\sigma_0^2 = 10^{-16}$, $S = 0.1$ W, $W = 2$ kHz, $W_l = 250$ kHz, $l \in \{1, 2, \ldots, L\}$, $L = 8$, $M = 64$, $p_n = 0.9$, and $T_s = 40\,\mu$s.

Figure 1 shows the ABR for OWF and PEPA as a function of p_a for $K = 6$ CRUs. For both cases, the ABR for OWF and PEPA decreases with p_a due to the reduced number of available subchannels. In Case A, the improvement of OWF over PEPA is 0.1% at $p_a = 0.1$ and 0.05% at $p_a = 0.9$. In Case B, the improvement of OWF over PEPA is 14% at $p_a = 0.1$ and 2% at $p_a = 0.99$. The difference between OWF and PEPA decreases with p_a because with a fixed total power, the average SNR for the available subchannels increases. The difference between OWF and PEPA is known to decrease with average SNR [6]. For both cases, the proposed upper bound for OWF is very close to the actual OWF curve and the difference decreases with p_a. The results show that the relative performance of PEPA depends on the activity level of the PUs and the variations in average subchannel gains among the CRUs.

The ABRs for OWF and PEPA were also determined as a function of the number (K) of CRUs. For both cases, the ABRs of OWF and PEPA increase with K as a result of multiuser diversity. The ABR difference between OWF and PEPA in Case A is negligible; in Case B, the improvement of OWF over PEPA is 9% for $K = 6$ and 2% for $K = 48$.
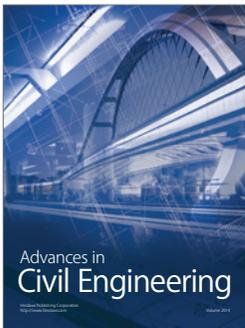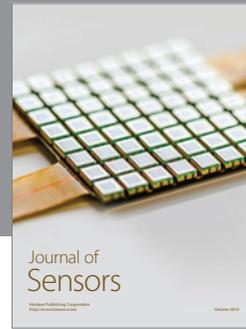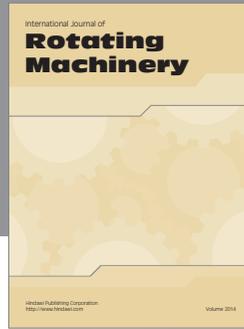
6. Conclusions

The performance difference between the PEPA and OWF subcarrier power allocation schemes in a multiuser OFDM-based CR system was studied. A proposed upper bound for OWF was shown to be tight. When the PU activity is high or