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Research Letter

Stochastic Processes with a Particular Type of Variograms

Chunsheng Ma

Department of Mathematics and Statistics, Wichita State University, Wichita, KS 67260, USA

Correspondence should be addressed to Chunsheng Ma, cma@math.wichita.edu

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This paper is concerned with a class of stochastic processes or random fields with second-order increments, whose variograms have a particular form, among which stochastic processes having orthogonal increments on the real line form an important subclass. A natural issue, how big this subclass is, has not been explicitly addressed in the literature. As a solution, this paper characterizes a stochastic process having orthogonal increments on the real line in terms of its variogram or its construction. Our findings are a little bit surprising: this subclass is big in terms of the variogram, and on the other hand, it is relatively "small" according to a simple construction. In particular, every such process with Gaussian increments can be simply constructed from Brownian motion. Using the characterizations we obtain a series expansion of the stochastic process with orthogonal increments.

1. INTRODUCTION

The variogram, or the structure function, was first employed by Kolmogorov [1, 2] to describe statistical properties of stochastic processes. It has been used as an important dependence measure for stochastic processes or random fields that have second-order increments, and as a useful structural tool in practice for analyzing space and/or time data (see, e.g., [3–12], among others).

Consider a real-valued stochastic process or random field $\{Z(\mathbf{x}),\ \mathbf{x} \in \mathcal{D}\}$ over an index set $\mathcal{D}$ that could be a temporal, spatial, or spatiotemporal domain. Its variogram or structure function is defined by half the variance of the increment:

$$\gamma(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{2} \operatorname{var}\{Z(\mathbf{x}_1) - Z(\mathbf{x}_2)\}, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}, \tag{1}$$

when the increments of $\{Z(\mathbf{x}),\ \mathbf{x} \in \mathcal{D}\}$ have second-order moments. In particular, for a second-order process $\{Z(\mathbf{x}),\ \boldsymbol{x} \in \mathcal{D}\}$ with mean $EZ(\mathbf{x})$ and covariance function

$$C(\mathbf{x}_1, \mathbf{x}_2) = E[\{Z(\mathbf{x}_1) - EZ(\mathbf{x}_1)\}\{Z(\mathbf{x}_2) - EZ(\mathbf{x}_2)\}], \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}, \tag{2}$$

its variogram is found to be

$$\gamma(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{2}\{C(\mathbf{x}_1, \mathbf{x}_1) + C(\mathbf{x}_2, \mathbf{x}_2)\} - C(\mathbf{x}_1, \mathbf{x}_2), \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}. \tag{3}$$

For a constant α with $0 < \alpha \leq 2$ and a real-valued function $g(\mathbf{x})$, $\mathbf{x} \in \mathcal{D}$, it is shown in Section 2 that

$$\gamma(\mathbf{x}_1, \mathbf{x}_2) = |g(\mathbf{x}_1) - g(\mathbf{x}_2)|^{\alpha}, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}, \tag{4}$$

is a variogram associated with, for instance, a Gaussian or elliptical contoured stochastic process. A well-known such example is the variogram of Brownian motion, with $g(x) = x$, $x \in \mathbb{R}$, and $\alpha = 1$, which has many interesting properties and particularly belongs to an important subclass of processes on $\mathbb{R}$ that have orthogonal increments. This motivates us to investigate the general variogram structure of stochastic processes having orthogonal increments in Section 3.

Let $\mathcal{D}$ be a temporal index set over the real line. A stochastic process $\{Z(x),\ x \in \mathcal{D}\}$ with second-order increments is said to have orthogonal (or uncorrelated) increments if

$$\operatorname{cov}\{Z(x_2) - Z(x_1),\ Z(x_4) - Z(x_3)\} = 0 \tag{5}$$

whenever $x_1 \leq x_2 \leq x_3 \leq x_4$, $x_k \in \mathcal{D}$ $(k = 1, \ldots, 4)$. White noise is a simple example of processes with orthogonal increments, since it is uncorrelated on its domain. A more interesting and also simple example is Brownian motion. For a stochastic process $\{Z(x),\ x \in \mathbb{R}\}$ having orthogonal increments, either its mean square derivative $Z'(x)$ does not exist, or it is a constant almost surely. Some important uses of this type of stochastic processes are in a spectral representation of a stationary process and in the formulation of the integral $\int_{-\infty}^{\infty} h(x)dZ(x)$, where $h(x)$ is a determinate function or a stochastic process.

This paper addresses a simple but interesting question that has not been explicitly addressed in the literature: how big is the class of stochastic processes having orthogonal increments on the real line? This leads us in Section 3 to characterize a stochastic process having orthogonal increments on an index set over the real line through its variogram (13) or its construction (14). As an application of construction (14), we obtain a series expansion of the stochastic process having orthogonal increments in Theorem 3, which is more transparent than the representation (2.8) of Cambanis [13] that provides a way of constructing every stochastic process with orthogonal Gaussian increments; see also [14] for series representations of second-order stochastic processes. The results here could be used, for instance, for numerical synthesis or simulation.

2. A CLASS OF VARIOGRAMS

The following theorem contains a type of variograms over a temporal, spatial, or spatiotemporal domain $\mathcal{D}$.

Theorem 1. *If* $g(\mathbf{x})$, $\mathbf{x} \in \mathcal{D}$, *is a real-valued function and* α *is a positive constant between 0 and 2, then*

$$\gamma(\mathbf{x}_1,\mathbf{x}_2) = |g(\mathbf{x}_1) - g(\mathbf{x}_2)|^{\alpha}, \quad \mathbf{x}_1,\mathbf{x}_2 \in \mathcal{D}, \tag{6}$$

is a variogram on $\mathcal{D}$.

Proof. Obviously, $\gamma(\mathbf{x}_1,\mathbf{x}_2) = \gamma(\mathbf{x}_2,\mathbf{x}_1) \geq 0$, $\mathbf{x}_1,\mathbf{x}_2 \in \mathcal{D}$, and $\gamma(\mathbf{x},\mathbf{x}) = 0$, $\mathbf{x} \in \mathcal{D}$. Let us start by proving that (6) is (conditionally) negative definite in a special case where $\alpha = 2$. In this case, for each integer $n \geq 2$, any $\mathbf{x}_1,\dots,\mathbf{x}_n \in \mathcal{D}$, and any real numbers $a_1,\dots,a_n$ with $\sum_{k=1}^{n} a_n = 0$, we have

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n} & a_i a_j |g(\mathbf{x}_i) - g(\mathbf{x}_j)|^2 \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \{g^2(\mathbf{x}_i) + g^2(\mathbf{x}_j) - 2g(\mathbf{x}_i)g(\mathbf{x}_j)\} \\ &= -2\Big\{\sum_{i=1}^{n} a_i g(\mathbf{x}_i)\Big\}^2 \leq 0;\end{aligned} \tag{7}$$

that is, $|g(\mathbf{x}_1) - g(\mathbf{x}_2)|^2$ is negative definite on $\mathcal{D}$.

Suppose now that $0 < \alpha < 2$, which we rewrite as $\alpha = 2\beta$. Notice that $0 < \beta < 1$ and

$$y^{\beta} = \frac{\beta}{\Gamma(1-\beta)}\int_0^{\infty} (1 - e^{-yu})\frac{du}{u^{\beta+1}}, \quad y \geq 0. \tag{8}$$

Since $|g(\mathbf{x}_1) - g(\mathbf{x}_2)|^2$ is negative definite on $\mathcal{D}$, $1 - \exp\{-|g(\mathbf{x}_1) - g(\mathbf{x}_2)|^2 u\}$ is also negative definite on $\mathcal{D}$ for any nonnegative constant u. So is

$$\begin{aligned}|g(\mathbf{x}_1) & - g(\mathbf{x}_2)|^{\alpha} \\ &= \{|g(\mathbf{x}_1) - g(\mathbf{x}_2)|^2\}^{\beta} \\ &= \frac{\beta}{\Gamma(1-\beta)}\int_0^{\infty} [1 - \exp\{-|g(\mathbf{x}_1) - g(\mathbf{x}_2)|^2 u\}]\frac{du}{u^{\beta+1}}, \\ & \qquad \mathbf{x}_1,\mathbf{x}_2 \in \mathcal{D}.\end{aligned} \tag{9}$$

□

A well-known example of (6) is the variogram of fractional Brownian motions, with $g(x) = x$, $x \in \mathbb{R}$, and $0 < \alpha \leq 2$.

As another example, in (6), taking $\alpha = 1$ and $g(\mathbf{x}) = \|\mathbf{x}\|$, $\mathbf{x} \in \mathbb{R}^d$, we obtain a variogram $|\|\mathbf{x}_1\| - \|\mathbf{x}_2\||$, which may be associated with a random field with covariance $\|\mathbf{x}_1\| + \|\mathbf{x}_2\| - |\|\mathbf{x}_1\| - \|\mathbf{x}_2\||$, $\mathbf{x}_1,\mathbf{x}_2 \in \mathbb{R}^d$, where $\|\mathbf{x}\|$ denotes the usual Euclidean norm. Interestingly, this covariance function is always positive except for the case where $\mathbf{x}_1 = \mathbf{0}$ or $\mathbf{x}_2 = \mathbf{0}$, and it is thus different from that of Brownian motion in $\mathbb{R}^d$, $\|\mathbf{x}_1\| + \|\mathbf{x}_2\| - \|\mathbf{x}_1 - \mathbf{x}_2\|$, of which a one-dimensional projection, $|x_1| + |x_2| - |x_1 - x_2|$, $x_1,x_2 \in \mathbb{R}$, vanishes when x_1 and x_2 locate at the opposite sides of the origin.

The third example of (6) is $|\boldsymbol{\theta}'(\mathbf{x}_1 - \mathbf{x}_2)|^{\alpha}$, which is obtained from (6) by letting $g(\mathbf{x}) = \boldsymbol{\theta}'\mathbf{x}$, $\mathbf{x} \in \mathcal{D}$, where $\boldsymbol{\theta}$ is a constant vector in $\mathcal{D}$.

It is interesting to see that $g(\mathbf{x})$ could be an arbitrary function on $\mathcal{D}$ in order to formulate the variogram (6). When it is nonnegative, we obtain the following type of covariances for nonstationary Gaussian processes or random fields.

Corollary 1. *For a real-valued function* $g(\mathbf{x})$, $\mathbf{x} \in \mathcal{D}$, *the function*

$$C(\mathbf{x}_1,\mathbf{x}_2) = \min\{g(\mathbf{x}_1), g(\mathbf{x}_2)\}, \quad \mathbf{x}_1,\mathbf{x}_2 \in \mathcal{D}, \tag{10}$$

is a covariance function if and only if $g(\mathbf{x})$ *is nonnegative on* $\mathcal{D}$.

Proof. Clearly, the function $g(\mathbf{x})$ in (10) is nothing but the variance of the corresponding process, and thus it is necessarily nonnegative.

On the other hand, let $g(\mathbf{x}) \geq 0$, $\mathbf{x} \in \mathcal{D}$. Moreover, suppose that there is an $\mathbf{x}_0 \in \mathcal{D}$ such that $g(\mathbf{x}_0) = 0$; otherwise, we choose an $\mathbf{x}_0 \in \mathcal{D}$ and modify the function $g(\mathbf{x})$ as follows:

$$\begin{aligned} &0, \quad \mathbf{x} = \mathbf{x}_0, \\ &g(\mathbf{x}), \quad \mathbf{x} \neq \mathbf{x}_0,\ \mathbf{x} \in \mathcal{D}, \end{aligned} \tag{11}$$

which is nonnegative on $\mathcal{D}$ as well. By Theorem 1, $\gamma(\mathbf{x}_1,\mathbf{x}_2) = (1/2)|g(x_1) - g(\mathbf{x}_2)|$, $\mathbf{x}_1,\mathbf{x}_2 \in \mathcal{D}$, is the variogram associated with a process, say, $\{Z(\mathbf{x}),\ \boldsymbol{x} \in \mathcal{D}\}$. Then, the covariance of the increment process $\{Z(\mathbf{x}) - Z(\mathbf{x}_0),\ \boldsymbol{x} \in \mathcal{D}\}$ is a Schoenberg-Lévy kernel [7]:

$$\begin{aligned}\operatorname{cov}(Z(\mathbf{x}_1)-Z(\mathbf{x}_0),\,Z(\mathbf{x}_2)-Z(\mathbf{x}_0))&=\gamma(\mathbf{x}_1,\mathbf{x}_0)+\gamma(\mathbf{x}_2,\mathbf{x}_0)-\gamma(\mathbf{x}_1,\mathbf{x}_2)\\&=\frac{g(\mathbf{x}_1)+g(\mathbf{x}_2)}{2}-\frac{|g(x_1)-g(\mathbf{x}_2)|}{2}\\&=\min\{g(\mathbf{x}_1),g(\mathbf{x}_2)\},\quad \mathbf{x}_1,\mathbf{x}_2\in\mathcal{D}.\end{aligned}\tag{12}$$

□

3. STOCHASTIC PROCESS WITH ORTHOGONAL INCREMENTS

In this section, $\mathcal{D}$ denotes an index set over the real line. The following theorem characterizes every stochastic process having orthogonal increments on $\mathcal{D}\subseteq\mathbb{R}$.

Theorem 2. *Assume that a stochastic process $\{Z(x),\ x\in\mathcal{D}\}$ has second-order increments and the variogram $\gamma(x_1,x_2)$. Then, the following statements are equivalent:*

(i) *$\{Z(x),\ x\in\mathcal{D}\}$ has orthogonal increments;*

(ii) *there exists a monotone function $g(x)$, $x\in\mathcal{D}$, such that*

$$\gamma(x_1,x_2)=|g(x_1)-g(x_2)|,\quad x_1,x_2\in\mathcal{D};\tag{13}$$

(iii) *there exists a monotone function $g(x)$, $x\in\mathcal{D}$, such that*

$$Z(x)=Z_0(g(x)),\quad x\in\mathcal{D},\tag{14}$$

where $\{Z_0(x),\ x\in R_g\}$ is a stochastic process with second-order increments and variogram

$$\gamma(x_1,x_2)=|x_1-x_2|,\quad x_1,x_2\in R_g,\tag{15}$$

and R_g is the range of the function $g(x)$, $x\in\mathcal{D}$.

As a naive interpretation of (13), the class of stochastic processes having orthogonal increments is as big as the class of real-valued monotone functions on $\mathcal{D}$. On the other hand, every such stochastic process can be simply obtained from the process $\{Z_0(x),\ x\in R_g\}$ via the construction (14), so that the class looks like being relatively "small".

Clearly, the function $g(x)$ in (13) or (14) is not unique; for instance, $-g(x)$ or $g(x)+c$ is another candidate, where c is a constant. This is comparable to what we know, that if $\{Z(x),\ x\in\mathcal{D}\}$ has orthogonal increments, then so do $\{-Z(x),\ x\in\mathcal{D}\}$, $\{Z(x)+c,\ x\in\mathcal{D}\}$, and $\{Z(x)+Y,\ x\in\mathcal{D}\}$, where Y is a random variable. Also, different stochastic processes with orthogonal increments could have the same variogram. As an important benefit of Theorem 2, the same techniques employed in the study of the process $\{Z_0(x),\ x\in\mathbb{R}\}$ can be used in studying all processes with orthogonal increments.

Before proving Theorem 2, notice that the following identity holds for any real numbers $a_1,\dots,a_4$:

$$\begin{aligned}(a_1-a_2)(a_3-a_4)&=\frac{1}{2}(a_1-a_4)^2+\frac{1}{2}(a_2-a_3)^2\\&\quad-\frac{1}{2}(a_1-a_3)^2-\frac{1}{2}(a_2-a_4)^2.\end{aligned}\tag{16}$$

In particular, in (16), choosing

$$a_i-a_j=Z(x_i)-Z(x_j)-E\{Z(x_i)-Z(x_j)\},\quad i,j\in\{1,\dots,4\},\tag{17}$$

and then taking the expectation of both sides of (16), we obtain

$$\operatorname{cov}\{Z(x_1)-Z(x_2),\ Z(x_3)-Z(x_4)\}=\gamma(x_1,x_4)+\gamma(x_2,x_3)-\gamma(x_1,x_3)-\gamma(x_2,x_4),\tag{18}$$

for a process $\{Z(x),\ x\in\mathcal{D}\}$ having second-order increments on $\mathcal{D}$, where $x_k\in\mathcal{D}$ $(k=1,\dots,4)$.

Proof of Theorem 2. (ii)⇒(i): suppose that (13) holds with $g(x)$ being a nondecreasing function on $\mathcal{D}$. Then for any $x_1\le x_2\le x_3\le x_4$, using formula (18) we obtain

$$\begin{aligned}\operatorname{cov}\{Z(x_2)-Z(x_1),Z(x_4)-Z(x_3)\}&=\gamma(x_1,x_4)+\gamma(x_2,x_3)-\gamma(x_1,x_3)-\gamma(x_2,x_4)\\&=|g(x_1)-g(x_4)|+|g(x_2)-g(x_3)|\\&\quad-|g(x_1)-g(x_3)|-|g(x_2)-g(x_4)|\\&=\{g(x_4)-g(x_1)\}+\{g(x_3)-g(x_2)\}\\&\quad-\{g(x_3)-g(x_1)\}-\{g(x_4)-g(x_2)\}=0,\end{aligned}\tag{19}$$

which means that $\{Z(x),\ x\in\mathcal{D}\}$ has orthogonal increments.

In case $g(x)$ is a nonincreasing function on $\mathcal{D}$, the orthogonal property of increments of $\{Z(x),\ x\in\mathcal{D}\}$ is obtained in a similar way.

(i)⇒(ii): let $\{Z(x),\ x\in\mathcal{D}\}$ have orthogonal increments, and choose a point $x_0\in\mathcal{D}$. Define

$$g(x)=\begin{cases}\dfrac{1}{2}\operatorname{var}\{Z(x)-Z(x_0)\}, & x\ge x_0,\\[2mm] -\dfrac{1}{2}\operatorname{var}\{Z(x)-Z(x_0)\}, & x<x_0.\end{cases}\tag{20}$$

To show that $g(x)$ is nondecreasing on $\mathcal{D}$ and $\gamma(x_1,x_2)$ is of the form (13), we consider three possibilities as follows.

Case 1 ($x_1\le x_0\le x_2$). In this case, it is obvious that $g(x_1)\le g(x_2)$ and

$$\begin{aligned}\gamma(x_1,x_2)&=\frac{1}{2}\operatorname{var}\{(Z(x_2)-Z(x_0))+(Z(x_0)-Z(x_1))\}\\&=\frac{1}{2}\operatorname{var}(Z(x_2)-Z(x_0))+\frac{1}{2}\operatorname{var}(Z(x_0)-Z(x_1))\\&\quad+2\operatorname{cov}\{Z(x_2)-Z(x_0),\ Z(x_0)-Z(x_1)\}\\&=g(x_2)-g(x_1),\end{aligned}\tag{21}$$

where the last equality follows from the orthogonal property of the increments and the definition of $g(x)$.

Case 2 ($x_0 \le x_1 \le x_2$). Now, the orthogonal property of the increments implies

$$
\begin{aligned}
g(x_2) - g(x_1) \\
&= \frac{1}{2}\operatorname{var}\{(Z(x_2) - Z(x_1)) + (Z(x_1) - Z(x_0))\} \\
&\quad - \frac{1}{2}\operatorname{var}(Z(x_1) - Z(x_0)) \\
&= \frac{1}{2}\operatorname{var}(Z(x_2) - Z(x_1)) \\
&\quad + E\{(Z(x_2) - Z(x_1))(Z(x_1) - Z(x_0))\} \\
&= \frac{1}{2}\operatorname{var}(Z(x_2) - Z(x_1)) \\
&= \gamma(x_1, x_2) \ge 0.
\end{aligned} \tag{22}
$$

Case 3 ($x_1 \le x_2 \le x_0$). This is similar to case 2.

If we replace $g(x)$ in its definition by $-h(x)$, then $h(x)$ is a nonincreasing function on $\mathcal{D}$ and (13) holds as well.

(ii)⇒(iii): consider the case where $g(x)$ is strictly increasing or decreasing on $\mathcal{D}$, while other cases may be treated with modification. In this case, $g^{-1}(x)$, the inverse function of $g(x)$, is well defined on R_g, so that we can define a new stochastic process:

$$Z_0(x) = Z(g^{-1}(x)), \quad x \in R_g. \tag{23}$$

Clearly, the variogram of $\{Z_0(x),\ x \in R_g\}$ is the same as (15), and in terms of $\{Z_0(x),\ x \in R_g\}$, $\{Z(x),\ x \in \mathcal{D}\}$ can be expressed as (14).

(iii)⇒(ii): the proof is obvious. □

The property (ii) of Theorem 2 is known in the particular case where $\{Z(x),\ x \in \mathbb{R}\}$ is a second-order process (see, e.g., [15, Subsection 37.5]).

Corollary 2. *A second-order stochastic process $\{Z(x),\ x \in \mathcal{D}\}$ has orthogonal increments if its covariance function is of the form*

$$C(x_1, x_2) = g(\min(x_1, x_2)), \quad x_1, x_2 \in \mathcal{D}, \tag{24}$$

where $g(x)$, $x \in \mathcal{D}$, is a nondecreasing and nonnegative function on $\mathcal{D}$.

The variogram corresponding to (24) is

$$\gamma(x_1, x_2) = \frac{1}{2}|g(x_1) - g(x_2)|, \quad x_1, x_2 \in \mathcal{D}, \tag{25}$$

which indicates that $g(x)$ in the expression (13) for the variogram of a second-order process having orthogonal increments is bounded from below. Generally speaking, however, this may not hold for other processes having orthogonal increments.

Unlike (13), the function $g(x)$ in (24) has to be nondecreasing on $\mathcal{D}$, because the Cauchy-Schwartz inequality implies that

$$
\begin{aligned}
&|\operatorname{cov}(Z(x_1), Z(x_2))| \\
&\quad \le \sqrt{\operatorname{var}(Z(x_1))}\sqrt{\operatorname{var}(Z(x_2))}, \quad x_1, x_2 \in \mathcal{D},
\end{aligned} \tag{26}
$$

or

$$g(\min(x_1, x_2)) \le \sqrt{g(\min(x_1, x_2))}\sqrt{g(\max(x_1, x_2))}; \tag{27}$$

in other words,

$$g(\min(x_1, x_2)) \le g(\max(x_1, x_2)), \quad x_1, x_2 \in \mathcal{D}. \tag{28}$$

The inverse of Corollary 2 is obtained under an additional assumption.

Corollary 3. *For a second-order stochastic process $\{Z(x),\ x \in D\}$ that satisfies*

$$\lim_{x \to x_0} \operatorname{var}(Z(x)) = 0, \tag{29}$$

where $x_0 = \min_{x \in \mathcal{D}} x$, if it has orthogonal increments, then its covariance function is given by (24), *where the nondecreasing and nonnegative function $g(x)$, $x \in \mathcal{D}$, takes the form*

$$g(x) = \operatorname{var}(Z(x)), \quad x \in \mathcal{D}. \tag{30}$$

In fact, condition (29) implies that for any $x_1 \le x_2$ and $x_1, x_2 \in \mathcal{D}$,

$$\operatorname{cov}(Z(x_0), Z(x_2) - Z(x_1)) = 0, \tag{31}$$

so that

$$
\begin{aligned}
&\operatorname{cov}(Z(x_1), Z(x_2) - Z(x_1)) \\
&\quad = \operatorname{cov}(Z(x_1) - Z(x_0), Z(x_2) - Z(x_1)) = 0,
\end{aligned} \tag{32}
$$

and moreover

$$
\begin{aligned}
&\operatorname{cov}(Z(x_1), Z(x_2)) \\
&\quad = \operatorname{cov}(Z(x_1), Z(x_2) - Z(x_1)) + \operatorname{cov}(Z(x_1), Z(x_1)) \\
&\quad = \operatorname{var}(Z(x_1)).
\end{aligned} \tag{33}
$$

Condition (29) in Corollary 3 is typical. For example, when defining a Wiener process on the interval $[0, 1]$, we often assume that $Z(0) \equiv 0$.

It would be of interest to find the most general form of the covariance function, just like (13) for the variogram, for a second-order stochastic process on $\mathcal{D}$ that has orthogonal increments. So far, what we know is that, according to the definition of orthogonal increments,

$$\operatorname{cov}\{Z(x_0) - Z(x_1),\ Z(x_2) - Z(x_0)\} = 0, \tag{34}$$

where x_0 is an arbitrary point between x_1 and x_2, which means that the covariance function must be of the form

$$C(x_1, x_2) = C(x_1, x_0) + C(x_2, x_0) - C(x_0, x_0), \quad x_1 \le x_0 \le x_2. \tag{35}$$

The function $g(x)$ in (13) or (24) is not necessarily bounded from both sides. When it is so, the next theorem provides a series expansion for the corresponding stochastic process with orthogonal increments, which may be employed to simulate the process (e.g., [16]) or for numerical synthesis.

Theorem 3. *If $\{Z(x),\ x \in \mathcal{D}\}$ is a stochastic process with orthogonal increments and $g(x)$ in the variogram (13) is bounded by $0 \le g(x) \le 1$, $x \in \mathcal{D}$, then*

$$Z(x) = \sqrt{2}\sum_{n=1}^{\infty} \varepsilon_n \frac{\sin\{(n-1/2)\pi g(x)\}}{(n-1/2)\pi}, \quad x \in \mathcal{D}, \tag{36}$$

where $\{\varepsilon_n, n = 1, 2, \dots\}$ is white noise with mean 0 and variance 1, and the series at the right-hand side of (36) converges to $Z(x)$ in mean square and uniformly for $x \in \mathcal{D}$.

Proof. For a stochastic process $\{Z_0(t),\ t \in [0,1]\}$ with variogram (15) on the interval $[0,1]$, it is known (see, e.g., [17, Section 1.4]) that $\{Z_0(t),\ t \in [0,1]\}$ has the Karhunen-Loéve expansion

$$Z_0(x) = \sqrt{2}\sum_{n=1}^{\infty} \varepsilon_n \frac{\sin\{(n-1/2)\pi x\}}{(n-1/2)\pi}, \quad 0 \le x \le 1, \tag{37}$$

where $\{\varepsilon_n, n = 1, 2, \dots\}$ is a sequence of uncorrelated random variables with mean 0 and variance 1. Substituting x with $g(x)$ in (37) yields (36). □

The boundedness of $g(x)$ is required for the representation (36), while $0 \le g(x) \le 1$ might not be necessary.

In the Gaussian case, $\{\varepsilon_n, n = 1, 2, \dots\}$ are independent, and thus for each x the series at the right-hand side of (36) converges almost surely. Obviously, the expansion (36) is more transparent than (2.8) of Cambanis [13] for processes with orthogonal Gaussian increments.

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