Research Article

Extended Gumbel Type-2 Distribution: Properties and Applications

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In this paper, we proposed a new four-parameter Extended Gumbel type-2 distribution which can further be split into the Lehman type I and type II Gumbel type-2 distribution by using a generalized exponentiated *G* distribution. The distributional properties of the proposed distribution have been studied. We derive the *p*th moment; thus, we generalize some results in the literature. Expressions for the density, moment-generating function, and *r*th moment of the order statistics are also obtained. We discuss estimation of the parameters by maximum likelihood and provide the information matrix of the developed distribution. Two life data, which consist of data on cancer remission times and survival times of pigs, were used to show the applicability of the Extended Gumbel type-2 distribution in modelling real life data, and we found out that the new model is more flexible than its submodels.

1. Introduction and Motivation

Statistical distributions are important tools in analysing and predicting real-world phenomena. Several distributions have been developed and studied. There is always room for development in statistical distributions to blend with the current situations which gives room for wider applications which can be achieved by inducing flexibility into the standard probability distribution to allow for fitting specific real-world scenarios. This has motivated researchers to work towards developing new and more flexible distributions. There are several ways to extend standard probability distributions, and one of the most popular methods is the use of distribution generators such as the exponentiated method by Lehmann [1]; the Marshall-Olkin method developed by Marshall and Olkin [2]; the beta distribution method proposed by Alexander et al. [3] and Eugene et al. [4]; the gamma distribution method by Cordeiro et al. [5], Ristic and Balakrishnan [6]), and Zografos and Balakrishnan [7]; the McDonald method proposed and studied by MacDonald [8]; and the exponentiated generalized method developed by De Andrade et al. [9]. The cubic rank transmutation map was proposed and studied by Granzotto et al. [10].

The Gumbel type-2 distribution is a very important distribution method from the theory of extreme value distribution. The distribution found applications in modelling extreme events in the field of meteorology, life testing, fracture roughness, seismology, and reliability analysis. It can also be used for modelling lifetime data sets with monotonic failure (or hazard) rates, most especially with a decreasing hazard rate. But in real-life data analysis, the hazard rate function of so many complex phenomena that are regularly encountered in practice are nonmonotonic and cannot be modelled by the Gumbel type-two (GTT) distribution. In other to improve the fit of (GTT) distribution, Okorie et al. [11] proposed and studied the properties of an exponentiated form of the GTT distribution of Lehman type I. Okorie et al. [12] investigated the properties of the Kumaraswamy *G* Exponentiated Gumbel type-two distribution. The cumulative distribution function (cdf) and probability density function (pdf) of the GTT distribution is given by

$$J(x) = \exp\left(-\alpha x^{-\nu}\right), \tag{1}$$

$$j(x) = \alpha \nu \exp\left(-\alpha x^{-\nu}\right), \tag{2}$$

where $x > 0$ and α and v are the scale and shape parameters, respectively.

Adding parameters to the GTT model may give rise to new, more flexible models for fitting real-life data. Therefore, we defined in this work an extension of the model above by using the methodology proposed by [6]. Given any given continuous baseline cdf $J(x)$ and $x \in R$, those authors defined the exponentiated generalized class of distributions with two extra shape parameters $a > 0$ and $b > 0$ with cdf $G(x)$ and pdf $g(x)$ given by

$$G(x) = [1 - \{1 - J(x)\}^a]^b, \tag{3}$$

$$g(x) = abj(x)\{1 - J(x)\}^{a-1}[1 - \{1 - J(x)\}^a]^{b-1}, \tag{4}$$

respectively, which have implicit dependence on the parameters of $J(x)$.

Here, $a > 0$ and $b > 0$ are the two additional shape parameters. It should be noted that there is no complicated function in (3), which is in contrast with the beta generalized family by Alexander et al. [3] and Eugene et al. [4]), which also includes two extra parameters but incorporates the beta incomplete function. Equation (3) has tractable properties especially for simulations, since its quantile function takes a simple form given by

$$x = Q_G\left\{1 - \left(1 - u^{1/b}\right)^{1/a}\right\}, \tag{5}$$

where $Q_G(u)$ is the baseline quantile function.

To illustrate the tractability and flexibility of the exponentiated generalized model, Cordeiro et al. [13] applied (3) to extend some well-known distributions such as the gamma, frechet, Gumbel, and normal distributions. Several properties for the exponentiated generalized class were discussed, which motivate the use of this generator. In fact, for $a = 1$ and also taking $b = 1$, (3) reduces to $G(x) = J(x)^b$ and $G(x) = 1 - J(x)^a$ which correspond to the cdf's of the Lehmann type I and II families, respectively. For this reason, the model proposed by Cordeiro et al. [13] contains both Lehmann type I and type II classes. For this reason, the exponentiated generalized family can be derived from a double transformation using these classes. The two extra parameters a and b in density (4) can control both tail weights, allowing the generation of flexible distributions, with heavier or lighter tails.

The above properties and many others have been examined and explored in recent works for the generator proposed and studied by Cordeiro et al. [14]: Silva et al. [15] examined the properties of the Dagum distribution, De Andrade et al. [9] investigated the properties of the exponential distribution, Elbatal and Muhammed [16] studied the properties of the Inverse Weibull distribution, De Andrade and Zea [17] studied the properties of the Gumbel model, Cordeiro et al. [14] studied the properties of the arcsine distribution, De Andrade et al. [18] examined the properties of the extended Pareto distribution, De Andrade et al. [19] studied the properties of the exponentiated generalized extended Gompertz distribution, etc.

2. Exponentiated Generalized Gumbel Type-Two (EGTT) Distribution

Given that a random variable X with support on the set of positive real numbers and $\text{EGTT}(a, b, v, \alpha)$ distribution, say, $X \sim \text{EGTT}(a, b, v, \alpha)$, is defined by inserting (1) in equation (3). Thus, the cdf of X is given by

$$G(x) = [1 - \{1 - \exp(-\alpha x^{-v})\}^a]^b, \tag{6}$$

where $a > 0$, $b > 0$, $\alpha > 0$, $v > 0$, and $x \geq 0$.

The graph of the cumulative distribution function is plotted in Figure 1 drawn below by taking the values of $b = 3.0$, $\alpha = 1.0$, and $v = 1.2$ and varying the value of a.

And the associated pdf is given by

$$g(x) = ab\alpha v x^{-(v+1)} \exp(-\alpha x^{-v}\{1 - \exp(-\alpha x^{-v})\}^a \cdot [1 - \{1 - \exp(-\alpha x^{-v})\}^a]^{b-1}, \tag{7}$$

where $a > 0$, $b > 0$, $\alpha > 0$, $v > 0$, and $x \geq 0$.

The graphs of the pdf of EGTT distribution are drawn in Figure 2 with various parameter values.

The graphs drawn in Figure 2 indicate that a and v control the skewness and kurtosis of EGTT distribution, and as such, the distribution can be used to model real-life data which are mesokurtic, leptokurtic, and platykurtic.

The reliability and hazard functions of EGGT, respectively, are given by

$$R(x) = 1 - [1 - \{1 - \exp(-\alpha x^{-v})\}^a]^b,$$

$$h(x) = \frac{ab\alpha v x^{-(v+1)} \exp(-\alpha x^{-v}\{1 - \exp(-\alpha x^{-v})\}^{a-1}[1 - \{1 - \exp(-\alpha x^{-v})\}^a]^{b-1}}{1 - [1 - \{1 - \exp(-\alpha x^{-v})\}^a]^b}. \tag{8}$$

Note that $R(x) = 1 - G(x)$ and $h(x) = (g(x)/R(x))$.

The plots of the reliability and hazard rate functions are shown in Figures 3 and 4, respectively.

The diagrams in Figures 3 and 4 indicate that the hazard rate function of the EGTT distribution could be shaped as unimodal, bathtub, or upside-down bathtub. These properties suggest that the EGTT distribution is suitable for modelling data sets with nonmonotonic hazard rate behaviour which are mostly encountered in practical situations.

2.1. Useful Expansion of the Probability Density Function. To motivate analytical derivation of some basic distributional properties of the EGTT distribution, we present the series representation of its pdf and cdf by using the generalized binomial expansion. If a is a positive real noninteger and $P \leq 1$, we consider the power series expansion given by

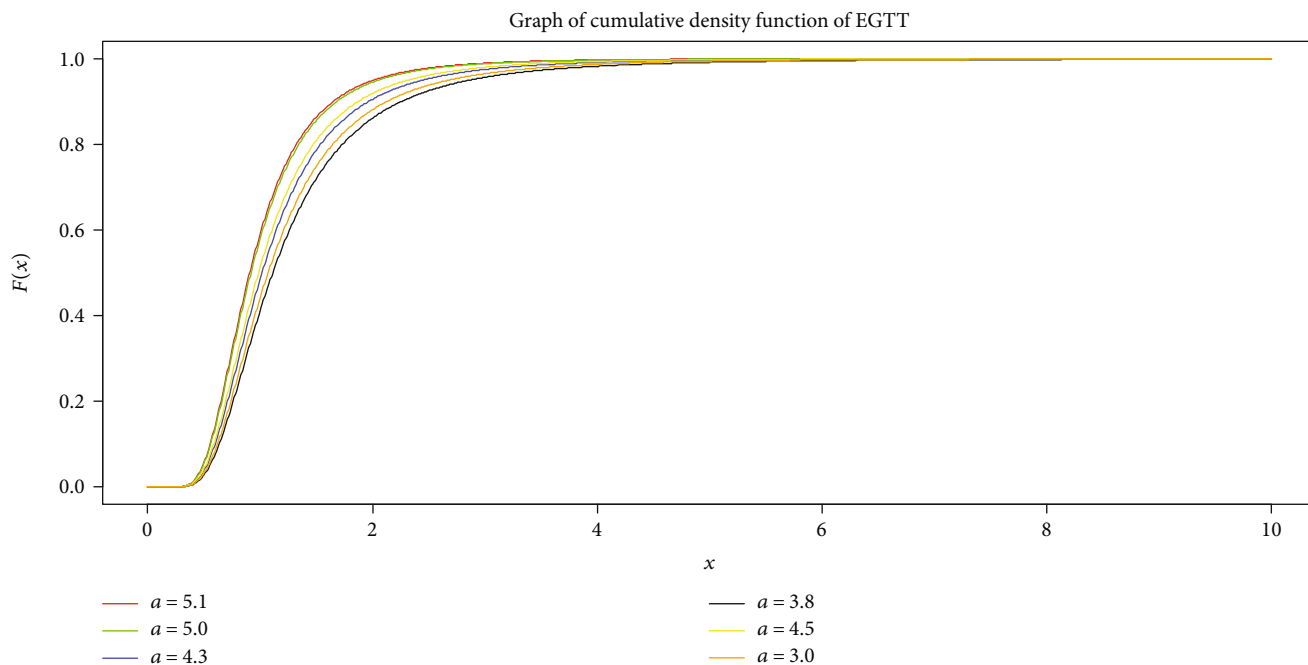$$(1 - P)^{a-1} = \sum_{i=0}^{\infty} (-1)^i \binom{a-1}{i} P^i. \tag{9}$$

Figure 1: This illustration indicates that the EGTT has a true pdf.

(a) $b = 3.3, v = 1.0$, and $\alpha = 1.2$
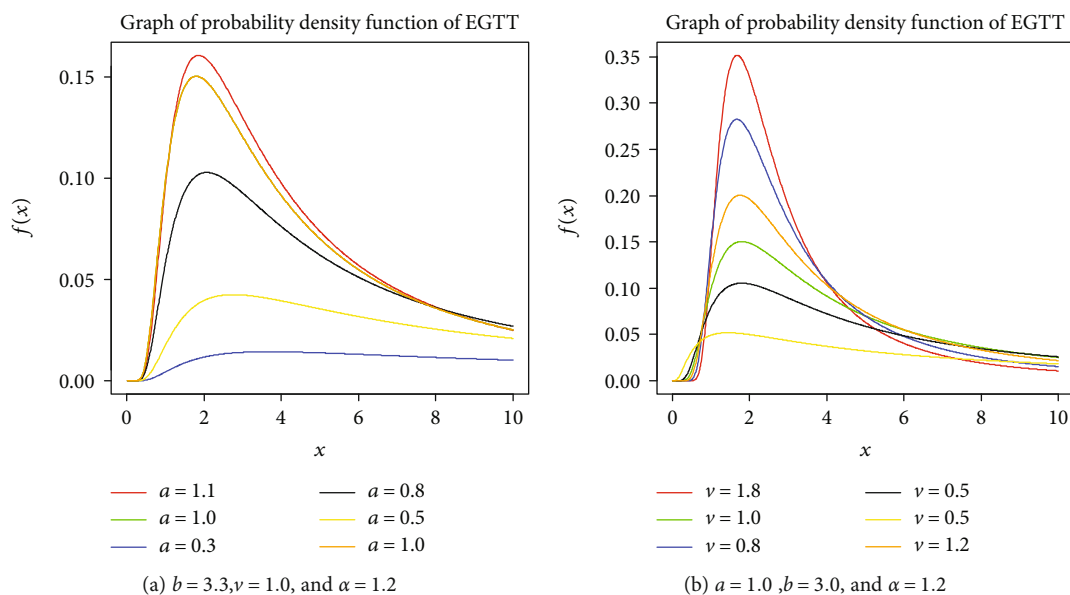
(b) $a = 1.0$, $b = 3.0$, and $\alpha = 1.2$

Figure 2: The graphs of the pdf of EGTT distribution.

Applying (9) in (4) and using the binomial expansion for a positive real power yield

$$g(x) = ab\alpha v x^{-(v+1)} \sum_{i,j=0}^{\infty} \sum_{k=0}^{aj} (-1)^{i+j+k}$$
$$\cdot \binom{a-1}{i} \binom{b-1}{j} \binom{aj}{k} e^{-\alpha x^{-v}(i+k+1)}. \quad (10)$$

Equation (10) can be rewritten as

$$g(x) = \hbar_{ijk} ab\alpha v x^{-(v+1)} e^{-\alpha x^{-v}(i+k+1)}. \quad (11)$$

Thus, we have

$$g(x) = \hbar_{ijk}(a,b) j(x \, ; i+k+1), \quad (12)$$
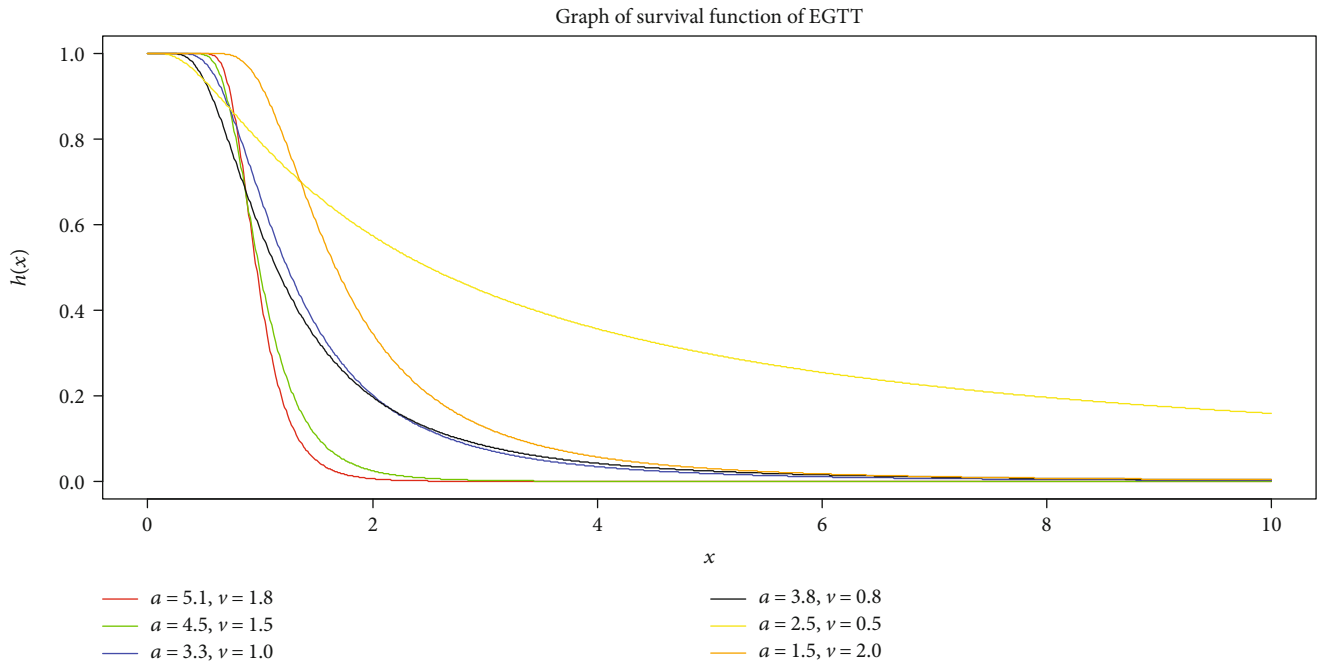
Figure 3: The graph of the reliability function of EGTT distribution.

(a) $b = 3.3$, $v = 1.0$, and $\alpha = 1.2$
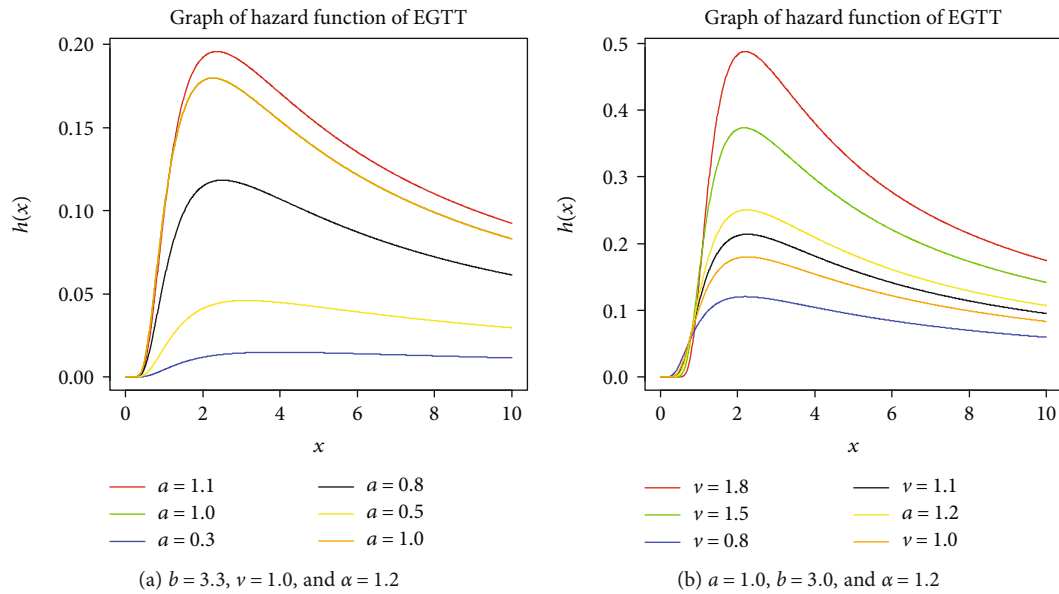
(b) $a = 1.0$, $b = 3.0$, and $\alpha = 1.2$

Figure 4: The graphs of the hazard function of EGTT distribution.

where

$$
\text{❓}_{ijk} = \sum_{i,j=0}^{\infty} \sum_{k=0}^{aj} (-1)^{i+j+k} \binom{a-1}{i} \binom{b-1}{j} \binom{aj}{k},
$$

(13)

are the weights and $j(x; i + k + 1)$ is the pdf of a Gumbel type-two distribution with a scale parameter of $(x; i + k + 1)$. Consequently, the Extended Gumbel type-two den-sity can be written as a linear combination of Gumbel type-two density functions. The mathematical properties of the EGTT follow directly from those of the GTT distribution.

3. Statistical Properties

Here, we study the statistical properties of the (EGTT) distribution, specifically quantile function, moments, and moment-generating function.

TABLE 1: Table of values of the Bowley skewness and the Moors kurtosis.

a, α, b, v	$X\left(\frac{1}{4}\right)$	$X\left(\frac{1}{2}\right)$	$X\left(\frac{3}{4}\right)$	$X\left(\frac{1}{8}\right)$	$X\left(\frac{3}{8}\right)$	$X\left(\frac{5}{8}\right)$	$X\left(\frac{7}{8}\right)$	S	κ
1.5, 0.5, 1.5, 2.0	1.0710	1.1710	1.3051	1.0144	1.1204	1.2292	1.4299	0.1451	0.4046
0.5, 0.5, 1.5, 0.3	1.2212	1.4723	1.9096	1.7044	1.3334	1.6472	2.43025	0.2705	1.6761
0.15, 0.25, 0.5, 0.25	0.8836	1.2105	2.2315	0.7751	1.0167	1.5399	4.4661	0.5149	1.9916
0.1, 0.05, 0.15, 0.1	0.4847	0.5646	0.7533	0.4482	0.5209	0.6283	1.2075	0.4056	1.8866

TABLE 2: Values for the first four moments, Var, and CV of EGTT distribution.

a, α, b, v	1st	2nd	3rd	4th	Var	CV
2.0, 2.0, 1.5, 3.0	1.3783	2.0778	3.3558	7.2449	0.1781	30.6188
2.5, 2.5, 5.5, 2.0	2.3227	6.1489	19.8775	95.5899	0.7539	106.7590
2.5, 2.5, 2.5, 3.0	1.5135	2.4285	4.1898	7.9541	0.1378	102.9643
2.5, 2.5, 2.5, 2.5	1.6252	2.9912	6.0721	14.6294	0.2515	104.4894

3.1. Quantile Function and Median.
The quantile function corresponding to (4) is given by

$$Q(u) = G^{-1}(u) = \left\{ -\frac{1}{\alpha} \left[\log \left(1 - \left[1 - u^{1/b} \right]^{1/a} \right) \right] \right\}^{-1/v}. \tag{14}$$

Taking U to be a uniform variate on the unit interval $(0, 1)$. Thus, by means of the inverse transformation method, we consider the random variable X given by

$$X(u) = \left\{ -\frac{1}{\alpha} \left[\log \left(1 - \left[1 - u^{1/b} \right]^{1/a} \right) \right] \right\}^{-1/v}. \tag{15}$$

The median in particular can be derived by taking the value of $u = 0.5$ in equation (14); then, we have

$$Q\left(\frac{1}{2}\right) = \left\{ -\frac{1}{\alpha} \left[\log \left(1 - \left[1 - \left(\frac{1}{2}\right)^{1/b} \right]^{1/a} \right) \right] \right\}^{-1/v}. \tag{16}$$

Expression for the lower quartiles and upper quartiles can also be developed by taking the value of u to be 1/4 and 3/4, respectively.

Invariably, the quantile function for certain fractile values provides an alternative measure of skewness and kurtosis such that the limitations of the classical measures are well known. The Bowley skewness due to Bowley [20] is given by

$$S = \frac{X(3/4) + X(1/4) - 2X(1/2)}{X(3/4) - X(1/4)}. \tag{17}$$

The Moors kurtosis due to Moors [21] is given by

$$\kappa = \frac{X(7/8) - X(5/8) - X(3/8) + X(1/8)}{X(6/8) - X(2/8)}. \tag{18}$$

The advantage of the Bowley skewness and the Moors kurtosis over the classical measures of skewness and kurtosis

is that they can be obtained even in situations where the moments of the distribution do not exist, and they are not reasonably affected by extreme values.

Table 1 gives the values of the Bowley skewness and the Moors kurtosis for various values of parameters.

From Table 1, it can be deduced that the EGTT distribution is positively skewed for various values of the parameters considered.

3.2. Moments.
Here, we consider the pth moment for (EGTT) distribution. Moments are important features in any statistical analysis, especially in applications. They can be used to study the characteristics of a distribution, e.g., dispersion, skewness, and kurtosis.

Theorem 1. If X has EGTT (a, b, α, v), then the pth moment of X is given by the following:

$$\mu_p' = ab\alpha \mathcal{K}_{ijk} [\alpha(i + j + k)]^{(p-v)/v} \Gamma\left(1 - \frac{p}{v} \right), \quad p < v. \tag{19}$$

Proof. Let X be a random variable with density function (11). The ordinary pth moment of the EGTT distribution is given by

$$\mu_p' = E(X^p) = \int_{-\infty}^{\infty} x^p f(x) dx = \mathcal{K}_{ijk} ab\alpha v \int_{-\infty}^{\infty} x^{p-(v-1)} e^{-\alpha x^{-v}(i+k+1)} dx, \tag{20}$$

$$\mu_p' = ab\alpha \mathcal{K}_{ijk} [\alpha(i + j + k)]^{(p-v)/v} \Gamma\left(1 - \frac{p}{v} \right). \tag{21}$$

This completes the proof.

Table 2 gives the first four moments, variance, and coefficient of variation (CV) for various values of the parameters of EGTT distribution.

Based on the first four moments of the EGTT distribution, the limitations of the classical kurtosis measure are known by the fact that some heavy-tailed distributions for which this measure is designed may be infinite. The

TABLE 3: Parameter estimate and standard error (parenthesis).

Distribution	Estimates			
EGTT	14.0571 (8.7149)	1.1494 (0.7270)	0.5629 (0.1486)	3.6017 (1.0458)
EGT	1.3020 (1.8073)	1.1747 (0.0844)	0.8173 (1.1345)	— (—)
GT	1.1760 (0.0842)	1.0709 (0.1328)	— (—)	— (—)

TABLE 4: Selection criterion statistics for pig data.

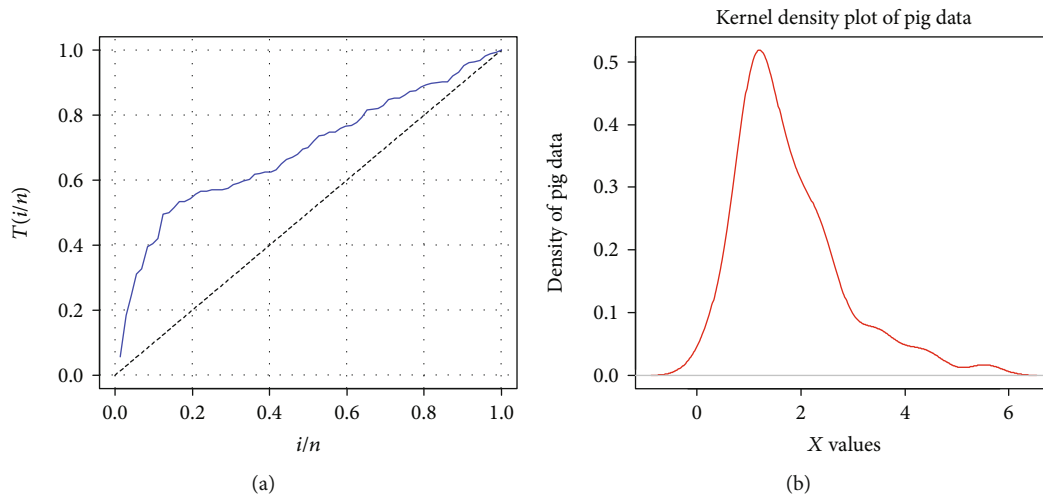| Distribution	$-l$	AIC	BIC	CAIC	HQIC	A^*	KS	P value
EGTT	98.063	204.126	213.331	204.723	207.758	0.8151	0.1192	0.2581
EGT	118.167	242.333	249.163	242.686	245.052	3.3560	0.1981	0.0070
GT	118.167	240.334	244.887	240.508	242.147	3.3656	0.1958	0.0080

(a)

(b)

FIGURE 5: Graph of TTT plot and the kernel density function of the cancer data.

Bowley skewness of Bowley [20] was based on quartiles of a distribution.

3.3. Moment-Generating Function. Here, we obtain an expression for the moment-generating function of (EGTT) distribution.

Theorem 2. *If X has (EGTT) distribution, then the moment-generating function $M_X(t)$ is given by*

$$M_x(t) = ab\alpha \sum_{P=0}^{\infty} \frac{t^p}{p!} \mathcal{K}_{ijk} [\alpha(i+j+k)]^{(p-v)/v} \Gamma\left(1 - \frac{p}{v}\right). \quad (22)$$

Proof. The moment-generating function of a random variable X can be obtained using a relation that exists between a moment and the moment-generating function given by

$$M_x(t) = \int_{p=0}^{\infty} \frac{t^p}{p!} E(x^p) dx. \quad (23)$$

By inserting equation (21) into (23), we complete the proof.

4. Order Statistics

Here, we derive closed form expressions for the pdf of the *i*th order statistic of (EGTT) distribution. Let $X_1, X_2, \cdots, X_n$ be a simple random sample from (EGTT) distribution with cdf and pdf given by (6) and (7), respectively. Let $x_{(1:n)} \leq x_{(2:n)} \leq \cdots \leq x_{(n:n)}$ denote the order statistics obtained from this sample. The probability density function of $x_{i:n}$ is given by

$$f_{i:n}(x) = \frac{1}{B(i, n-i+1)} [G(x, \psi)]^{i-1} [1 - G(x, \psi)]^{n-1} g(x, \psi), \quad (24)$$

where $G(x, \psi)$ and $g(x, \psi)$ are the cdf and the pdf of the EGGT distribution given in equations (6) and (7), respectively. Thus, we have

$$f_{i:n}(x) = \frac{ab}{B(i, n-i+1)} j(x)[1-J(x)]^{a-1} \left\{1-[1-J(x)]^{a-1}\right\}^{bi-1}$$
$$\cdot \left\{1 - [1 - \{1 - J(x)\}^a]\right\}^{n-1}. \tag{25}$$

Using the binomial series expansion in (9), $f_{i:n}(x)$ can be expressed as

$$f_{i:n}(x) = \sum_{k=0}^{n-i} (-1)^k \binom{n-i}{k} \frac{ab}{B(i, n-i+1)} j(x)[1-J(x)]^{a-1}$$
$$\cdot \left\{1 - [1 - J(x)]^a\right\}^{b(i+k)-1}. \tag{26}$$

Also applying (9) to the last term, we obtain

$$f_{i:n}(x) = \sum_{k=0}^{n-i} \sum_{l=0}^{\infty} (-1)^{k+l} \binom{n-i}{k} \binom{b(i+k)-1}{l} \frac{ab}{B(i, n-i+1)} j(x)$$
$$\cdot [1-J(x)]^{a(l+1)-1}. \tag{27}$$

Finally, we have

$$f_{i:n}(x) = \sum_{k=0}^{n-i} \sum_{l=0}^{\infty} \sum_{r=0}^{\infty} (-1)^{k+l+r} \binom{n-i}{k} \binom{b(i+k)-1}{l}$$
$$\cdot \binom{a(l+1)-1}{r} \frac{ab}{B(i, n-i+1)} j(x)[J(x)]^r,$$

$$f_{i:n}(x) = Z_{klr} \frac{ab}{B(i, n-i+1)} j(x)[J(x)]^r, \tag{28}$$

where

$$Z_{klr} = \sum_{k=0}^{n-i} \sum_{l=0}^{\infty} \sum_{r=0}^{\infty} (-1)^{k+l+r} \binom{n-i}{k} \binom{b(i+k)-1}{l}$$
$$\cdot \binom{a(l+1)-1}{r} \tag{29}$$

By substituting equations (1) and (2), we obtain the ith order statistic of the EGTT distribution.

5. Estimation and Inference

Here, we obtain the maximum likelihood estimates (MLEs) of the parameters of the EGTT distribution from complete samples only. Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from EGTT(ψ, x), $\psi = (a, b, \alpha, v)$. The log likelihood function for the vector of parameters $\psi = (a, b, \alpha, v)$ can be written as

$$\text{Log}L = n \log (ab\alpha v) + (a-1) \sum_{i=0}^{n} \log\left(1 - e^{-\alpha x_i^{-v}}\right)$$
$$+ (b+1) \sum_{i=0}^{n} \log\left(1 - \left(1 - e^{-\alpha x_i^{-v}}\right)^a\right) \tag{30}$$
$$- (v+1) \sum_{i-0}^{n} \log(x_i) - \alpha \sum_{i=0}^{n} x_i^v.$$

The log likelihood can be maximized either directly or by solving the nonlinear likelihood equations obtained by differentiating (30). The components of the score vector are given by

$$\frac{\partial l}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n} \log\left(1 - e^{-\alpha x_i^{-v}}\right) - (b-1) \sum_{i=1}^{n}$$
$$\cdot \left[\frac{\left(1 - e^{-\alpha x_i^{-v}}\right)^a \log\left(1 - e^{-\alpha x_i^{-v}}\right)}{[1 - (1 - e^{-\alpha x_i^{-v}})^a]}\right], \tag{31}$$

$$\frac{\partial l}{\partial b} = \frac{n}{b} + \sum_{i=0}^{n} \log\left(1 - \left(1 - e^{-\alpha x_i^{-v}}\right)^a\right) - \sum_{i=0}^{n} \log(x), \tag{32}$$

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} + (a-1) \sum_{i=0}^{n} \left[\frac{x_i^{-v} e^{-\alpha x_i^{-v}}}{1 - e^{-\alpha x_i^{-v}}}\right] + (b-1) \sum_{i=0}^{n}$$
$$\cdot \left[\frac{ax_i^{-v} e^{-\alpha x_i^{-v}} \left(1 - e^{-\alpha x_i^{-v}}\right)^a}{1 - (1 - e^{-\alpha x_i^{-v}})^a}\right] - \sum_{i=0}^{n} x_i^{-v}, \tag{33}$$

$$\frac{\partial l}{\partial v} = \frac{n}{v} - (a-1)\alpha \sum_{i=0}^{n} \left[\frac{x_i^{-v} \log(x) e^{-\alpha x_i^{-v}}}{1 - e^{-\alpha x_i^{-v}}}\right] + a(b-1)\alpha \sum_{i=0}^{n}$$
$$\cdot \left[\frac{x_i^{-v} \log(x) e^{-\alpha x_i^{-v}} \left(1 - e^{-\alpha x_i^{-v}}\right)^a}{1 - (1 - e^{-\alpha x_i^{-v}})^a}\right]. \tag{34}$$

We can obtain the estimates of the unknown parameters by the maximum likelihood method by setting these above nonlinear equations (31), (32), (33), and (24) to zero and solving them simultaneously. Therefore, statistical software can be employed in obtaining the numerical solution to the nonlinear equations such as R and MATLAB. For the four-parameter Extended Gumbel type-two pdf, all the second-order derivatives can be obtained. Thus, the inverse dispersion matrix is given by

$$\begin{pmatrix} \hat{a} \\ \hat{b} \\ \hat{\alpha} \\ \hat{v} \end{pmatrix} \sim N \left[\begin{pmatrix} a \\ b \\ \alpha \\ v \end{pmatrix} \begin{pmatrix} \hat{U}_{aa} & \hat{U}_{ab} & \hat{U}_{a\alpha} & \hat{U}_{av} \\ \hat{U}_{ba} & \hat{U}_{bb} & \hat{U}_{b\alpha} & \hat{U}_{bv} \\ \hat{U}_{\alpha a} & \hat{U}_{\alpha b} & \hat{U}_{\alpha\alpha} & \hat{U}_{\alpha v} \\ \hat{U}_{va} & \hat{U}_{vb} & \hat{U}_{v\alpha} & \hat{U}_{vv} \end{pmatrix} \right],$$

$$U^{-1} = -E \begin{pmatrix} U_{aa} & U_{ab} & U_{a\alpha} & U_{av} \\ U_{ba} & U_{bb} & U_{b\alpha} & U_{bv} \\ U_{\alpha a} & U_{\alpha b} & U_{\alpha\alpha} & U_{\alpha v} \\ U_{va} & U_{vb} & U_{v\alpha} & U_{vv} \end{pmatrix},$$
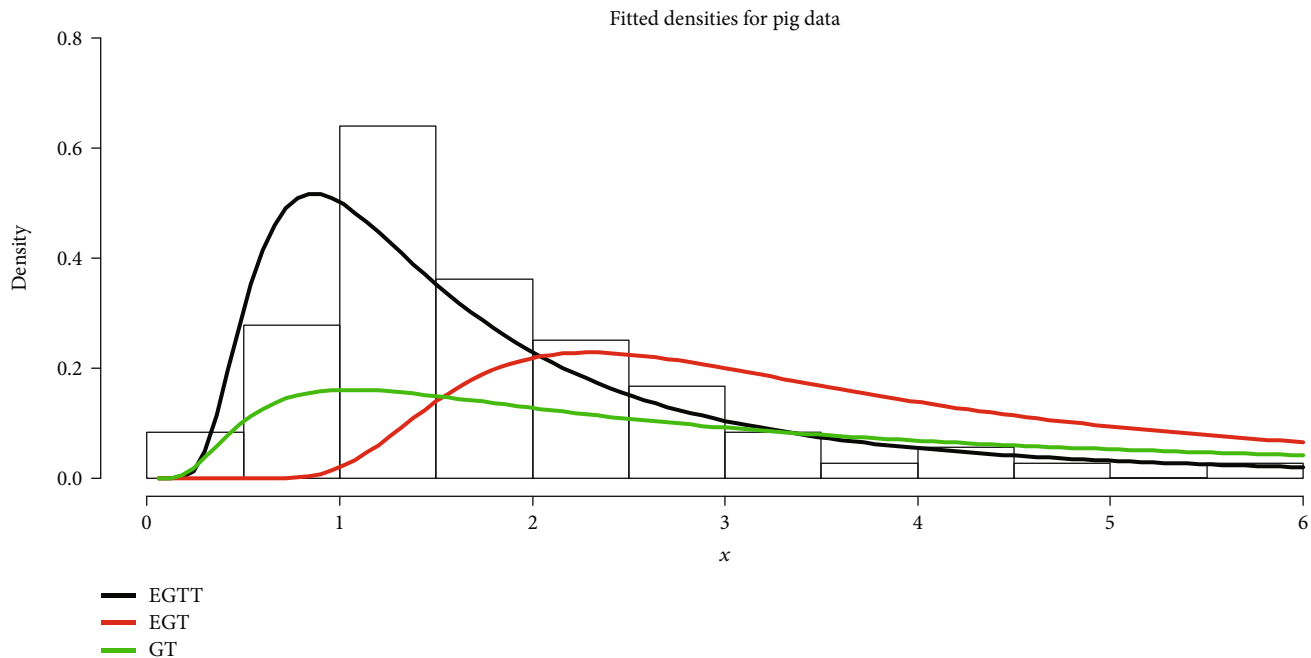
$$\tag{35}$$

Figure 6: Fitted densities for pig data.

Table 5: Parameter estimate and standard error (parenthesis) for cancer data.

Distribution	MLE estimates			
EGGT	11.7689 (6.4231)	0.8918 (0.2526)	0.3674 (0.0641)	5.5657 (1.0114)
EGT	2.2915 (1.9970)	0.7512 (0.0425)	1.0551 (0.9195)	— (—)
GT	0.7528 (0.0424)	2.4389 (0.2200)	— (—)	— (—)

Table 6: Selection criterion statistics for cancer data.

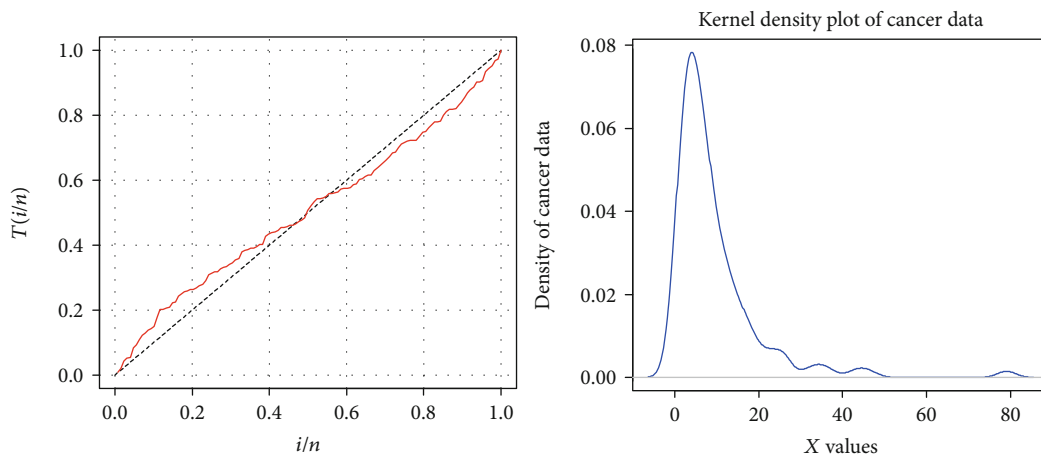| Distribution	$-l$	AIC	BIC	CAIC	HQIC	A^*	KS	P value
EGTT	415.856	839.712	851.120	840.037	844.347	0.8731	0.0828	0.3434
EGT	444.003	894.005	902.561	894.199	897.482	4.4534	0.1404	0.0129
GT	118.167	892.003	897.707	892.099	894.321	4.5548	0.1410	0.0124

Figure 7: Graph of TTT plot and the kernel density function of the cancer data.
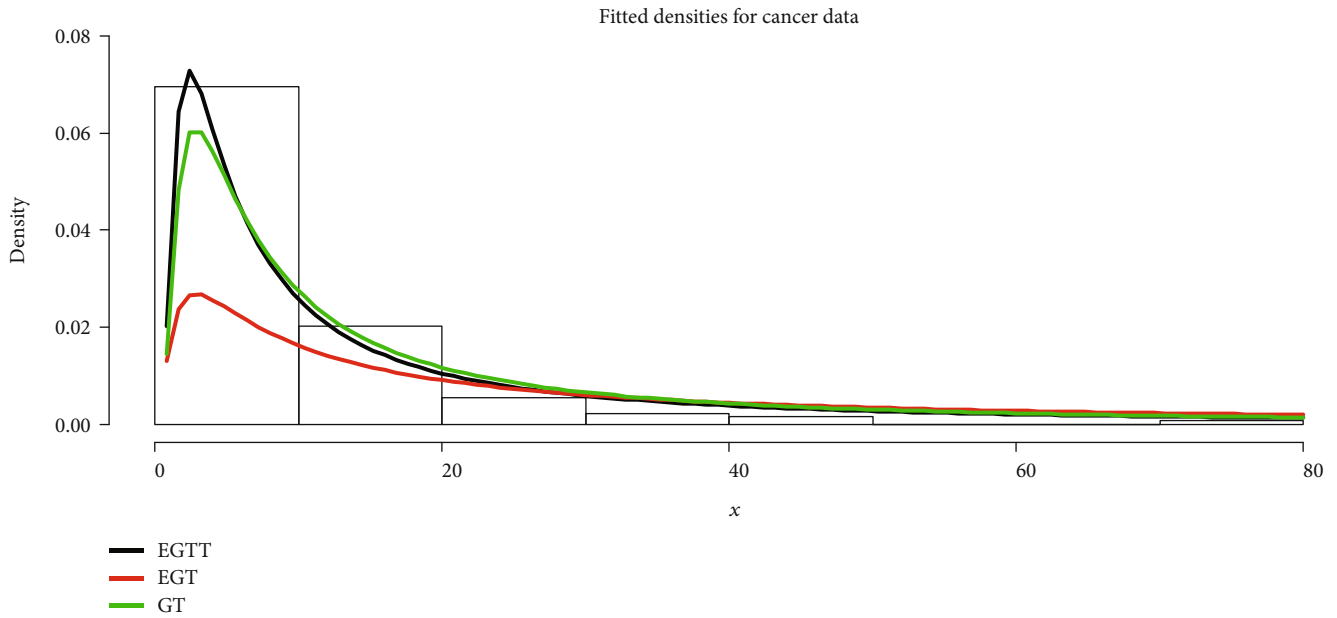
FIGURE 8: Fitted densities of the cancer data.

where $\widehat{U}_{ij} = U_{ij} \Big|_{w = w_i} = (a, b, \alpha, \eta)$ with $\{J_{ij}\} = [-l_{ij}]^{-1} = [\partial^2 l / \partial w_i \partial w_j]$. This gives the approximate variance covariance matrix. By solving for the inverse of the dispersion matrix, the solution will give the asymptotic variance and covariance of the MLs for $\widehat{a}$, $\widehat{b}$, $\widehat{\alpha}$, and $\widehat{v}$. The approximate $100(1 - v)\%$ confidence intervals for a, b, α, and, η can be obtained, respectively, as

$$
\begin{aligned}
&\widehat{a} \pm z_{(\eta/2)} \sqrt{\widehat{U}_{aa}}, \\
&\widehat{b} \pm z_{(\eta/2)} \sqrt{\widehat{U}_{bb}}, \\
&\widehat{\alpha} \pm z_{(\eta/2)} \sqrt{\widehat{U}_{\alpha\alpha}}, \\
&\widehat{v} \pm z_{(\eta/2)} \sqrt{\widehat{U}_{vv}}.
\end{aligned}
\tag{36}
$$

5.1. Percentile Estimator (PE). Let $X_1, X_2, \cdots, X_n$ be a random sample for the EGTT distribution and let $x_{(1)}, x_{(2)}, \cdots, x_{(n)}'$ be the corresponding order statistics. Based on the PR method of estimation technique, the estimators of the set of parameters $\mathscr{Z} = (a, b, \alpha, v)^T$ can be obtained by minimizing the following:

$$
\sum_{i-1}^{n} \left[\ln \left(P_i - \ln \left\{ \left[1 - \left\{ 1 - \exp \left(-\alpha x^{-v} \right\}^a \right]^b \right\} \right) \right) \right]^2,
\tag{37}
$$

with respect to $\mathscr{Z}$, where P_i denotes some estimates of $G(x_i, \mathscr{Z})$, and $P_i = i/n + 1$.

5.2. The Cramer-von Mises Minimum Distance Estimators. The CV estimator is a type of minimum distance estimator which is based on the difference between the estimate of the cdf and the empirical cdf, as shown by D'Agostino and Stephens [22] and Luceno [23]. The CV estimators are obtained by minimizing

$$
C(\mathscr{Z}) = \frac{1}{12n} + \sum_{i=1}^{n} \left[\left\{ \left[1 - \left\{ 1 - \exp \left(-\alpha x^{-v} \right\}^a \right]^b \right\} - \frac{2i-1}{2n} \right]^2.
\tag{38}
$$

D'Agostino and Stephens [22] mentioned that the choice of CV method type minimum distance estimators provides the empirical evidence that the bias of the estimator is smaller than the other minimum distance estimators.

6. Applications to Lifetime Data

Here, we present two examples that demonstrate the flexibility and the applicability of the EGTT distribution in modelling real-world data. We fit the density functions of the EGTT distribution and its submodels such as the Exponentiated Gumbel type-two (EGTT) distribution and the Gumbel type-two (GTT) distribution. The first data set represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal [24]. The starting point of the iterative processes for the guinea pig data set is ($1 : 0$; $0 : 009$; $10 : 0$; $0 : 1$; and $0 : 1$). For the survival times (in days) of guinea pigs infected with virulent tubercle bacilli, the following data are given: 0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, and 5.55. Table 3 gives the MLEs, and Table 4 gives the selection criterion statistics for the pig data.

Figure 5 gives the graph of the total time on test plot and the graph of the kernel density of the pig data. Figure 6 gives the fitted densities of the pig data.

The element of the information matrix is given by

$$U^{-1} = \begin{pmatrix} 2.959117e-04 & -1.797274e-03 & -1.771022e-07 & -5.468235e-06 \\ -1.797274e-03 & 5.567826e-05 & -9.014037e-06 & -3.564958e-06 \\ -1.771022e-07 & -5.167826e-05 & 9.014037e-06 & -3.564958e-06 \\ -5.468235e-06 & -5.658267e-04 & -3.564958e-06 & 9.666310e-03 \end{pmatrix}. \tag{39}$$

6.1. Cancer Remission Time Data. The second data set consists of data of cancer patients. The data represents the remission times (in months) of a random sample of 128 bladder cancer patients from Lee and Wang [25]. The starting point of the iterative processes for the cancer patient data set is $(1:0; 0:009; 10:0; 0:1; 0:1)$. The following data are given: 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, and 22.69. Table 5 gives the MLEs, and Table 6 gives the selection criterion statistics for the cancer data. Figure 7 gives the graph of the total time on test plot and the graph of the kernel density of the cancer data. Figure 8 gives the fitted densities to the cancer data.

The element of the information matrix is given by

$$U^{-1} = \begin{pmatrix} 6.618289e-09 & 1.186986e-05 & 1.903495e-06 & -6.504090e-06 \\ 1.186986e-05 & 3.565997e-04 & 6.052337e-05 & -2.096251e-04 \\ 1.903495e-06 & 6.052337e-05 & 4.449735e-04 & -3.880824e-04 \\ -6.504090e-06 & -2.096251e-04 & -3.880824e-04 & 9.585634e-03 \end{pmatrix}. \tag{40}$$

We employ statistical tools for model comparison which includes Kolmogorov-Smirnov (KS) statistics, Anderson Darling (A^*) statistics, probability value (P value), Akaike Information Criterion (AIC), Hannan-Quinine Information Criterion (HQIC), Consistent Akaike Information Criterion (CAIC), and Bayesian Information Criterion (BIC). The selection procedure will be based on the lowest AIC, BIC, CAIC, HQIC, A^*, and KS, and the largest P value will be considered as the best model.

7. Conclusion

In this work, we studied a four-parameter distribution named the Extended Gumbel type-two distribution which is an extension of the Gumbel type-two distribution. This work also provided several mathematical properties of the Extended Gumbel type-two distribution including explicit expressions for the density and quantile functions, ordinary moments, and order statistics. We employed the maximum likelihood estimation method to estimate the model parameters and also to obtain the elements of the information matrix. To evaluate the performance/flexibility of the new distribution, two real-life data applications which include data on cancer remission times of patients and also survival times of pigs clearly illustrate the potential of the Extended Gumbel type-2 distribution in fitting the two survival data because it possesses the lowest AIC, BIC, CAIC, HQIC, A^*, and KS as well as the largest P value.

Data Availability

The data used for this research are commonly used in the area of research.

Additional Points

Contribution to Knowledge. This work has been able to provide the authors a more flexible distribution of work, most especially in modelling real-life data which include failure rate increases and decreases and nonmonotonic failure. This is often encountered in real-life phenomenon, where failure is considered throughout the entire life span.

Conflicts of Interest

The authors declare that they have no conflicts of interests.

Acknowledgments

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