Hindawi
International Journal of Differential Equations
Volume 2020, Article ID 6383916, 16 pages
https://doi.org/10.1155/2020/6383916

Research Article

Existence, Uniqueness, and Mittag–Leffler–Ulam Stability Results for Cauchy Problem Involving ψ-Caputo Derivative in Banach and Fréchet Spaces

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Received 6 July 2020; Revised 26 August 2020; Accepted 16 September 2020; Published 14 October 2020

Academic Editor: Mayer Humi

Our aim in this paper is to investigate the existence, uniqueness, and Mittag–Leffler–Ulam stability results for a Cauchy problem involving ψ-Caputo fractional derivative with positive constant coefficient in Banach and Fréchet Spaces. The techniques used are a variety of tools for functional analysis. More specifically, we apply Weissinger's fixed point theorem and Banach contraction principle with respect to the Chebyshev and Bielecki norms to obtain the uniqueness of solution on bounded and unbounded domains in a Banach space. However, a new fixed point theorem with respect to Meir–Keeler condensing operators combined with the technique of Hausdorff measure of noncompactness is used to investigate the existence of a solution in Banach spaces. After that, by means of new generalizations of Grönwall's inequality, the Mittag–Leffler–Ulam stability of the proposed problem is studied on a compact interval. Meanwhile, an extension of the well-known Darbo's fixed point theorem in Fréchet spaces associated with the concept of measures of noncompactness is applied to obtain the existence results for the problem at hand. Finally, as applications of the theoretical results, some examples are given to illustrate the feasibility of the main theorems.

1. Introduction

Fractional differential equations gained much attention due to their applications in various fields of science and engineering (see, for instance, [1–7] and the references therein). For more information about the basic theory of fractional differential equations, we can refer to the monographs [8–11] and references cited therein. Besides the classical and fractional-order differential and integral operators, there is another kind of fractional derivatives that appears in the literature called ψ-Caputo fractional derivative, which was introduced by Almeida in [12], where the kernel operator contains a special function of an arbitrary exponent. According to this idea, a wide class of well-known fractional derivatives are obtained like Caputo and Caputo–Hadamard for particular choices of $\psi(t)$. Additionally, some interesting details about the ψ-fractional derivatives and integrals can be found in [13–23]. Moreover, fixed point theory is a very useful tool in the theory of the existence of solutions to functional and differential equations; the reader is advised to see references [24–29] in which many scholars turned to the existence and uniqueness of solutions for differential equations involving different kinds of fractional derivatives under various boundary conditions. On the contrary, the notion of measure of noncompactness was first introduced by Kuratowski [30] in 1930 which was further extended to general Banach spaces by Banás and Goebel [31]. Later Darbo formulated his celebrated fixed point theorem in 1955 for the case of the Kuratowski measure of noncompactness (cf. [32]) which generalizes both the classical Schauder fixed point principle and (a special variant of) Banach's contraction mapping principle. After that, the Darbo fixed point theorem has been generalized in many different directions; we suggest some works for reference [33–36]. The reader may also consult [37–43] and references therein where several applications of the measure of noncompactness can be found.

Very recently Dudek [44] proved a new fixed point theorem using the concept of measures of noncompactness in Fréchet spaces which generalize the famous Darbo's fixed point theorem. To see more applications about the usefulness of this new fixed point theorem to prove the existence of solutions for certain classes of functional integral equations in Fréchet spaces, the reader can refer to [45–50].

On the contrary, one of the important parts of the qualitative theory of linear and nonlinear differential equations is the Ulam–Hyers stability, first formulated by Hyers and Ulam in 1940 [51–53]. Furthermore, the fractional Ulam stability was introduced by Wang et al. [54]. For some recent results of stability analysis by different types of fractional derivative operator, we refer the reader to articles [55–64], as well as to the recent book by Abbas et al. [65] and the references cited therein. More recently, some authors explored another form of stability known as Mittag–Leffler–Ulam–Hyers for the solutions of fractional differential equations [66–73].

Inspired by the above works, our goal is to extend the studies in [29, 37, 45, 72]. More precisely, we consider first the problem of the existence, uniqueness, and Mittag-Leffler–Ulam–Hyers stability for the following initial value problem of the fractional differential equation with constant coefficient $\lambda > 0$ in Banach spaces of the form:

$$\begin{cases} {}^{c}\mathscr{D}_{a^+}^{\alpha;\psi} x(t) = \lambda x(t) + f(t, x(t)), & t \in J := [a,b], \\ x(a) = \phi_0, \end{cases} \tag{1}$$

where ${}^{c}\mathscr{D}_{a^+}^{\alpha;\psi}$ is the ψ-Caputo fractional derivatives such that $0 < \alpha \leq 1$, $f\colon J \times \mathbb{X} \longrightarrow \mathbb{X}$ is a given function satisfying some assumptions that will be specified later, $\mathbb{X}$ is a Banach space with norm $\|\cdot\|$, and $\phi_0 \in \mathbb{X}$. Moreover, we also extend the above problem to give a uniqueness results on unbounded domains in a Banach space via Banach contraction principle coupled with Bielecki-type norm.

Next, we turn our attention to the existence of solutions for the same problem (1) in the Fréchet spaces. In precise terms, we investigate the existence of solutions for the following problem:

$$\begin{cases} {}^{c}\mathscr{D}_{a^+}^{\alpha;\psi} x(t) = \lambda x(t) + f(t, x(t)), & t \in J' := [a,\infty), \\ x(a) = \phi_0. \end{cases} \tag{2}$$

The structure of the present work is organized as follows: in Section 2, we collect some basic concepts on the fractional integrals and derivatives, auxiliary results, lemmas and notions of measures of noncompactness, and fixed point theorems that are used throughout this paper. In Section 3, based on Weissinger's fixed point theorem combined with the Chebyshev norm, we give a uniqueness result for problem (1) on a compact interval in a Banach space. In Section 4, using the ideas of Hausdorff measure of noncompactness and Meir–Keeler condensing operator, we present the existence of solutions of IVP (1) in Banach spaces. In Section 5, we discuss the Mittag–Leffler–Ulam stability results for the problem at hand. In Section 6, we apply the Banach fixed point theorem coupled with a Bielecki-type norm to derive the uniqueness of solution on unbounded domains in a Banach space. In Section 7, we look into the existence of solutions for the IVP (2) in the Fréchet spaces via Darbo's fixed point theorem. The last section provides a couple of examples to illustrate the applicability of the results developed.

2. Preliminaries and Background Materials

In this section, we present some basic notations, definitions, and preliminary results, which will be used throughout this paper.

Let $J := [a,b]\,(0 < a < b < \infty)$ be a finite interval and $\psi\colon J \longrightarrow \mathbb{R}$ be an increasing function with $\psi'(t) \neq 0$, for all $t \in J$, and let $C(J,\mathbb{X})$ be the Banach space of all continuous functions x from J into $\mathbb{X}$ with the supremum (uniform) norm:

$$\|x\|_\infty = \sup_{t\in J} \|x(t)\|. \tag{3}$$

A measurable function $x\colon J \longrightarrow \mathbb{X}$ is Bochner integrable if and only if $\|x\|$ is Lebesgue integrable.

By $L^1(J,\mathbb{X})$, we denote the space of the Bochner integrable functions $x\colon J \longrightarrow \mathbb{X}$, with the norm

$$\|x\|_1 = \int_a^b \|x(t)\| \mathrm{d}t. \tag{4}$$

Now, we define the Hausdorff measure of noncompactness and give some of its important properties.

Definition 1 (see [31]). Let $\mathbb{X}$ be a Banach space and B a bounded subset of $\mathbb{X}$. Then the Hausdorff measure of noncompactness of B is defined by

$$\chi(B) = \inf\{\varepsilon > 0\colon B \text{ can be covered by finitely many balls with radius} < \varepsilon\}. \tag{5}$$

To discuss the problem in this paper, we need the following lemmas.

Lemma 1. *Let $A, B \subset \mathbb{X}$ be bounded. Then the Hausdorff measure of noncompactness has the following properties. For more details and the proof of these properties see [31]:*

(1) $\chi(A) = 0 \Longleftrightarrow A$ *is relatively compact*

(2) $A \subset B \Longrightarrow \chi(A) \leq \chi(B)$

(3) $\chi(A \cup B) = \max\{\chi(A), \chi(B)\}$

(4) $\chi(A) = \chi(\overline{A}) = \chi(conv(A))$, *where* $\overline{A}$ *and* $convA$ *represent the closure and the convex hull of* A, *respectively*

(5) $\chi(A + B) \leq \chi(A) + \chi(B)$, *where* $A + B = \{x + y : x \in A, y \in B\}$

(6) $\chi(\beta A) \leq |\beta| \chi(A)$, *for any* $\beta \in \mathbb{R}$

Now, we recall some fixed point theorems that will be used later

Theorem 1 (Weissinger's fixed point theorem [74]). *Assume (E, d) to be a nonempty complete metric space and let $\beta_j \geq 0$ for every $j \in \mathbb{N}$ such that $\sum_{j=0}^{\infty} \beta_j$ converges. Furthermore, let the mapping $T: E \longrightarrow E$ satisfy the following inequality:*

$$d\left(T^j u, T^j v\right) \leq \beta_j d(u, v), \quad (6)$$

for every $j \in \mathbb{N}$ and every $u, v \in E$. Then, $\mathbb{T}$ has a unique fixed point u^*. Moreover, for any $v_0 \in E$, the sequence $\{\mathbb{T}^j v_0\}_{j=1}^{\infty}$ converges to this fixed point u^*.

On the contrary, in 1969, the concepts of the Meir–Keeler contraction mapping were introduced by Meir and Keeler.

Definition 2 (see [75]). Let (E, d) be a metric space. Then a mapping $\mathcal{T}$ on E is said to be a Meir–Keeler contraction (MKC, for short); if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq d(u, v) < \varepsilon + \delta \Longrightarrow d(\mathcal{T}u, \mathcal{T}v) < \varepsilon, \quad \forall u, v \in E. \quad (7)$$

In [34], the authors defined the notion of the Meir–Keeler condensing operator on a Banach space and gave some fixed point results.

Definition 3 (see [34]). Let C be a nonempty subset of a Banach space $\mathbb{X}$ and μ arbitrary measure of noncompactness on $\mathbb{X}$. We say that an operator $\mathcal{T}: C \longrightarrow C$ is a Meir–Keeler condensing operator if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq \mu(\Omega) < \varepsilon + \delta \Longrightarrow \mu(\mathcal{T}\Omega) < \varepsilon, \quad (8)$$

for any bounded subset Ω of C.

The following fixed point theorem with respect to the Meir–Keeler condensing operator which is introduced by Aghajani et al. [34] plays a key role in the proof of our main results.

Theorem 2 (see [34]). *Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space $\mathbb{X}$. Also, let μ be an arbitrary measure of noncompactness on $\mathbb{X}$. If $\mathcal{T}: \Omega \longrightarrow \Omega$ is a continuous and Meir–Keeler condensing operator, then $\mathcal{T}$ has at least one fixed point and the set of all fixed points of $\mathcal{T}$ in Ω is compact.*

The following lemmas are needed in our argument.

Lemma 2 (see [76]). *Let $\mathbb{X}$ be a Banach space. If $B \subset C(J, \mathbb{X})$ is bounded, then $\chi(B(t)) \leq \chi_C(B)$ for any $t \in J$, where $B(t) = \{x(t): x \in B\}$, $t \in J$, and χ_C is the Hausdorff measure of noncompactness defined on the bounded sets of $C(J, \mathbb{X})$. Furthermore if B is equicontinuous, then $t \longrightarrow \chi(B(t))$ is continuous on J, and*

$$\chi_C(B) = \max_{t \in J} \chi(B(t)). \quad (9)$$

Lemma 3 (see [77]). *Let $\mathbb{X}$ be a Banach space and let $B \subset \mathbb{X}$ be bounded. Then for each ε, there is a sequence $\{x_n\}_{n=1}^{\infty} \subset B$, such that*

$$\chi(B) \leq 2\chi(\{x_n\}_{n=1}^{\infty}) + \varepsilon. \quad (10)$$

We call $B \subset L^1(J, \mathbb{X})$ uniformly integrable if there exists $\eta \in L^1(J, \mathbb{R}^+)$ such that

$$\|x(t)\| \leq \eta(t), \quad \text{for all } x \in B \text{ and a.e. } t \in J. \quad (11)$$

Lemma 4 (see [78]). *If $\{x_n\}_{n=1}^{\infty} \subset L^1(J, \mathbb{X})$ is uniformly integrable, then $t \longmapsto \chi(\{x_n(t)\}_{n=1}^{\infty})$ is measurable, and*

$$\chi\left(\left\{\int_a^t x_n(s)\mathrm{d}s\right\}_{n=1}^{\infty}\right) \leq 2\int_a^t \chi(\{x_n(s)\}_{n=1}^{\infty})\mathrm{d}s. \quad (12)$$

Before introducing the basic facts on fractional operators, we recall three types of functions that are important in fractional calculus: the gamma, beta, and Mittag–Leffler functions

Definition 4 (see [79]). The gamma function, or the second-order Euler integral, denoted $\Gamma(\cdot)$ is defined as

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-t} t^{\alpha-1}\mathrm{d}t, \quad \alpha > 0. \quad (13)$$

Definition 5 (see [79]). The beta function, or the first-order Euler function, can be defined as

$$B(\alpha, \beta) = \int_0^1 (1-t)^{\alpha-1} t^{\beta-1}\mathrm{d}t, \quad \alpha, \beta > 0. \quad (14)$$

We use the following formula which expresses the beta function in terms of the gamma function:

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad \alpha, \beta > 0. \quad (15)$$

The next function is a direct generalization of the exponential series.

Definition 6 (see [79]). The one-parameter Mittag–Leffler function $\mathbb{E}_\alpha(\cdot)$ is defined as

$$\mathbb{E}_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad (z \in \mathbb{R}, \alpha > 0). \quad (16)$$

For $\alpha = 1$, this function coincides with the series expansion of e^z, i.e.,

$$\mathbb{E}_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z. \quad (17)$$

Definition 7 (see [79]). The two-parameter Mittag–Leffler function $\mathbb{E}_{\alpha,\beta}(\cdot)$ is defined as

$$\mathbb{E}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \ \alpha, \beta > 0 \text{ and } z \in \mathbb{R}. \quad (18)$$

Now, we give some results and properties from the theory of fractional calculus. We begin by defining ψ-Riemann–Liouville fractional integrals and derivatives, in what follows.

Definition 8 (see [2, 12]). For $\alpha > 0$, the left-sided ψ-Riemann–Liouville fractional integral of order α for an integrable function $x\colon J \longrightarrow \mathbb{R}$ with respect to another function $\psi\colon J \longrightarrow \mathbb{R}$ that is an increasing differentiable function such that $\psi'(t) \neq 0$, for all $t \in J$, is defined as follows:

$$\mathcal{I}_{a^+}^{\alpha;\psi} x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} x(s) \mathrm{d}s, \quad (19)$$

where Γ is the gamma function.

Note that equation (19) is reduced to the Riemann–Liouville and Hadamard fractional integrals when $\psi(t) = t$ and $\psi(t) = \ln t$, respectively.

The integer order of the differential operator $x_{\psi}^{[1]}$ with respect to another function $\psi\colon J \longrightarrow \mathbb{R}$ that is an increasing differentiable function such that $\psi'(t) \neq 0$, for all $t \in J$ is defined by

$$x_{\psi}^{[1]}(t) = \frac{1}{\psi'(t)} \frac{d}{\mathrm{d}t} x(t). \quad (20)$$

Furthermore, for $n \in \mathbb{N}$, we use the symbol $x_{\psi}^{[n]}$ to indicate the n-th composition of $x_{\psi}^{[1]}$ with itself; that is, we put

$$x_{\psi}^{[n]}(t) \coloneqq x_{\psi}^{[1]} x_{\psi}^{[n-1]}(t) = \left(\frac{1}{\psi'(t)} \frac{\mathrm{d}}{\mathrm{d}t}\right)^n x(t), \quad n \geq 2. \quad (21)$$

Definition 9 (see [12]). Let $n \in \mathbb{N}$ and let $\psi, x \in C^n(J, \mathbb{R})$, be two functions such that ψ is increasing and $\psi'(t) \neq 0$, for all $t \in J$. The left-sided ψ–Riemann–Liouville fractional derivative of a function x of order α is defined by

$$\begin{aligned} \mathcal{D}_{a^+}^{\alpha;\psi} x(t) &= x_{\psi}^{[n]} \mathcal{I}_{a^+}^{n-\alpha;\psi} x(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\psi'(t)} \frac{\mathrm{d}}{\mathrm{d}t}\right)^n \\ &\quad \cdot \int_a^t \psi'(s)(\psi(t) - \psi(s))^{n-\alpha-1} x(s) \mathrm{d}s, \end{aligned} \quad (22)$$

where $n = [\alpha] + 1$.

Definition 10 (see [12]). Let $n \in \mathbb{N}$ and let ψ, $x \in C^n(J, \mathbb{R})$, be two functions such that ψ is increasing and $\psi'(t) \neq 0$, for all $t \in J$. The left-sided ψ-Caputo fractional derivative of x of order α is defined by

$${}^c\mathcal{D}_{a^+}^{\alpha;\psi} x(t) = \mathcal{I}_{a^+}^{n-\alpha;\psi} x_{\psi}^{[n]}(t), \quad (23)$$

where $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$ and $n = \alpha$ for $\alpha \in \mathbb{N}$.

From the definition, it is clear that

$${}^c\mathcal{D}_{a^+}^{\alpha;\psi} x(t) = \begin{cases} \displaystyle\int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{n-\alpha-1}}{\Gamma(n-\alpha)} x_{\psi}^{[n]}(s) \mathrm{d}s, & \text{if } \alpha \notin \mathbb{N}, \\ x_{\psi}^{[n]}(t), & \text{if } \alpha \in \mathbb{N}. \end{cases} \quad (24)$$

This generalization (24) yields the Caputo fractional derivative operator when $\psi(t) = t$. Moreover, for $\psi(t) = \ln t$, it gives the Caputo–Hadamard fractional derivative.

Some basic properties are listed in the following lemma.

Lemma 5 (see [2, 12]). *Let $\alpha, \beta > 0$, and $x \in C(J, \mathbb{R})$. Then for each $t \in J$, we have*

(1) ${}^c\mathcal{D}_{a^+}^{\alpha;\psi} \mathcal{I}_{a^+}^{\alpha;\psi} x(t) = x(t)$

(2) $\mathcal{I}_{a^+}^{\alpha;\psi} {}^c\mathcal{D}_{a^+}^{\alpha;\psi} x(t) = x(t) - x(a)$, $0 < \alpha \leq 1$

(3) $\mathcal{I}_{a^+}^{\alpha;\psi} (\psi(t) - \psi(a))^{\beta-1} = (\Gamma(\beta)/\Gamma(\beta+\alpha))(\psi(t) - \psi(a))^{\beta+\alpha-1}$

(4) ${}^c\mathcal{D}_{a^+}^{\alpha;\psi} (\psi(t) - \psi(a))^{\beta-1} = (\Gamma(\beta)/\Gamma(\beta-\alpha))(\psi(t) - \psi(a))^{\beta-\alpha-1}$

(5) ${}^c\mathcal{D}_{a^+}^{\alpha;\psi} (\psi(t) - \psi(a))^k = 0$, *for all* $k \in \{0, \ldots, n-1\}$, $n \in N$

Remark 1. Note that for an abstract function $x\colon J \longrightarrow \mathbb{X}$, the integrals which appear in the previous definitions are taken in Bochner's sense (see, for instance, [80]).

In the sequel, we will make use of the following generalizations of Grönwall's lemmas

Theorem 3 (see [23]). *Let u, v be two integrable functions and w continuous, with domain J. Let $\psi \in C^1(J, \mathbb{R}_+)$ be an increasing function such that $\psi'(t) \neq 0$, $\forall t \in J$. Assume that*

(1) u and v are nonnegative

(2) w is nonnegative and nondecreasing.

If

$$u(t) \leq v(t) + w(t) \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} u(s) \mathrm{d}s, \ t \in J, \quad (25)$$

then

$$u(t) \leq v(t) + \int_a^t \sum_{n=0}^{\infty} \frac{(w(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)} \psi'(s)(\psi(t) - \psi(s))^{n\alpha-1} v(s) \mathrm{d}s, \quad t \in J. \quad (26)$$

Corollary 1 (see [23]). *Under the hypotheses of Theorem 3, let v be a nondecreasing function on J. Then, we have*

$$u(t) \leq v(t) \mathbb{E}_{\alpha}(\Gamma(\alpha) w(t) (\psi(t) - \psi(a))^{\alpha}), \quad t \in J. \quad (27)$$

where $\mathbb{E}_{\alpha}(\cdot)$ is a Mittag–Leffler function with one parameter.

Lemma 6 (see [14]). *Let $\alpha, \beta > 0$. Then for all $t \in J$, we have*

$$\mathcal{I}_{a^+}^{\alpha;\psi}\left(\mathbb{E}_\alpha\left(\beta(\psi(t)-\psi(a))^\alpha\right) = \frac{1}{\beta}\left(\mathbb{E}_\alpha\left(\beta(\psi(t)-\psi(a))^\alpha - 1\right)\right). \quad (28)$$

Remark 2. Observe that from Lemma 6 if $\beta = 1$, we can get the following inequality:

$$\mathcal{I}_{a^+}^{\alpha;\psi}\left(\mathbb{E}_\alpha\left((\psi(t)-\psi(a))^\alpha\right) \le \mathbb{E}_\alpha\left((\psi(t)-\psi(a))^\alpha\right)\right). \quad (29)$$

For the existence of solutions for the problem (1), we need the following lemma.

Lemma 7. *Let $f \in C(J \times \mathbb{R}, \mathbb{R})$. Then $x \in C(J, \mathbb{R})$ is the solution of*

$$\begin{cases} {}^c\mathcal{D}_{a^+}^{\alpha;\psi} x(t) = \lambda x(t) + f(t, x(t)), & t \in J, \\ x(a) = \phi_0 \in \mathbb{R}, \end{cases} \quad (30)$$

if and only if it is the solution of the integral equation:

$$\begin{aligned} x(t) &= \phi_0 + \mathcal{I}_{a^+}^{\alpha;\psi}(\lambda x(t) + f(t, x(t))) \\ &= \phi_0 + \lambda \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} x(s) \mathrm{d}s \\ &\quad + \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s)) \mathrm{d}s. \end{aligned} \quad (31)$$

Proof. Let $x(t)$ be a solution of the problem (30). Define $h(t) = \lambda x(t) + f(t, x(t))$. Then,

$${}^c\mathcal{D}_{a^+}^{\alpha;\psi} x(t) = h(t), \quad 0 < \alpha \le 1. \quad (32)$$

Since

$${}^c\mathcal{D}_{a^+}^{\alpha;\psi} x(t) = \mathcal{I}_{a^+}^{1-\alpha;\psi}\left(\frac{1}{\psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}x\right)(t) = h(t), \quad 0 < \alpha \le 1, \quad (33)$$

taking the ψ-Riemann–Liouville fractional integral of order α to the above equation, we get

$$\mathcal{I}_{a^+}^{1;\psi}\left(\frac{1}{\psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}x\right)(t) = \mathcal{I}_{a^+}^{\alpha;\psi} h(t), \quad 0 < \alpha \le 1. \quad (34)$$

Since

$$\begin{aligned} \mathcal{I}_{a^+}^{1;\psi}\left(\frac{1}{\psi'(t)}\frac{\mathrm{d}}{\mathrm{d}t}x\right)(t) &= \mathcal{I}_{a^+}^{1}\left(\frac{\mathrm{d}}{\mathrm{d}t}x\right)(t) \\ &= x(t) - x(a), \end{aligned} \quad (35)$$

we get

$$x(t) = \phi_0 + \mathcal{I}_{a^+}^{\alpha;\psi} h(t). \quad (36)$$

Using the definition of $h(t)$, we obtain equation (31). Conversely, suppose that $x(t)$ is the solution of the equation (31). Then, it can be written as

$$x(t) = \phi_0 + \mathcal{I}_{a^+}^{\alpha;\psi} h(t), \quad (37)$$

where $h(t) = \lambda x(t) + f(t, x(t))$. Since $h(t)$ is continuous and ϕ_0 is constant, operating the ψ-Caputo fractional differential operator ${}^c\mathcal{D}_{a^+}^{\alpha;\psi}$ on both sides of equation (37), we obtain

$${}^c\mathcal{D}_{a^+}^{\alpha;\psi} x(t) = {}^c\mathcal{D}_{a^+}^{\alpha;\psi}\phi_0 + {}^c\mathcal{D}_{a^+}^{\alpha;\psi}\mathcal{I}_{a^+}^{\alpha;\psi} h(t). \quad (38)$$

Using Lemma 5, the following is obtained:

$${}^c\mathcal{D}_{a^+}^{\alpha;\psi} x(t) = \lambda x(t) + f(t, x(t)). \quad (39)$$

From equation (37), we get $x(a) = \phi_0$. This proves that $x(t)$ is the solution of the Cauchy problem (30) which completes the proof.

Now, we are ready to present our main results. □

3. Uniqueness Result with respect to the Chebyshev Norm and Weissinger's Fixed Point Theorem

First of all, we define what we mean by a solution of equation (1).

Definition 11. A function $x \in C(J, \mathbb{X})$ is said to be a solution of equation (1) if x satisfies the equation ${}^c\mathcal{D}_{a^+}^{\alpha;\psi} x(t) = \lambda x(t) + f(t, x(t))$ on J and the condition $x(a) = \phi_0$.

Theorem 4. *Let the following assumptions hold:*

(H1) The function $f\colon [a, b] \times \mathbb{X} \longrightarrow \mathbb{X}$ is continuous.

(H2) There exists a constant $\mathbb{L} > 0$ such that

$$\|f(t, x) - f(t, y)\| \le \mathrm{L}\|x - y\|, \quad (40)$$

for any $x, y \in \mathbb{X}$ and $t \in J$.

Then there exists a unique solution of equation (1) on J.

Proof. In view of Lemma 7, we introduce an operator $\mathcal{T}\colon C(J, \mathbb{X}) \longrightarrow C(J, \mathbb{X})$ associated with equation (1) as follows:

$$\begin{aligned} \mathcal{T}x(t) &= \phi_0 + \lambda \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} x(s) \mathrm{d}s \\ &\quad + \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} f(s, x(s)) \mathrm{d}s. \end{aligned} \quad (41)$$

Clearly, the fixed points of the operator $\mathcal{T}$ are solutions of equation (1). Weissinger's fixed point theorem will be used to prove that $\mathcal{T}$ has a fixed point. For this reason, we shall show that $\mathcal{T}$ is a contraction. Let $x, y \in C(J, \mathbb{X})$. Then, for every $n \in \mathbb{N}$ and $t \in J$, using (H2), we have

$$\begin{aligned}\left\|\mathcal{T}^{n}x(t)-\mathcal{T}^{n}y(t)\right\| &= \left\|\|\mathcal{T}\left(\mathcal{T}^{n-1}x(t)\right)-\mathcal{T}\left(\mathcal{T}^{n-1}y(t)\right)\right\| \\ &\leq \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}\left\|f\left(s,\mathcal{T}^{n-1}x(s)\right)-f\left(s,\mathcal{T}^{n-1}y(s)\right)\right\|ds \\ &\quad+\lambda\int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}\left\|\mathcal{T}^{n-1}x(s)-\mathcal{T}^{n-1}y(s)\right\|ds \\ &\leq (\mathbb{L}+\lambda)\int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}\left\|\mathcal{T}^{n-1}x(s)-\mathcal{T}^{n-1}y(s)\right\|ds \\ &\leq (\mathbb{L}+\lambda)^2\int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{2\alpha-1}}{\Gamma(2\alpha)}\left\|\mathcal{T}^{n-2}x(s)-\mathcal{T}^{n-2}y(s)\right\|ds \\ &\ \ \vdots \\ &\leq (\mathbb{L}+\lambda)^n\int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{n\alpha-1}}{\Gamma(n\alpha)}\|x(s)-y(s)\|ds.\end{aligned} \tag{42}$$

Therefore, we conclude

$$\left\|\mathcal{T}^{n}x-\mathcal{T}^{n}y\right\|_{\infty} \leq \frac{\left((\mathrm{L}+\lambda)(\psi(b)-\psi(a))^{\alpha}\right)^{n}}{\Gamma(n\alpha+1)}\|x-y\|_{\infty}, \tag{43}$$

for each $n \in \mathbb{N}$ and all $x, y \in C(J, \mathbb{X})$.

Now let

$$\beta_n = \frac{\left((\mathbb{L}+\lambda)(\psi(b)-\psi(a))^{\alpha}\right)^{n}}{\Gamma(n\alpha+1)}. \tag{44}$$

By Definition (6), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\beta_n &= \sum_{n=0}^{\infty}\frac{\left((\mathbb{L}+\lambda)(\psi(b)-\psi(a))^{\alpha}\right)^{n}}{\Gamma(n\alpha+1)} \\ &= \mathrm{E}_{\alpha}\left((\mathbb{L}+\lambda)(\psi(b)-\psi(a))^{\alpha}\right).\end{aligned} \tag{45}$$

Therefore, the existence of the unique fixed point of $\mathcal{T}$ follows from Weissinger's fixed point theorem. That is, (1) fd1 has a unique solution. This completes the proof. □

4. Existence Result via Meir–Keeler Condensing Operators

In this section, we can weaken the condition (H2) to a linear growth condition. But now Theorem 2 that we apply will only guarantee the existence not also the uniqueness of the solution.

Theorem 5. *Assume that the hypothesis (H1) holds. Furthermore, we impose the following:*

(H3) There exist continuous functions μ, ν: $J \longrightarrow \mathbb{R}_+$ such that

$$\|f(t,x)\| \leq \mu(t)+\nu(t)\|x\|, \tag{46}$$

for any $x \in \mathbb{X}$ and $t \in J$.

(H4) For each bounded set $B \subset \mathbb{X}$, and each $t \in J$, the following inequality holds:

$$\chi(f(t,B)) \leq \nu(t)\chi(B). \tag{47}$$

Then the problem (1) has at least one solution defined on J provided that

$$4\left(\nu^{*}+\lambda\right)\ell_{\alpha,\psi} < 1, \tag{48}$$

where

$$\begin{aligned}\nu^{*} &= \sup_{t\in J}\nu(t), \\ \ell_{\alpha,\psi} &= \frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}.\end{aligned} \tag{49}$$

Proof. Consider the operator $\mathcal{T}$ defined by equation (41) and define a bounded closed convex set

$$\Omega_r = \left\{x \in C(J,\mathbb{X}) \colon \|x\|_{\infty} \leq r\right\}. \tag{50}$$

with

$$\begin{aligned} r &\geq \frac{\left\|\phi_0\right\|+\ell_{\alpha,\psi}\mu^{*}}{1-\left(\nu^{*}+\lambda\right)\ell_{\alpha,\psi}}, \\ \mu^{*} &= \sup_{t\in J}\mu(t).\end{aligned} \tag{51}$$

We shall show that the operator $\mathcal{T}$ satisfies all the assumptions of Theorem 2. We split the proof into four steps:

Step 1. The operator $\mathcal{T}$ maps the set Ω_r into itself. By the assumption (H3), we have

$$
\begin{aligned}
\|\mathcal{T}x(t)\| &\leq \|\phi_0\| + \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \|f(s,x(s))\| \mathrm{d}s \\
&\quad + \lambda \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \|x(s)\| \mathrm{d}s \\
&\leq \|\phi_0\| + \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} (\mu(s) + (\nu(s)+\lambda)\|x(s)\|) \mathrm{d}s \\
&\leq \|\phi_0\| + (\mu^* + (\nu^*+\lambda)\|x\|_\infty) \int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \mathrm{d}s \\
&\leq \|\phi_0\| + \frac{(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)} (\mu^* + (\nu^*+\lambda)r) \\
&= \|\phi_0\| + \ell_{\alpha,\psi}(\mu^* + (\nu^*+\lambda)r) \\
&\leq r.
\end{aligned} \tag{52}
$$

Thus

$$\|\mathcal{T}x\|_\infty \leq r. \tag{53}$$

This proves that $\mathcal{T}$ transforms the ball Ω_r into itself.

Step 2. The operator $\mathcal{T}$ is continuous. Suppose that $\{x_n\}$ is a sequence such that $x_n \longrightarrow x$ in Ω_r as $n \longrightarrow \infty$. It is easy to see that $f(s, x_n(s)) \longrightarrow f(s, x(s))$, as $n \longrightarrow +\infty$ due to the continuity of f. On the contrary, taking (H3) into consideration, we get the following inequality:

$$
\begin{aligned}
&\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\|f(s,x_n(s)) - f(s,x(s))\| \leq 2(\mu(s) \\
&\quad + \nu(s)r)\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}.
\end{aligned} \tag{54}
$$

We notice that since the function $s \longmapsto 2(\mu(s) + \nu(s)r)\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}$ is the Lebesgue integrable over $[a,t]$. This fact together with the Lebesgue dominated convergence theorem implies that

$$
\begin{aligned}
&\int_a^t \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} \\
&\quad \cdot \|f(s,x_n(s)) - f(s,x(s))\| \mathrm{d}s \longrightarrow 0 \text{ as } n \longrightarrow +\infty.
\end{aligned} \tag{55}
$$

It follows that $\|\mathcal{T}x_n - \mathcal{T}x\|_\infty \longrightarrow 0$ as $n \longrightarrow +\infty$, which implies the continuity of the operator $\mathcal{T}$.

Step 3. $\mathcal{T}(\Omega_r)$ is equicontinuous.

For any $a < t_1 < t_2 < b$ and $x \in \Omega_r$, we get

$$
\begin{aligned}
\|\mathcal{T}x(t_2) - \mathcal{T}x(t_1)\| &\leq \int_a^{t_1} \frac{\psi'(s)\left[(\psi(t_1)-\psi(s))^{\alpha-1} - (\psi(t_2)-\psi(s))^{\alpha-1}\right]}{\Gamma(\alpha)} (\lambda\|x(s)\| + \|f(s,x(s))\|) \mathrm{d}s \\
&\quad + \int_{t_1}^{t_2} \frac{\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)} (\lambda\|x(s)\| + \|f(s,x(s))\|) \mathrm{d}s \\
&\leq \frac{\mu^* + (\nu^*+\lambda)r}{\Gamma(\alpha+1)} \left[(\psi(t_1)-\psi(a))^\alpha + 2(\psi(t_2)-\psi(t_1))^\alpha - (\psi(t_2)-\psi(a))^\alpha\right] \\
&\leq 2\frac{\mu^* + (\nu^*+\lambda)r}{\Gamma(\alpha+1)} (\psi(t_2)-\psi(t_1))^\alpha,
\end{aligned} \tag{56}
$$

where we have used the fact that $(\psi(t_1)-\psi(a))^{\alpha}-(\psi(t_2)-\psi(a))^{\alpha}\leq 0$. Therefore,

$$\|\mathcal{T}x(t_2)-\mathcal{T}x(t_1)\|\leq 2\frac{\mu^*+(\nu^*+\lambda)r}{\Gamma(\alpha+1)}(\psi(t_2)-\psi(t_1))^{\alpha}. \tag{57}$$

As $t_2 \longrightarrow t_1$, the right-hand side of the above inequality tends to zero independently of $x\in\Omega_r$. Hence, we conclude that $\mathcal{T}(\Omega_r)\subseteq C(J,\mathbb{X})$ is bounded and equicontinuous.

Step 4. Now, we prove that $\mathcal{T}:\Omega_r\longrightarrow\Omega_r$ is a Meir–Keeler condensing operator. To do this, suppose $\varepsilon>0$ is given. We will prove that there exists $\delta>0$ such that

$$\varepsilon\leq\chi_C(B)<\varepsilon+\delta\Longrightarrow\chi_C(\mathcal{T}B)<\varepsilon,\quad \text{for any } B\subset\Omega_r. \tag{58}$$

For every bounded and equicontinuous subset $B\subset\Omega_r$ and $\varepsilon'>0$ using Lemma 3 and the properties of χ, there exist sequences $\{x_n\}_{n=1}^{\infty}\subset B$ such that

$$\chi(\mathcal{T}(B)(t))\leq 2\chi\left\{\int_a^t\frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}\cdot\left(\lambda\{x_n(s)\}_{n=1}^{\infty}+f\left(s,\{x_n(s)\}_{n=1}^{\infty}\right)\right)\mathrm{d}s\right\}+\varepsilon'. \tag{59}$$

Next, by Lemma 4 and (H4), we have

$$\begin{aligned}\chi(\mathcal{T}(B)(t))&\leq 4\int_a^t\frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}\left(\lambda\chi\left(\{x_n(s)\}_{n=1}^{\infty}\right)+\chi\left(f\left(s,\{x_n(s)\}_{n=1}^{\infty}\right)\right)\right)\mathrm{d}s+\varepsilon'\\&\leq\frac{4(\nu^*+\lambda)}{\Gamma(\alpha)}\int_a^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\chi\left(\{x_n(s)\}_{n=1}^{\infty}\right)\mathrm{d}s+\varepsilon'\\&\leq 4\frac{(\nu^*+\lambda)(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)}\chi_C(B)+\varepsilon'\\&=4(\nu^*+\lambda)\ell_{\alpha,\psi}\chi_C(B)+\varepsilon'.\end{aligned} \tag{60}$$

As the last inequality is true, for every $\varepsilon'>0$, we infer

$$\chi(\mathcal{T}(B)(t))\leq 4(\nu^*+\lambda)\ell_{\alpha,\psi}\chi_C(B). \tag{61}$$

Since $\mathcal{T}(B)\subset\Omega_r$ is bounded and equicontinuous, we know from Lemma 3 that

$$\chi_C(\mathcal{T}(B))=\max_{t\in J}\chi(\mathcal{T}(B)(t)). \tag{62}$$

Therefore, we have

$$\chi_C(\mathcal{T}(B))\leq 4(\nu^*+\lambda)\ell_{\alpha,\psi}\chi_C(B). \tag{63}$$

Observe that from the last estimates

$$\chi_C(\mathcal{T}(B))\leq 4(\nu^*+\lambda)\ell_{\alpha,\psi}\chi_C(B)<\varepsilon\Longrightarrow\chi_C(B)<\frac{1}{4(\nu^*+\lambda)\ell_{\alpha,\psi}}\varepsilon. \tag{64}$$

Let us now take

$$\delta=\frac{1-4(\nu^*+\lambda)\ell_{\alpha,\psi}}{4(\nu^*+\lambda)\ell_{\alpha,\psi}}\varepsilon, \tag{65}$$

so we get

$$\varepsilon\leq\chi(B)<\varepsilon+\delta. \tag{66}$$

which means that $\mathcal{T}:\Omega_r\longrightarrow\Omega_r$ is a Meir–Keeler condensing operator. It follows from Theorem 2 that the operator $\mathcal{T}$ defined by (41) has at least one fixed point $x\in\Omega_r$, which is just the solution of the initial value problem (1). This completes the proof of Theorem 5. □

5. Mittag–Leffler–Ulam–Hyers Stability Analysis

In this section, we discuss the Mittag–Leffler–Ulam–Hyers stability analysis of the solutions to the proposed problem (1).

Now, we consider the Mittag–Leffler–Ulam–Hyers stability for problem (1).

Let $\varepsilon, \lambda > 0$ and $\Phi: J \longrightarrow \mathbb{R}^+$ be a continuous function. We consider the following inequalities:

$$\left\| {}^c\mathscr{D}_{a^+}^{\alpha;\psi} x(t) - \lambda x(t) - f(t, x(t)) \right\| \leq \varepsilon \mathbb{E}_\alpha\left((\psi(t) - \psi(a))^\alpha \right), \quad t \in J, \tag{67}$$

$$\left\| {}^c\mathscr{D}_{a^+}^{\alpha;\psi} x(t) - \lambda x(t) - f(t, x(t)) \right\| \leq \varepsilon \Phi(t) \mathbb{E}_\alpha\left((\psi(t) - \psi(a))^\alpha \right), \quad t \in J. \tag{68}$$

Definition 12 (see [71, 73]). Equation (1) is Mittag–Leffler–Ulam–Hyers stable, with respect to $\mathbb{E}_\alpha((\psi(t) - \psi(a))^\alpha)$ if there exists a real number $c > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in C(J, \mathbb{X})$ of the inequality (67), there exists a solution $x \in C(J, \mathbb{X})$ of equation (1) with

$$\| y(t) - x(t) \| \leq c\varepsilon \mathbb{E}_\alpha\left((\psi(t) - \psi(a))^\alpha \right), \quad t \in J. \tag{69}$$

Definition 13 (see [71, 73]). Equation (1) is generalized Mittag–Leffler–Ulam–Hyers stable, with respect to $\mathbb{E}_\alpha((\psi(t) - \psi(a))^\alpha)$ if there exists $\omega: C(\mathbb{R}_+, \mathbb{R}_+)$ with $\omega(0) = 0$ such that for each $\varepsilon > 0$ and for each solution $y \in C(J, \mathbb{X})$ of the inequality (67), there exists a solution $x \in C(J, \mathbb{X})$ of equation (1) with

$$\| y(t) - x(t) \| \leq \omega(\varepsilon) \mathbb{E}_\alpha\left((\psi(t) - \psi(a))^\alpha \right), \quad t \in J. \tag{70}$$

Remark 3 (see [71, 73]). It is clear that Definition 12 $\Longrightarrow$ Definition 13,

Remark 4 (see [71, 73]). A function $y \in C(J, \mathbb{X})$ is a solution of the inequality (67) if and only if there exists a function $z \in C(J, \mathbb{X})$ (which depends on solution y) such that

(i) $\|z(t)\| \leq \varepsilon \mathbb{E}_\alpha((\psi(t) - \psi(a))^\alpha)$, $t \in J$

(ii) ${}^c\mathscr{D}_{a^+}^{\alpha;\psi} y(t) = \lambda y(t) + f(t, y(t)) + z(t)$, $t \in J$

Now we are ready to state our Mittag–Leffler–Ulam–Hyers stability of solution to the problem (1). The arguments are based on the Grönwall inequality equation (27).

Theorem 6 Assume that (H1) and (H2) hold. *Then problem (1) is Mittag–Leffler–Ulam–Hyers stable on J and consequently generalized Mittag–Leffler–Ulam–Hyers stable.*

Proof. Let $\varepsilon, \lambda > 0$ and let $y \in C(J, \mathbb{X})$ be a function which satisfies the inequality (67) and let $x \in C(J, \mathbb{X})$ be the unique solution of the following problem:

$$\begin{cases} {}^c\mathscr{D}_{a^+}^{\alpha;\psi} x(t) = \lambda x(t) + f(t, x(t)), & t \in J := [a, b], \\ x(a) = \phi_0. \end{cases} \tag{71}$$

By Lemma 7, we have

$$x(t) = \phi_0 + \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} (\lambda x(s) + f(s, x(s)))\mathrm{d}s. \tag{72}$$

Since we have assumed that y is a solution of the inequality (67), we have the following by Remark 4:

$$\begin{cases} {}^c\mathscr{D}_{a^+}^{\alpha;\psi} y(t) = \lambda y(t) + f(t, y(t)) + z(t), & t \in J := [a, b], \\ y(a) = \phi_0. \end{cases} \tag{73}$$

Again by Lemma 7, we have

$$y(t) = \phi_0 + \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} (\lambda y(s) + f(s, y(s)) + z(s))\mathrm{d}s. \tag{74}$$

On the contrary, we have, for each $t \in J$,

$$\begin{aligned} \| y(t) - x(t) \| \leq & \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} \|z(s)\| \mathrm{d}s \\ & + \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} \cdot \| f(s, y(s)) - f(s, x(s)) \| \mathrm{d}s \\ & + \lambda \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} \| y(s) - x(s) \| \mathrm{d}s. \end{aligned} \tag{75}$$

Hence, using Remark 2 and part (i) of Remark 4 and (H2), we can get

$$\begin{aligned} \| y(t) - x(t) \| \leq & \varepsilon \mathbb{E}_\alpha\left((\psi(t) - \psi(a))^\alpha \right) \\ & + (\mathbb{L} + \lambda) \int_a^t \frac{\psi'(s)(\psi(t) - \psi(s))^{\alpha-1}}{\Gamma(\alpha)} \cdot \| y(s) - x(s) \| \mathrm{d}s. \end{aligned} \tag{76}$$

Applying Corollary 1 (the Grönwall inequality equation (27)) to the above inequality with $u(t) = \| y(t) - x(t) \|$, $v(t) = \varepsilon \mathbb{E}_\alpha((\psi(t) - \psi(a))^\alpha)$, and $w(t) = \mathbb{L} + \lambda/\Gamma(\alpha)$. Since $v(t)$ is a nondecreasing function on J, we conclude that

$$\| y(t) - x(t) \| \leq v(t) \mathbb{E}_\alpha\left((\mathbb{L} + \lambda)(\psi(t) - \psi(a))^\alpha \right), \quad t \in J, \tag{77}$$

which yields

$$\| y(t) - x(t) \| \leq \mathbb{E}_\alpha\left((\psi(t) - \psi(a))^\alpha \right) \mathbb{E}_\alpha\left((\mathbb{L} + \lambda)(\psi(b) - \psi(a))^\alpha \right) \varepsilon, \quad t \in J. \tag{78}$$

Taking for simplicity

$$c = \mathbb{E}_\alpha\left((\mathbb{L}+\lambda)(\psi(b)-\psi(a))^\alpha\right), \tag{79}$$

then (78) becomes

$$\|y(t)-x(t)\| \le c\varepsilon \mathbb{E}_\alpha\left((\psi(t)-\psi(a))^\alpha\right), \quad t \in J. \tag{80}$$

In consequence, it follows that

$$\|y-x\|_\infty \le c\varepsilon \mathbb{E}_\alpha\left((\psi(t)-\psi(a))^\alpha\right). \tag{81}$$

Thus, the problem (1) is Mittag–Leffler–Ulam–Hyers stable. Furthermore, if we set $\omega(\varepsilon) = c\varepsilon; \omega(0) = 0$, then the problem (1) is generalized Mittag–Leffler–Ulam–Hyers stable. This completes the proof. □

6. Uniqueness Result with respect to the Bielecki Norms and Banach's Fixed Point Theorem on an Unbounded Domain

In this section, we present an uniqueness result concerning the problem (1) on unbounded domain, i.e., in the case $J' = [a, +\infty)$.

Theorem 7. *If $f: J' \times \mathbb{X} \longrightarrow \mathbb{X}$ be a continuous function that satisfies the Lipschitz condition with respect to the second variable, i.e., there exists a positive constant $\mathbb{M}$ such that*

$$\|f(t,x)-f(t,y)\| \le \mathbb{M}\|x-y\|, \tag{82}$$

for all $t \in J'$ and each $x, y \in \mathbb{X}$.

Then, the problem (1) possesses a unique solution defined on $J' = [a, +\infty)$.

Proof. Consider the Banach space $C(J', \mathbb{X})$ equipped with a Bielecki norm type $\|\cdot\|_{\mathscr{B}}$ defined as below:

$$\|x\|_{\mathscr{B}} := \sup_{t \in J'} \frac{\|x(t)\|}{\mathbb{E}_\alpha\left(\beta(\psi(t)-\psi(a))^\alpha\right)}, \tag{83}$$

where $\beta > 0$ will be chosen later and $\mathbb{E}_\alpha(\cdot)$ is the Mittag–Leffler function which is given in Definition 6. (for more properties on the Bielecki-type norm, see [25, 28]).

Consider the operator $\mathscr{A}: C(J', \mathbb{X}) \longrightarrow C(J', \mathbb{X})$ defined by

$$\mathscr{A}x(t) = \phi_0 + \int_a^t \mathscr{G}_\psi^\alpha(t,s) f(s, x(s)) \mathrm{d}s. \quad t \in J', \tag{84}$$

where

$$\mathscr{G}_\psi^\alpha(t,s) = \frac{\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}}{\Gamma(\alpha)}. \tag{85}$$

Now, we prove that the operator $\mathscr{A}$ is a contraction mapping on $C(J', \mathbb{X})$ with respect to the Bielecki norm. To this end, we apply the Banach fixed point theorem to prove that $\mathscr{A}$ has a fixed point. Given $x, y \in C(J', \mathbb{X})$ and $t \in J'$, using the inequality (82), and Lemma 6, we can get

$$\begin{aligned}
\|\mathscr{A}x(t)-\mathscr{A}y(t)\| \le & \int_a^t \mathscr{G}_\psi^\alpha(t,s) \frac{\mathbb{M}\|x(s)-y(s)\|}{\mathbb{E}_\alpha\left(\beta(\psi(s)-\psi(a))^\alpha\right)} \mathbb{E}_\alpha\left(\beta(\psi(s)-\psi(a))^\alpha\right) \mathrm{d}s \\
& + \lambda \int_a^t \mathscr{G}_\psi^\alpha(t,s) \frac{\|x(s)-y(s)\|}{\mathbb{E}_\alpha\left(\beta(\psi(s)-\psi(a))^\alpha\right)} \mathbb{E}_\alpha\left(\beta(\psi(s)-\psi(a))^\alpha\right) \mathrm{d}s \\
& \le \frac{\mathbb{M}+\lambda}{\beta}\left(\mathbb{E}_\alpha\left(\beta(\psi(t)-\psi(a))^\alpha - 1\right)\|x-y\|_{\mathscr{B}}\right).
\end{aligned} \tag{86}$$

Hence, we have

$$\|\mathscr{A}x-\mathscr{A}y\|_{\mathscr{B}} \le \frac{\mathbb{M}+\lambda}{\beta}\left(1 - \frac{1}{\mathbb{E}_\alpha\left(\beta(\psi(t)-\psi(a))^\alpha\right)}\right)\|x-y\|_{\mathscr{B}}. \tag{87}$$

Note that $\mathbb{E}_\alpha(\cdot)$ is a monotone increasing function on J', then we get

$$\|\mathscr{A}x-\mathscr{A}y\|_{\mathscr{B}} \le \frac{\mathbb{M}+\lambda}{\beta}\|x-y\|_{\mathscr{B}}. \tag{88}$$

Since we can choose $\beta > 0$ sufficiently large such that

$$\frac{\mathbb{M}+\lambda}{\beta} < 1, \tag{89}$$

it follows that the mapping $\mathscr{A}$ is a contraction with respect to the Bielecki norm. Hence, by the Banach fixed point theorem, $\mathscr{A}$ has a unique fixed point which is a unique solution of the initial value problem (1) in the space $C(J', \mathbb{X})$. This completes the proof. □

7. An Existence Result in Fréchet Spaces via Darbo's Fixed Point Theorem

By using Darbo's fixed point theorem, we give in this section our last existence theorem concerning the IVP (2) in the Fréchet spaces.

Firstly, we need to fix the notation. Let $J' = [a, +\infty)$ and let $J_n := [a, n]$, $n \in \mathbb{N}^*$.

In this section, we let $\mathrm{E} := C(J', \mathbb{X})$ to be the Fréchet space of all continuous functions x from J' into $\mathbb{X}$, equipped with the family of seminorms:

$$\|x\|_n = \sup_{t \in J_n} \|x(t)\|, \quad n \in \mathbb{N}^*. \tag{90}$$

Definition 14. A nonempty subset $B \subset E$ is said to be bounded if

$$\sup_{x \in B} \|x\|_n < \infty, \quad n \in \mathbb{N}^*. \tag{91}$$

Next we present some facts concerning the notion of a sequence of measures of noncompactness in the Fréchet spaces [44, 50].

In what follows, F will be a real Fréchet space. If B is a nonempty subset of F, then $\overline{B}$ and Conv$B$ denote the closure and the closed convex closure of B, respectively. Also, we denote by $m_{\mathbb{F}}$ the family of all nonempty and bounded subsets of $\mathbb{F}$ and $n_{\mathbb{F}}$ the family of all relatively compact subsets of $\mathbb{F}$.

Definition 15. A family of functions $\{\mu_n\}_{n \in \mathbb{N}}$ where $\mu_n: m_{\mathbb{F}} \longrightarrow \mathbb{R}_+ = [0, \infty)$ is said to be a family of measures of noncompactness in the real Fréchet space $\mathbb{F}$ if it satisfies the following conditions:

(1) The family $\text{Ker}\mu_n = \{B \in m_{\mathbb{F}}; \mu_n(B) = 0, \text{ for } n \in \mathbb{N}\}$ is nonempty and $\text{Ker}\mu_n \in n_{\mathbb{F}}$
(2) $B_1 \subset B_2 \Longrightarrow \mu_n(B_1) \le \mu_n(B_2)$, for all $B_1, B_2 \in m_{\mathbb{F}}$ and $n \in \mathbb{N}^*$
(3) $\mu_n(\text{Conv}B) = \mu_n(B)$, for all $B \in m_{\mathbb{F}}$ and $n \in \mathbb{N}$
(4) If (B_i) is a sequence of closed subsets of $m_{\mathbb{F}}$ such that $B_{i+1} \subset B_i$ for $(i = 1, 2, 3, \ldots)$, and $\lim_{i \longrightarrow \infty} \mu_n(B_i) = 0$ for each $n \in \mathbb{N}$, then $B_\infty := \cap_{i=1}^{\infty} B_i \neq \emptyset$

The following lemmas are needed in our argument.

Lemma 8 (see [77]). *If B is a bounded subset of a Fréchet space $\mathbb{F}$, then for each $\varepsilon > 0$, there is a sequence $\{x_k\}_{k=1}^{\infty} \subset B$, such that*

$$\mu_n(B) \le 2\mu_n(\{x_k\}_{k=1}^{\infty}) + \varepsilon, \quad \text{for each } n \in \mathbb{N}^*. \tag{92}$$

Lemma 9 (see [78]). *If $\{x_k\}_{k=1}^{\infty} \subset L^1(J_n, \mathbb{X})$ is uniformly integrable, then $\mu_n(\{x_k\}_{k=1}^{\infty})$ is measurable, for $n \in \mathbb{N}^*$ and*

$$\mu_n\left(\left\{\int_a^t x_k(s)\mathrm{d}s\right\}_{k=1}^{\infty}\right) \le 2\int_a^t \mu_n(\{x_k(s)\}_{k=1}^{\infty})\mathrm{d}s, \tag{93}$$

for each $t \in J_n$.

Definition 16. Let Ω be a nonempty subset of a Fréchet space $\mathbb{F}$, and let $\mathscr{A}: \Omega \longrightarrow \mathbb{F}$ be a continuous operator which transforms bounded subsets of Ω onto bounded ones. One says that $\mathscr{A}$ satisfies the Darbo condition with constants $\{k_n\}_{n \in \mathbb{N}^*}$ with respect to a family of measures of noncompactness $\{\mu_n\}_{n \in \mathbb{N}^*}$, if

$$\mu_n(\mathscr{A}(B)) \le k_n \mu_n(B), \tag{94}$$

for each bounded set $B \subset \Omega$ and $n \in \mathbb{N}^*$.

If $k_n < 1, n \in \mathbb{N}^*$, then $\mathscr{A}$ is called a contraction with respect to $\{\mu_n\}_{n \in \mathbb{N}^*}$.

The following generalization of the classical Darbo fixed point theorem for the Fréchet spaces plays a key role in the proof of our main results.

Theorem 8. *(see [44]). Let Ω be a nonempty, bounded, closed, and convex subset of a Fréchet space $\mathbb{F}$ and let $\mathscr{A}: \Omega \longrightarrow \Omega$ be a continuous mapping. Suppose that $\mathscr{A}$ is a contraction with respect to a family of measures of noncompactness $\{\mu_n\}_{n \in \mathbb{N}^*}$. Then $\mathscr{A}$ has at least one fixed point in the set Ω.*

Definition 17. A function $x \in C(J', \mathbb{X})$ is said to be a solution of equation (2) if x satisfies the equation ${}^c\mathscr{D}_{a^+}^{\alpha;\psi} x(t) = f(t, x(t))$ on J', and the condition $x(a) = \phi_0$.

Now, we shall prove the following theorem concerning the existence of solutions of problem (2).

Theorem 9. *Let $f: J' \times \mathbb{X} \longrightarrow \mathbb{X}$ be a continuous function such that the following assumptions hold:*

(H5) There exists a continuous functions $p: J' \longrightarrow \mathbb{R}_+$ such that

$$\|f(t, x)\| \le p(t)(1 + \|x\|), \quad \text{for all } t \in J', \text{ and each } x \in \mathbb{X}. \tag{95}$$

(H6) For each bounded set $D \subset \mathbb{X}$ and for each $t \in J'$, we have

$$\mu(f(t, D)) \le p(t)\mu(D), \tag{96}$$

where μ is a measure of noncompactness on the Banach space $\mathbb{X}$.

For $n \in \mathbb{N}^*$, let

$$p_n^* = \sup_{t \in J_n} p(t), \qquad \ell_{n,\psi} = \frac{(\psi(n) - \psi(a))^\alpha}{\Gamma(\alpha + 1)}. \tag{97}$$

Define on $\mathbb{E}$ the family of measures of noncompactness by

$$\mu_n(D) = \sup_{t \in J_n} \mu(D(t)), \tag{98}$$

where D is a bounded and equicontinuous set of $\mathbb{E}$, and $D(t) = \{x(t) \in \text{X}: x \in D\}$, $t \in J_n$.

If

$$4(p_n^* + \lambda)\ell_{n,\psi} < 1, \tag{99}$$

for each $n \in \mathbb{N}^*$, then the problem (2) has at least one solution.

Proof. Consider the operator $\mathscr{A}$ defined by (84), but in the Fréchet space $\text{E} := C(J', \mathbb{X})$.

Note that, the fixed points of the operator $\mathscr{A}$ are solutions of the problems (2). For any $n \in \mathbb{N}^*$, let R_n be a positive real number with

$$R_n \ge \frac{\|\phi_0\| + \ell_{n,\psi} p_n^*}{1 - (p_n^* + \lambda)\ell_{n,\psi}}, \tag{100}$$

and we consider the ball

$$B_{R_n} = \{x \in \mathbb{E}: \|x\|_n \leq R_n\}. \quad (101)$$

Notice that B_{R_n} is closed, convex, and bounded subset of the Fréchet space $\mathbb{E}$. We shall show that the operator $\mathcal{A}$ satisfies all the assumptions of Theorem 8. We split the proof into three steps.

Step 1. The operator $\mathcal{A}$ maps the set B_{R_n} into itself. For any $n \in \mathbb{N}^*$, and each $x \in B_{R_n}$ and $t \in J_n$, by (H5), we have

$$\begin{aligned}
\|\mathcal{A}x(t)\| &\leq \|\phi_0\| + \int_a^t \mathcal{G}_\psi^\alpha(t,s)(p(s)(1+\|x(s)\|) + \lambda\|x(s)\|)ds \\
&\leq \|\phi_0\| + \frac{(\psi(n)-\psi(a))^\alpha}{\Gamma(\alpha+1)}(p_n^* + (p_n^* + \lambda)\|x\|_n) \\
&\leq \|\phi_0\| + \ell_{n,\psi}(p_n^* + (p_n^* + \lambda)R_n) \\
&\leq R_n.
\end{aligned} \quad (102)$$

Thus

$$\|\mathcal{A}x\|_n \leq R_n. \quad (103)$$

This proves that $\mathcal{A}$ transforms the ball B_{R_n} into itself.

Step 2. The operator $\mathcal{A}$ is continuous. Suppose that $\{x_k\}_{k\in\mathbb{N}}$ is a sequence such that $x_k \longrightarrow x$ in B_{R_n} as $k \longrightarrow \infty$. Then for each $t \in J_n$, we have

$$\begin{aligned}
\|\mathcal{A}x_n(t) - \mathcal{A}(t)\| \leq \int_a^t \mathcal{G}_\psi^\alpha(t,s)\|f(s,x_k(s)) - f(s,x(s))\|ds \\
+ \lambda\int_a^t \mathcal{G}_\psi^\alpha(t,s)\|x_k(s) - x(s)\|ds.
\end{aligned} \quad (104)$$

Since $x_k \longrightarrow x$ as $k \longrightarrow +\infty$, the Lebesgue dominated convergence theorem implies that

$$\|\mathcal{A}x_k - \mathcal{A}x\|_n \longrightarrow 0 \text{ as } k \longrightarrow +\infty, \quad (105)$$

which implies the continuity of the operator $\mathcal{A}$.

Step 3. Our aim in this step is to show that $\mathcal{A}$ is μ_n contraction on B_{R_n}. For every bounded equicontinuous subset $D \subset B_{R_n}$ and $\varepsilon > 0$ using Lemma 8, there exist sequences $\{x_k\}_{k=1}^\infty \subset D$ such that for all $t \in J_n$, we have

$$\mu(\mathcal{A}(D)(t)) \leq 2\mu\left\{\int_a^t \mathcal{G}_\psi^\alpha(t,s)(\lambda\{x_n(s)\}_{n=1}^\infty + f(s,\{x_n(s)\}_{n=1}^\infty))ds\right\} + \varepsilon. \quad (106)$$

Next, by Lemma 9 and (H6), we have

$$\begin{aligned}
\mu(\mathcal{A}(D)(t)) &\leq 4\left\{\int_a^t \mathcal{G}_\psi^\alpha(t,s)\mu(\lambda\{x_n(s)\}_{n=1}^\infty + f(s,\{x_n(s)\}_{n=1}^\infty))ds\right\} + \varepsilon \\
&\leq 4\left\{\int_a^t \mathcal{G}_\psi^\alpha(t,s)(p(s)+\lambda)\mu(\{x_k(s)\}_{k=1}^\infty)ds\right\} + \varepsilon \\
&\leq \frac{4p_n^*(\psi(n)-\psi(a))^\alpha}{\Gamma(\alpha+1)}\mu_n(D) + \varepsilon \\
&= 4(p_n^* + \lambda)\ell_{n,\psi}\mu_n(D) + \varepsilon.
\end{aligned} \quad (107)$$

As the last inequality is true, for every $\varepsilon > 0$, we infer

$$\mu(\mathcal{A}(D)(t)) \leq 4(p_n^* + \lambda)\ell_{n,\psi}\mu_n(D). \quad (108)$$

Thus

$$\mu_n(\mathcal{A}(D)) \leq 4(p_n^* + \lambda)\ell_{n,\psi}\mu_n(D). \quad (109)$$

Using the condition (99), we claim that $\mathcal{A}$ is a k_n contraction on B_{R_n}. It follows from Theorem 8 that the operator $\mathcal{A}$ defined by (84) has at least one fixed point $x \in B_{R_n}$, which is just the solution of initial value problem (2). This completes the proof of Theorem 9. □

8. Examples

In this section, we give a couple of examples to illustrate the usefulness of our main result.

Let

$$X = c_0 = \{x = (x_1, x_2, \ldots, x_k, \ldots): x_k \longrightarrow 0\,(k \longrightarrow \infty)\}. \quad (110)$$

be the Banach space of real sequences converging to zero, endowed its usual norm

$$\|x\|_\infty = \sup_{k\geq 1}|x_k|. \tag{111}$$

Example 1. Consider the following initial value problem of a fractional differential posed in c_0:

$$\begin{cases} {}^{CH}\mathscr{D}^{0.5}_{1^+}x(t) = 0.5x(t) + f(t, x(t)), \\ t \in [1, e], \\ u(1) = (0.5, 0.25, \ldots, 0.5^n, \ldots). \end{cases} \tag{112}$$

Note that, this problem is a particular case of IVP (1), where

$$\begin{aligned} \alpha &= 0.5, \\ a &= 1, \\ b &= e, \\ \lambda &= 0.5, \\ \psi(t) &= \ln(t), \end{aligned} \tag{113}$$

and $f\colon [1, e] \times c_0 \longrightarrow c_0$ given by

$$f(t, x) = \left\{\frac{1}{(t+1)}\left(\frac{1}{k^2} + \arctan\left(|x_k|\right)\right)\right\}_{k\geq 1}, \quad \text{for } t \in [1, e], x = \{x_k\}_{k\geq 1} \in c_0. \tag{114}$$

It is clear that condition (H1) holds. Moreover, for any $x, y \in c_0$ and $t \in [1, e]$, we have

$$\|f(t, x) - f(t, y)\| \leq \frac{1}{2}\|x - y\|, \tag{115}$$

where we have used the trigonometric identity $\arctan(u) - \arctan(v) = \arctan(u - v/1 + uv)$ and $\arctan(u) \leq u, \forall u, v \in \mathbb{R}^+$. Hence condition (H2) holds with $L = 1/2$. Therefore, by Theorem 4, the IVP (112) has a unique solution $x \in C([1, e], c_0)$. Moreover, the inequality appearing in (67) has the following expression:

$$\left\|{}^{CH}\mathscr{D}^{0.5}_{1^+}x(t) - 0.5x(t) - f(t, x(t))\right\| \leq \varepsilon \mathrm{E}_{0.5}(\sqrt{\ln t}), \quad \varepsilon > 0, t \in [1, e]. \tag{116}$$

By Theorem 6, the IVP (112) is Mittag–Leffler–Ulam–Hyers stable with

$$\|y - x\|_\infty \leq c\varepsilon \mathbb{E}_\alpha\left((\psi(t) - \psi(a))^\alpha\right), \tag{117}$$

where

$$c = \mathbb{E}_{0.5}(1) \simeq 5.0090. \tag{118}$$

Let now

$$\mathbb{X} = \ell^1 = \left\{x = (x_1, x_2, \ldots, x_k, \ldots), \sum_{k=1}^{\infty}|x_k| < \infty\right\}, \tag{119}$$

be the Banach space with the norm

$$\|x\| = \sum_{k=1}^{\infty}|x_k|. \tag{120}$$

Example 2. Consider the following initial value problem of a fractional differential posed in ℓ^1:

$$\begin{cases} {}^{c}\mathscr{D}^{0.9}_{0^+}x(t) = 0.1x(t) + f(t, x(t)), \quad t \in [0, 1], \\ x(0) = (1, 0, \ldots, 0, \ldots). \end{cases} \tag{121}$$

In this case, we take

$$\begin{aligned} \alpha &= 0.9, \\ a &= 0, \\ b &= 1, \\ \lambda &= 0.1, \\ \psi(t) &= t, \end{aligned} \tag{122}$$

and $f\colon [0, 1] \times \ell^1 \longrightarrow \ell^1$ given by

$$f(t, x) = \left\{\frac{1}{e^t + 9}\left(\frac{1}{2^k} + \frac{x_k}{\|x\| + 1}\right)\right\}_{k\geq 1}, \quad \text{for } t \in [0, 1], x = \{x_k\}_{k\geq 1} \in \ell^1. \tag{123}$$

It is clear that condition (H1) holds and as

$$\|f(t, x)\| \leq \frac{1}{e^t + 9}(1 + \|x\|), \quad x, y \in \ell^1. \tag{124}$$

Hence condition (H3) holds with $\mu(t) = \nu(t) = 1/e^t + 9$; we get easily $\mu^* = \nu^* = 0.1$. On the contrary, for any bounded set $B \subset \ell^1$, we have

$$\chi(f(t, B)) \leq \frac{1}{e^t + 9}\chi(B), \quad \text{for any } t \in [0, 1]. \tag{125}$$

Hence, (H4) is satisfied. Now, we shall check that condition (48) is satisfied. Indeed

$$\begin{aligned} &4(\nu^* + \lambda)\ell_{\alpha,\psi} \simeq 0.8318 < 1, \\ &r \geq \frac{\|\phi_0\| + \ell_{\alpha,\psi}\mu^*}{1 - (\nu^* + \lambda)\ell_{\alpha,\psi}} \simeq 1.3938. \end{aligned} \tag{126}$$

Then r can be chosen as $r = 1.4$. Consequently, all the hypothesis of Theorem 5 are satisfied, and we conclude that the IVP (121) has at least one solution $x \in C([0, 1], \ell^1)$.

Example 3. Consider the following scalar initial value problem:

$$\begin{cases} {}^{c}\mathscr{D}^{0.5,\psi}_{0^+}x(t) = 2x(t) + \dfrac{1}{t^2 + 1}(1 + \sin x(t)), \\ t \in [0, +\infty), \\ u(0) = 2, \end{cases} \tag{127}$$

where

$$\alpha = 0.5,$$
$$a = 0,$$
$$\lambda = 2,$$
$$\psi(t) = t^2,$$
$$X = R,$$
$$f(t,x) = \frac{1}{t^2+1}(1+\sin x), \quad \text{for } t \in [0,+\infty), x \in \mathbb{R}. \tag{128}$$

Clearly, $f\colon [0,+\infty)\times\mathbb{R} \longrightarrow \mathbb{R}$ is continuous. On the contrary, for any $t \in [0,+\infty), x, y \in \mathbb{R}$, we have

$$|f(t,x) - f(t,y)| \leq |x-y|. \tag{129}$$

Hence the inequality (82) holds with $\mathbb{M} = 1$. Moreover, if we choose, $\beta > 3$, it follows that the mapping $\mathscr{A}$ is a contraction. Hence by Theorem 7, the IVP (127) has a unique solution on $[0,+\infty)$.

Example 4. Let us consider problem (2) with specific data:

$$\begin{aligned}\alpha &= \frac{1}{2},\\ a &= 0,\\ \lambda &= \frac{1}{8},\\ \mathbb{X} &= c_0,\\ \phi_0 &= (1,0,\ldots,0,\ldots),\end{aligned} \tag{130}$$

and $\mathbb{E} := C[0,+\infty)$, $\mathbb{X}$ be the Fréchet space of all continuous functions x from $[0,+\infty)$ into $\mathbb{X}$, equipped with the family of seminorms:

$$\|x\|_n = \sup_{t\in[0,n]} \|x(t)\|, \quad n \in \mathbb{N}^*. \tag{131}$$

In order to illustrate Theorem 9, we take $\psi(t) = \sigma(t)$ where $\sigma(t)$ is the sigmoid function [19] which can be expressed as in the following form:

$$\sigma(t) = \frac{1}{1+e^{-t}}, \tag{132}$$

and a convenience of the sigmoid function is its derivative:

$$\sigma'(t) = \sigma(t)(1-\sigma(t)). \tag{133}$$

Taking also $f\colon [0,+\infty)\times c_0 \longrightarrow c_0$ given by

$$f(t,x) = \left\{\frac{\sigma(t)}{8}\left(\frac{1}{k+1} + \ln\left(1+|x_k(t)|\right)\right)\right\}_{k\geq 1}, \quad \text{for } t \in [0,+\infty), x = \{x_k\}_{k\geq1} \in c_0. \tag{134}$$

The hypothesis (H5) is satisfied with $p(t) = \sigma(t)/8$. So, for any $n \in \mathbb{N}^*$, we have

$$\begin{aligned}p_n^* &= \sup_{t\in[0,n]} p(t)\\ &= \frac{\sigma(n)}{8},\end{aligned} \tag{135}$$

$$4(p_n^* + \lambda)\ell_{n,\psi} = (\sigma(n)+1)\sqrt{\frac{\sigma(n)-1/2}{\pi}} \leq \sqrt{\frac{2}{\pi}} < 1.$$

It follows from Theorem 9 that the problem (2) with the data (130), (132), and (134) has at least one solution defined on $\mathbb{R}_+$.

Data Availability

There are no data used in this work.

Conflicts of Interest

The authors declare they have no conflict of interest.

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