Research Article

Modeling and Control of the Public Opinion: An Agree-Disagree Opinion Model

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In this paper, we aim to investigate optimal control to a new mathematical model that describes agree-disagree opinions during polls, which we presented and analyzed in Bidah et al., 2020. We first present the model and recall its different compartments. We formulate the optimal control problem by supplementing our model with a objective functional. Optimal control strategies are proposed to reduce the number of disagreeing people and the cost of interventions. We prove the existence of solutions to the control problem, we employ Pontryagin's maximum principle to find the necessary conditions for the existence of the optimal controls, and Runge–Kutta forward-backward sweep numerical approximation method is used to solve the optimal control system, and we perform numerical simulations using various initial conditions and parameters to investigate several scenarios. Finally, a global sensitivity analysis is carried out based on the partial rank correlation coefficient method and the Latin hypercube sampling to study the influence of various parameters on the objective functional and to identify the most influential parameters.

1. Introduction

A prominent assertion in public opinion research is that when citizens are exposed to both party clues and political information, they follow easier-to-use party clues and ignore information on politicians. The theoretical basis for this assertion stems from dual-process change attitude models [1]. These models suggest that people use simple decision rules and clues when they lack the motivation or the ability to process information systematically. Since most citizens lack knowledge and there is a lack of interest in politics, party signals are traditionally considered to have the most effect on citizens' opinion [2].

When a person agrees to take a divergent position, a biased analysis of the arguments on the issue occurs. The person becomes temporarily motivated to find favorable information on the side to defend and to suppress thoughts unfavorable to the problem. This biased search for information likely increases the likelihood of a change in attitude [1]. To attract the attention of citizens to participate in political life, candidate parties must provide certain benefits because this task requires time, knowledge, motivation, and money. Citizen participation is then guaranteed if there are proportional benefits with political mobilizations and individual economic, psychological, and social costs [3].

Candidates and the media often use opinion polls before and during election campaigns to determine which candidates are first and which are likely to emerge victorious. These polls are intention polls for a sample of potential voters, which reveal the expected voting quotas. Then, future results allow candidates to negotiate and discuss opinions based on a particular outcome. Public opinion polls are usually used to determine people's position in the political life, votes, belonging, and political orientations by answering questions about their opinions, personal characteristics, and their activities. These answers are collected, counted, mathematically analyzed, and interpreted. These analyses allow the candidate parties to improve their campaign strategies in order to change their positions in the public opinion in the long term [4].

Opinion polls play a key role in political campaigns by leading to an update of party performance. These updates attract considerable media attention, providing a good opportunity to work on a new political commentary in the

weeks leading up to the polling day [5, 6], knowing that opinion polls are widely criticized for having influenced voters' perceptions and for leaving politics and focusing on popularity. Then, through strategic communication of the results of opinion polls, candidates can define the vision of electoral competition and encourage popular mobilization [7]. At any time, party leaders can choose to speak of candidates for their party or other election candidates, focusing on a variety of political and gender issues, depending on polling results. Voting in opinion polls can be used to balance different elements of the speech of party leaders [8].

There are many cases involving a bilateral decision. When there are only two candidates, the chances of creating a strong political polarization are likely to increase. Some previous studies show that the U.S. elections are attracting the polarization in the internet, where blogs and Twitter have a strong political polarization [9, 10]. The electoral processes of the second round also require a bilateral decision. In the first round, citizens voted among a large list of political parties, but in the last round, they can only vote for two final candidates. This second round increases political polarization in the country [11]. An example of a bilateral decision in the second round of the 2017 presidential election in Chile is being studied, and the resulting political polarization is measured in [12].

Mathematical models play a crucial role in describing true phenomena. In [13], the authors developed a mathematical model that describes the evolution of the public opinions during polls by considering the classes of agreeing, disagreeing, and ignorant people. They performed several computational and statistical experiments to validate their theoretical results, and they provided more interesting insights about the most influential parameters of the model on the equilibria thresholds.

In this article, we study how the opinion controls can spin the course of events and the preferences and expectations of voters are in competition and how they convince themselves in the face of the information of the candidates in opinion polls. We start by presenting the mathematical model that describes the evolution of opinions during public opinion polls. Then, we introduce two control functions that represent the positive and negative effects of the media and the publicity on people's attitudes. We prove mathematically the existence of these controls, and we characterize the optimal controls aiming to minimize an objective function using Pontryagin's maximum principle. The development of more efficient models and the accuracy of the modeling process require reliable statistical methods. Sensitivity analysis is one of the most recently used methods [14–17], which is used in development decisions, recommendations, understanding and quantification of systems, verification of the validity and accuracy of the model, and even to identify the important parameter for other studies [18]. Sensitivity analysis is a tool to evaluate the effect of changes in the value of the input parameter on the output value of a simulation model. Finally, we perform a global sensitivity analysis to determine the relationship between the model's parameters and the cost function. Numerical simulations are provided to illustrate the efficiency of the obtained results.

The paper is organized as follows: Section 2 introduces the model, giving some details about different compartments and parameters of the model. In Section 3, we present the optimal control problem and we derive the sufficient conditions for the existence of controls and the necessary optimality conditions. Section 4 provides numerical results and discussion of several scenarios. In Section 5, global sensitivity analysis is performed to identify the most important parameter in the proposed control model. Section 6 concludes the paper.

2. Presentation of the Model

Here, we propose a mathematical model from [13] describing the evolution of agree and disagree opinions during polls, and the types of surveys we consider are surveys that can be answered in agreement, disagreement, or otherwise. Thus, the targeted population by the poll is regrouped into three groups, and the model herewith has been formulated using compartments. Three compartments have been considered in this model which are described as follows:

(1) Indifferent (I) individuals: undecided or ambivalent people, or people who do not know about the poll yet or those who abstain from voting for personal reasons. This category has weak or nonexistent attitudes about the ideas, parties, or candidates and lacked any strong positive or negative associations [19].

(2) Agree (A): people in agreement with the idea being studied.

(3) Disagree (D): people in disagreement with the idea being studied.

For the modeling processes, a set of assumptions has been used. They are as follows:

(1) The targeted population is well mixed, that is, the indifferent individuals are homogeneously spread throughout the entire population

(2) Recruitment and mortality are negligible under the temporal scale consideration; therefore, no individual is recruited, and no individual dies during the poll

(3) Individuals have the right to communicate with each other and can thus convince themselves

(4) People who are unsure of their opinion are indifferent

(5) People who abstain from voting are indifferent

Everyone has their own reasons for agreement or disagreement; indifferent individuals can be convinced by reasons of agree people at a rate β_1 or by reasons of disagree people at a rate β_2. Agree individuals can be convinced by disagree people at a rate α_2, or disagree people convinced by agree individuals at a rate α_1. People can abstain from voting or lose the interest without any direct contact with individuals from the opposite opinion group; then, agree people become indifferent at the rate γ_1, and disagree people become indifferent at the rate γ_2. All contacts are modeled by

the standard incidence rate. A flowchart describing different interactions between the compartments of the model is presented in Figure 1.

All these assumptions and considerations are written as the following system of ordinary differential equations:

$$I' = -\beta_1 \frac{AI}{N} - \beta_2 \frac{DI}{N} + \gamma_1 A + \gamma_2 D, \tag{1}$$

$$A' = \beta_1 \frac{AI}{N} + \alpha_1 \frac{AD}{N} - \alpha_2 \frac{AD}{N} - \gamma_1 A, \tag{2}$$

$$D' = \beta_2 \frac{DI}{N} + \alpha_2 \frac{AD}{N} - \alpha_1 \frac{AD}{N} - \gamma_2 D, \tag{3}$$

where $I(0) \geq 0$, $A(0) \geq 0$, and $D(0) \geq 0$. Also, $N = I + A + D$, and note that $N' = I' + A' + D' = 0$; thus, the population size N is considered as a constant in time. A summary of parameters' description is given in Table 1.

3. Optimal Control Problem

3.1. Presentation of the Model with Controls. Opinion polls play a significant role in contemporary political campaigns, where party leaders use polls on the campaign trail to mobilize voters and to fine-tune their campaign strategies, because party performance updates receive significant attention from news agencies and often serve as the basis for political comment in the weeks leading up to the election day. It is known that learning about the positions of the electorate can shape the behavior of voters, and it has often been criticized for its effects on their perceptions [20, 21]. This explains that, at the turn of the century, over thirty democracies around the world had embargoes on the publication of opinion polls close to the election day [8].

The success of the election campaign depends to a large extent on its ability to reformulate new information in its favor, and the way the party leaders respond to opinion polls is one of the most important pillars of this process.

Here, we investigate the impact of media programs and publicity in changing people's opinions during opinion polls. To do this, we introduce two control variables: the first u_1 represents the effect of publicity and positive media programs to attract more people in the positive opinion group based on real facts and providing people with more accurate and realistic information in an easy way that all people use such as WhatsApp, Facebook, and Twitter. Thus, this control targets the indifferent group to bring them to the agreeing group, that is, an indifferent individual becomes agreeing at a rate $u_1 I$.

The second control u_2 represents the effect of negative media programs against competitors. This control targets the disagreeing and abstaining people to change their mind by providing them with negative information about the competitor or information clarifying certain ambiguities to at least motivate them, not to abstain. For instance, partisans shift their opinions away from their party's positions when policy information provides a compelling reason for doing so [2]. Thus, a disagreeing (abstaining) individual becomes again indifferent at a rate $u_2 D$. Therefore, the controlled model takes the following form:

$$I' = -\beta_1 \frac{AI}{N} - \beta_2 \frac{DI}{N} + \gamma_1 A + \gamma_2 D - u_1 I + u_2 D, \tag{4}$$

$$A' = \beta_1 \frac{AI}{N} + \alpha_1 \frac{AD}{N} - \alpha_2 \frac{AD}{N} - \gamma_1 A + u_1 I, \tag{5}$$

$$D' = \beta_2 \frac{DI}{N} + \alpha_2 \frac{AD}{N} - \alpha_1 \frac{AD}{N} - \gamma_2 D - u_2 D, \tag{6}$$

where $I(0) \geq 0$, $A(0) \geq 0$, and $D(0) \geq 0$.

3.2. Optimal Control Problem. Now, we consider an optimal control problem to minimize the objective functional

$$J(u_1, u_2) = \int_0^{t_f} \left(c_1 I(t) - c_2 A(t) + c_3 D(t) + \frac{K_1}{2} u_1^2(t) + \frac{K_2}{2} u_2^2(t) \right) dt, \tag{7}$$

where c_1, c_2, and c_3 are small positive constants to keep a balance in the size of $I(t)$, $A(t)$, and $D(t)$, respectively. The positive constants K_1 and K_2 balance the size of quadratic control terms. The reason behind considering a finite time horizon is that the control period is usually restricted to a limited time window. The objective of our work here is to minimize the indifferent and disagree groups by using possible minimal costs of applying control variables $u_1(t)$ and $u_2(t)$ attempting to increase the number of agreeing people.

We seek an optimal control pair (u_1^*, u_2^*) such that

$$J(u_1^*, u_2^*) = \min\{J(u_1, u_2) | (u_1, u_2) \in U\}, \tag{8}$$

subject to (4)–(6), where

$$U = \{(u_1, u_2) | u_1, u_2 \text{ measurables}, 0 \leq u_1 \leq 1, 0 \leq u_2 \leq 1, t \in [0, t_f]\}. \tag{9}$$

In order to find an optimal solution, first we find the Lagrangian and Hamiltonian for our optimal control problem. In fact, the Lagrangian of the optimal problem is given by

$$\mathcal{L}(I, A, D, u_1, u_2) = c_1 I(t) - c_2 A(t) + c_3 D(t)$$
$$+ \frac{K_1}{2} u_1^2(t) + \frac{K_2}{2} u_2^2(t). \tag{10}$$

3.3. Existence of an Optimal Solution. To prove that there is an optimal solution of problem (8), we will use a result, Theorem 1 in the following, which ensures the existence of the solution for optimal control problems contained in Theorem III.4.1 and Corollary III.4.1 in [22]. Problem (8) is an optimal control problem in Lagrange form

$$J(x, u) = \int_{t_0}^{t_1} \mathscr{L}(t, x(t), u(t)) dt \longrightarrow \min,$$

$$\begin{cases} x'(t) = f(t, x(t), u(t)), \\ x(t_0) = x_0, \end{cases} \quad (11)$$

$$x(\cdot) \in AC([t_0, t_1]; \mathbb{R}^n),$$

$$u(\cdot) \in L^1([t_0, t_1]; U \in \mathbb{R}^m).$$

In the above context, we say that a pair $(x, u) \in AC([t_0, t_1]; \mathbb{R}^n) \times L^1([t_0, t_1]; U \in \mathbb{R}^m)$ is feasible if it satisfies the Cauchy problem in (11). We denote the set of all feasible pairs by $\mathscr{F}$. Next, we recall the following.

Theorem 1. (see [22]). For problem (8), suppose that f and $\mathscr{L}$ are continuous and there exist positive constants C_1 and C_2 such that, for $t \in \mathbb{R}$, $x, x_1, x_2 \in \mathbb{R}^n$, and $u \in \mathbb{R}^m$, we have

(1) $\|f(t, x(t), u(t))\| \le C_1(1 + \|x\| + \|u\|)$

(2) $\|f(t, x_1(t), u(t)) - f(t, x_2(t), u(t))\| \le C_2\|x_1 - x_2\|(1 + \|u\|)$

(3) $\mathscr{F}$ is a nonempty set

(4) U is closed

(5) There is a compact set S such that $x(t_1) \in S$ for any state variable x

(6) U is convex, $f(t, x, u) = \alpha(t, x) + \beta(t, x)u$, and $\mathscr{L}(t, x)$ is convex on U

(7) $\mathscr{L}(t, x, u) \ge c_1|u|^\beta - c_2$, for some $c_1 > 0$ and $\beta > 1$

Then, there exist (x^*, u^*) minimizing J on $\mathscr{F}$.

Applying Theorem 1 to our problem, we obtain the following result:

Theorem 2. *There exists an optimal control pair (u_1^*, u_2^*) and a corresponding solution of the initial value problem in (8), (I^*, A^*, D^*), which minimizes the cost functional J in (8) over $L^1([0, t_f]; [0, 1] \times [0, 1]))$.*

Proof. First of all, note that if we sum equations (4)–(6), we conclude that the total population is constant: $N(t) = I(0) + A(0) + D(0) = N(0)$. Thus,

$$I(t), A(t), D(t) \le N(0). \quad (12)$$

Additionally, $((A(t)I(t))/N(t)) \le A(t) \le N(0)$, $((A(t)D(t))/N(t)) \le D(t) \le N(0)$, and $((D(t)I(t))/N(t)) \le I(t) \le N(0)$. We immediately obtain (1) and (2).

Conditions (3) and (4) are immediate from the definition of $\mathscr{F}$ since $U = [0, 1] \times [0, 1]$. We conclude that all the state variables are in the compact set

$$\{(x, y, z) \in (\mathbb{R}^+)^3 : 0 \le x + y + z = N(0)\}, \quad (13)$$

and condition (5) follows. Since the state equations are linearly dependent on the controls and $\mathscr{L}$ is quadratic in the controls, we obtain (6). Finally,

$$\mathscr{L} = c_1 I(t) - c_2 A(t) + c_3 D(t) + \frac{K_1}{2}u_1^2(t) + \frac{K_2}{2}u_2^2(t)$$

$$\ge \min\left\{\frac{K_1}{2}, \frac{K_2}{2}\right\}\|(u_1, u_2)\|^2, \quad (14)$$

and we establish (7) with $c_1 = \min\{(K_1/2), (K_2/2)\}$. Therefore, the result follows from Theorem 1. $\quad\square$

3.4. Characterization of the Optimal Controls. We seek the minimal value of the Lagrangian. To accomplish this, we define the Hamiltonian $\mathscr{H}$ as follows:

$$\mathscr{H} = \mathscr{L}(I, A, D, u_1, u_2)$$

$$+ \lambda_1(t)\left[-\beta_1\frac{A(t)I(t)}{N} - \beta_2\frac{D(t)I(t)}{N} + \gamma_1 A(t) + \gamma_2 D(t) - u_1(t)I(t) + u_2(t)D(t)\right]$$

$$+ \lambda_2(t)\left[\beta_1\frac{A(t)I(t)}{N} + \alpha_1\frac{A(t)D(t)}{N} - \alpha_2\frac{A(t)D(t)}{N} - \gamma_1 A(t) + u_1(t)I(t)\right] \quad (15)$$

$$+ \lambda_3(t)\left[\beta_2\frac{D(t)I(t)}{N} + \alpha_2\frac{A(t)D(t)}{N} - \alpha_1\frac{A(t)D(t)}{N} - \gamma_2 D(t) - u_2(t)D(t)\right].$$

To find the optimal solution, we apply Pontryagin's maximum principle to the Hamiltonian [23], and we obtain the following theorem.

Theorem 3. *Let $I^*(t)$, $A^*(t)$, and $D^*(t)$ be optimal state solutions with associated optimal control variables $u_1^*(t)$ and $u_2^*(t)$ for optimal control problem (8). Then, there exist adjoint variables $\lambda_1(t), \lambda_2(t)$, and $\lambda_3(t)$ that satisfy*

$$\dot{\lambda}_1(t) = -\left[c_1 - \lambda_1(t)\left(u_1(t) + \frac{A(t)\beta_1}{N} + \frac{D(t)\beta_2}{N}\right) + \lambda_2\left(u_1(t) + \frac{A(t)\beta_1}{N}\right) + \frac{D(t)\beta_2\lambda_3(t)}{N}\right],$$

$$\dot{\lambda}_2(t) = -\left[-c_2 + \lambda_1(t)\left(\gamma_1 - \frac{I(t)\beta_1}{N}\right) - \lambda_3(t)\left(\frac{D(t)\alpha_1}{N} - \frac{D(t)\alpha_2}{N}\right) - \lambda_2(t)\left(\gamma_1 - \frac{D(t)\alpha_1}{N} + \frac{D(t)\alpha_2}{N} - \frac{I(t)\beta_1}{N}\right)\right]$$

$$\dot{\lambda}_3(t) = -\left[c_3 - \lambda_3(t)\left(\gamma_2 + u_2(t) + \frac{A(t)\alpha_1}{N} - \frac{A(t)\alpha_2}{N} - \frac{I(t)\beta_2}{N}\right) + \lambda_2(t)\left(\frac{A(t)\alpha_1}{N} - \frac{A(t)\alpha_2}{N}\right) + \lambda_1(t)\left(\gamma_2 + u_2(t) - \frac{I(t)\beta_2}{N}\right)\right],$$

$$(16)$$

with the transversality conditions $\lambda_i(t_f) = 0$, $i = 1, 2, 3$. Furthermore, the optimal controls $u_1^*(t)$ and $u_2^*(t)$ are given by

$$u_1^*(t) = \max\left\{\min\left\{\frac{I(t)(\lambda_1(t) - \lambda_2(t))}{K_1}, 1\right\}, 0\right\},$$

$$u_2^*(t) = \max\left\{\min\left\{\frac{D(t)(\lambda_3(t) - \lambda_1(t))}{K_2}, 1\right\}, 0\right\}. \tag{17}$$

Proof. To determine the adjoint equations and the transversality conditions, we use the Hamiltonian $\mathcal{H}$ defined by (15). From setting $I(t) = I^*(t)$, $A(t) = A^*(t)$, and $D(t) = D^*(t)$ and differentiating $\mathcal{H}$ with respect to $I(t)$, $A(t)$, and $D(t)$, we obtain

$$\dot{\lambda}_1(t) = -\frac{\partial\mathcal{H}}{\partial I}$$

$$= -\left[c_1 - \lambda_1(t)\left(u_1(t) + \frac{A(t)\beta_1}{N} + \frac{D(t)\beta_2}{N}\right) + \lambda_2\left(u_1(t) + \frac{A(t)\beta_1}{N}\right) + \frac{D(t)\beta_2\lambda_3(t)}{N}\right],$$

$$\dot{\lambda}_2(t) = -\frac{\partial\mathcal{H}}{\partial A}$$

$$= -\left[-c_2 + \lambda_1(t)\left(\gamma_1 - \frac{I(t)\beta_1}{N}\right) - \lambda_3(t)\left(\frac{D(t)\alpha_1}{N} - \frac{D(t)\alpha_2}{N}\right) - \lambda_2(t)\left(\gamma_1 - \frac{D(t)\alpha_1}{N} + \frac{D(t)\alpha_2}{N} - \frac{I(t)\beta_1}{N}\right)\right],$$

$$\dot{\lambda}_3(t) = -\frac{\partial\mathcal{H}}{\partial D}$$

$$= -\left[c_3 - \lambda_3(t)\left(\gamma_2 + u_2(t) + \frac{A(t)\alpha_1}{N} - \frac{A(t)\alpha_2}{N} - \frac{I(t)\beta_2}{N}\right) + \lambda_2(t)\left(\frac{A(t)\alpha_1}{N} - \frac{A(t)\alpha_2}{N}\right) + \lambda_1(t)\left(\gamma_2 + u_2(t) - \frac{I(t)\beta_2}{N}\right)\right].$$

$$(18)$$

By the optimality conditions, we have

$$\frac{\partial\mathcal{H}}{\partial u_1} = I\lambda_2 - I\lambda_1 + K_1u_1 = 0, \tag{19}$$

and then

$$u_1(t) = \frac{I(t)(\lambda_1(t) - \lambda_2(t))}{K_1}, \tag{20}$$

and from

$$\frac{\partial\mathcal{H}}{\partial u_2} = D\lambda_1 - D\lambda_3 + K_2u_2 = 0, \tag{21}$$

we have

$$u_2(t) = \frac{D(t)(\lambda_3(t) - \lambda_1(t))}{K_2}. \tag{22}$$

As our controls are bounded below by 0 and above by 1, we have

$$u_1^*(t) = \max\left\{\min\left\{\frac{I(t)(\lambda_1(t) - \lambda_2(t))}{K_1}, 1\right\}, 0\right\},$$

$$u_2^*(t) = \max\left\{\min\left\{\frac{D(t)(\lambda_3(t) - \lambda_1(t))}{K_2}, 1\right\}, 0\right\}. \tag{23}$$

□

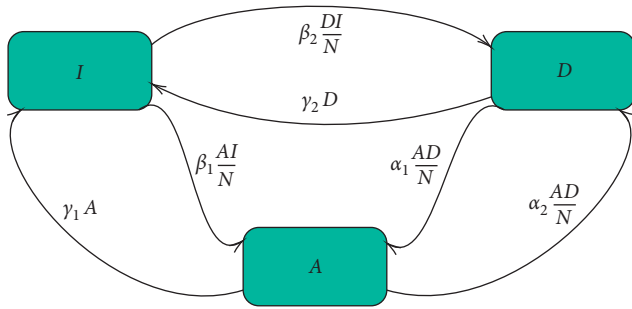Figure 1: Flowchart for the model (1)–(3).

Table 1: Parameters' descriptions and values.

Parameter	Description	Values	
		Data 1	Data 2
β_1	Indifferent to agree transmission rate	0.91	0.1200
β_2	Indifferent to disagree transmission rate	0.5816	0.8900
α_1	Disagree to agree transmission rate	0.3346	0.4858
α_2	Agree to disagree transmission rate	0.2459	0.3362
γ_1	Interest loss factor of agree individuals	0.2579	0.0898
γ_2	Interest loss factor of disagree individuals	0.0938	0.0001

4. Numerical Simulation

We now present numerical simulations which are associated with our optimal system derived from the previous mathematical model. We wrote a code in MATLAB™ and simulated our results using different data. We solve the optimality systems using an iterative method with a progressive-regressive Runge–Kutta fourth-order scheme. Such numerical procedures are called forward-backward sweep methods, where the state system with an initial guess is solved forward in time, and then the adjoint system is solved backward in time. First, starting with an initial guess for the adjoint variables λ_1, λ_2, and λ_3, we solve the state equations by a forward Runge–Kutta fourth-order procedure in time. Then, those state values are used to solve the adjoint equations by a backward Runge–Kutta fourth-order procedure because of the transversality conditions [24–27]. Afterwards, we updated the optimal control values using the values of state and costate variables obtained in the previous steps. Finally, we execute the previous steps until a tolerance criterion is reached.

In order to show the importance of our work, here we consider two examples of data given in Table 1.

4.1. Example 1. In the following simulations, we use parameter values of data 1 given in Table 1. We chose these parameters here because we know the poll result, and in such situation, the control intervention is needed, see Figure 2. The survey targets 252 people, including 250 indifferent, one person agrees and one person disagrees. The interrogation period is 5 days, and users are allowed to change their minds.

Figure 2 depicts the state variables I, A, and D of the model (1)–(3) when there is no control intervention. It can be seen that. from the beginning of the poll until about 17 hours, the number of people approving increases significantly and then begins to decrease continuously until the end of the survey, while the number of people who disagree increases rapidly by about 17 hours, and it continues to increase until the end of the survey. The number of indifferent people decreases very rapidly in 17 hours, from 250 to about 70, to begin to decrease slightly.

Around 90 hours, we can predict the result of this survey, where the number of people in disagreement continues to increase, and the number of people in agreement continues to decrease. However, implementing control strategies may take some time to influence the outcome, and therefore, it is recommended that controls be introduced early.

Figure 3 depicts the state variables of the model (1)–(3) when controls are applied from the beginning of the survey. It can be seen that the number of people who approved increased and stabilized at around 200 people until the end of the survey. And the number of indifferent people decreased and stabilized at around 50 people until the end of the survey, while the number of disapproving people takes about zero values. This control strategy gives satisfactory results within 5 hours.

Note that the control functions decrease quickly at first and settle at constant values until the end of the interrogation time, see Figure 3(b). The u_2 control takes zero values from 110 hours, which means that there is no longer need for this type of control, but the control u_1 is maintained until the end of the survey. This simulation shows the effectiveness of optimal controls to reduce the number of people who refuse and increase the number of people who approve.

Here, we also consider a different scenario in which the control strategy is implemented after detecting a decrease in the number of people approving, that is, after 40 hours from the start of the poll. Figure 4 depicts the state variables of the model (4)–(6) in the case when the controls are introduced at instant $t = 40$. We can see that, after discovering the decrease in the number of people in agreement, between $t = 16$ and $t = 40$, controls were established to control the situation in order to increase the number of people in agreement and thus changing the increase in the number of people who did not agree to a decrease towards zero, but this time, the u_1 control is also maintained at a fixed value until the end, while the u_2 control tends to zero by about 100 hours, compared to the case where the controls are introduced at the start of the poll.

The simulation of this scenario shows that it is possible to effectively control the survey results even if the controls are not implemented at the start of the survey. In some cases, the authorities have to wait to see the course of events without controls because it is possible to obtain satisfactory results without any type of control, but this is not always the case. Obviously, this second scenario is less expensive than the first scenario because applying the controls from the beginning of the survey may require more effort and cost.

As a different scenario, we discuss hereafter the use of one control instead of two controls. Figure 5(a) depicts the
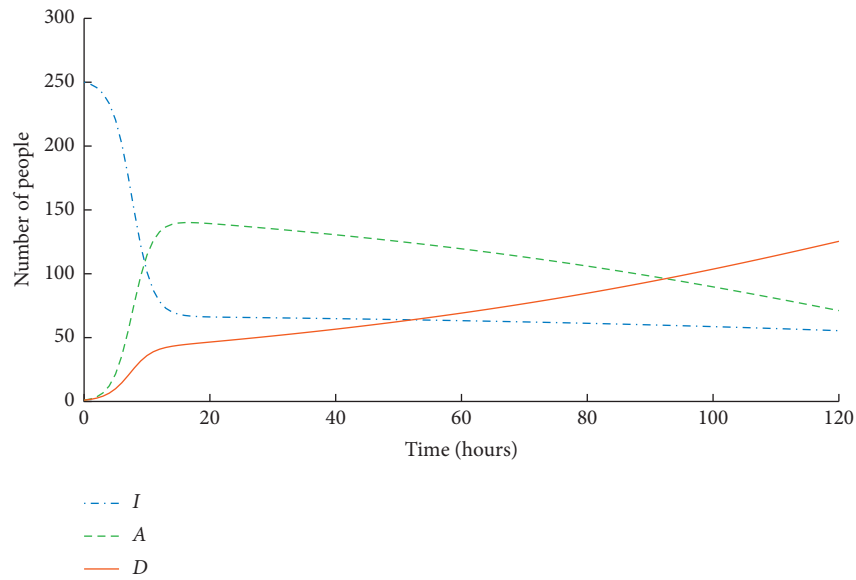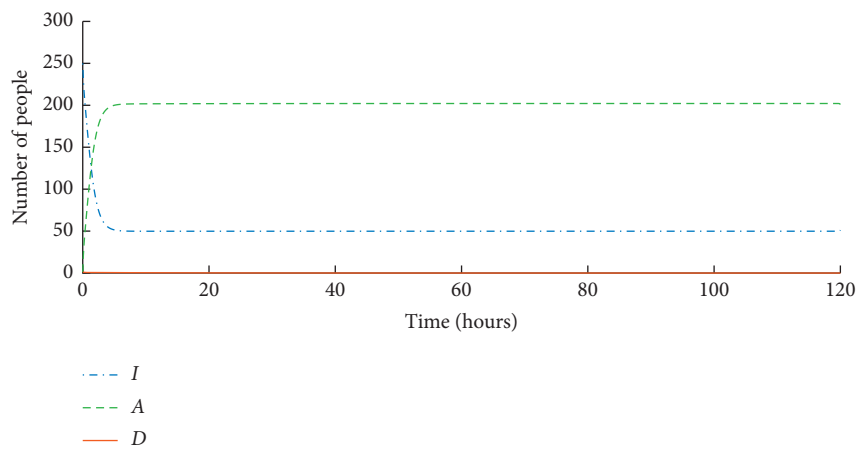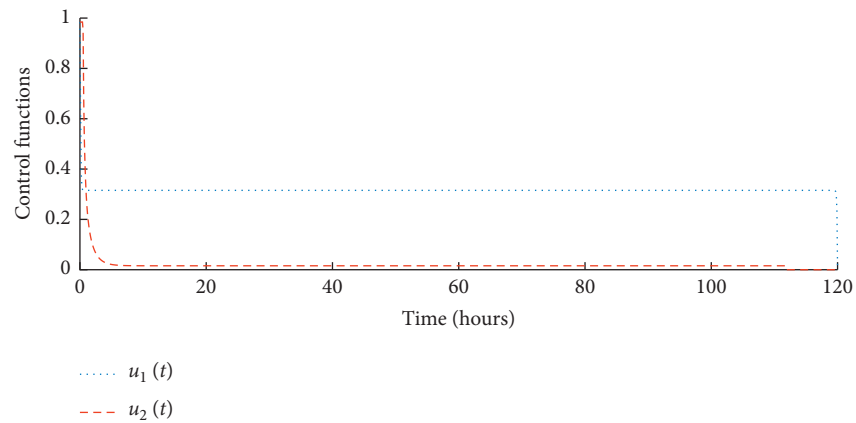
FIGURE 2: State variables of the model (1)–(3) without controls.

(a)

(b)

FIGURE 3: State variables of the model (4)–(6) with controls.
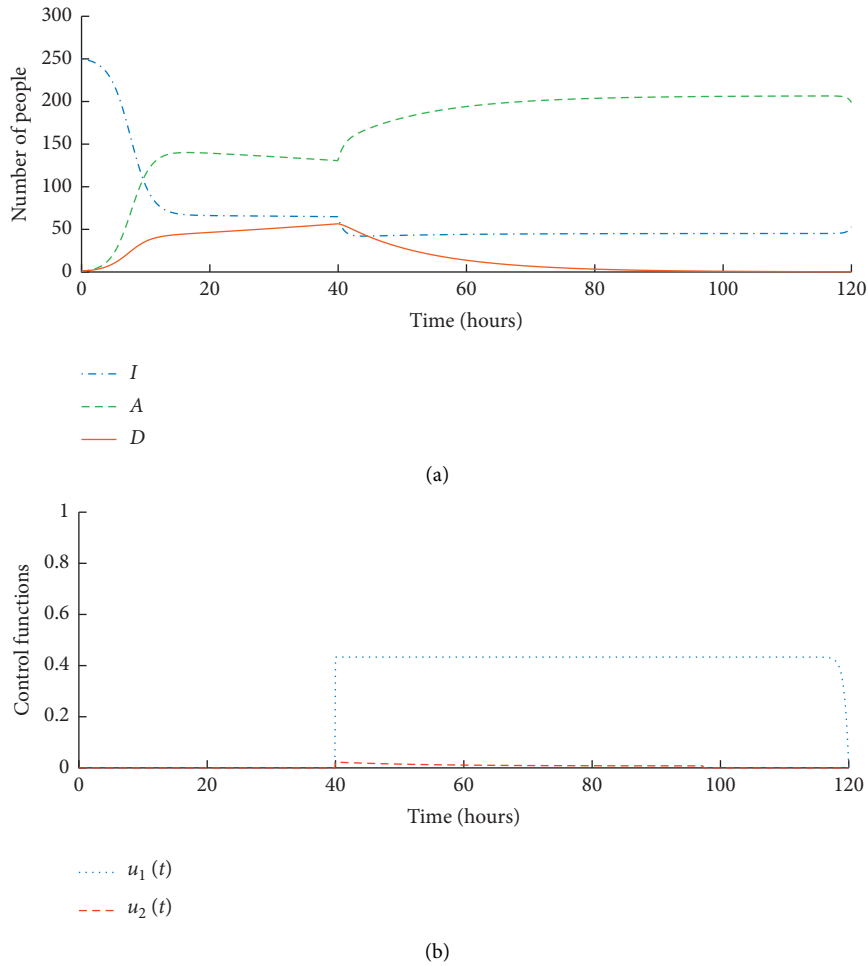
(a)

(b)

Figure 4: State variables of the model (4)–(6) with controls. The control strategy is implemented after 40 hours.

state variables of the model (4)–(6) when only the control u_1 is used. It can be seen that this strategy of control is also effective in reducing the number of disapproving people and increasing the number of approving individuals by about 5 hours.

It can be seen that the number of people approving increases to about 210 individuals and then begins to decrease slightly towards 200 by the end of the poll, while the number of indifferent people decreases to about 48 individuals and then begins to increase towards 50 individuals by the end of the survey. Note that the number of disapproved people takes approximately zero values.

We can see that the control variable u_1 takes the maximum value of 1 until about 8 hours to begin decreasing, see Figure 5(b). This control variable takes big values compared to the case when both controls are used. Thus, in this control strategy, an additional effort of u_1 is necessary to obtain good results.

Figure 6(a) depicts the state variables of the model (4)–(6) when only the control u_2 is used. We can see that the number of people approving increases to about 175 individuals and then stabilizes at this number until the end of the poll, while the number of indifferent people decreases to 65

individuals and then stabilizes at this number until the end of the survey. Note that the number of disapproved people also takes a small value of 10 individuals.

It can also be seen that the control variable u_2 takes the maximum value of 1 until about 4 hours to begin decreasing, see Figure 6(b). This control variable u_2 takes approximately the same values compared to the case when both controls are used. Therefore, more effort of u_2 at the beginning is needed to obtain good results.

In the scenario of using only the u_2 control, the state variables of the model stabilize in around 18 hours, while they stabilize in 5 hours when we use the u_1 control in the second scenario. Simulating these scenarios shows that a single control is enough to get a good result, but it requires more energy, effort, and more costs.

4.2. Example 2. In the following simulations, we use parameter values of data 2 given in Table 1. In this case, we also know the poll result, where the control intervention is needed, see Figure 7.

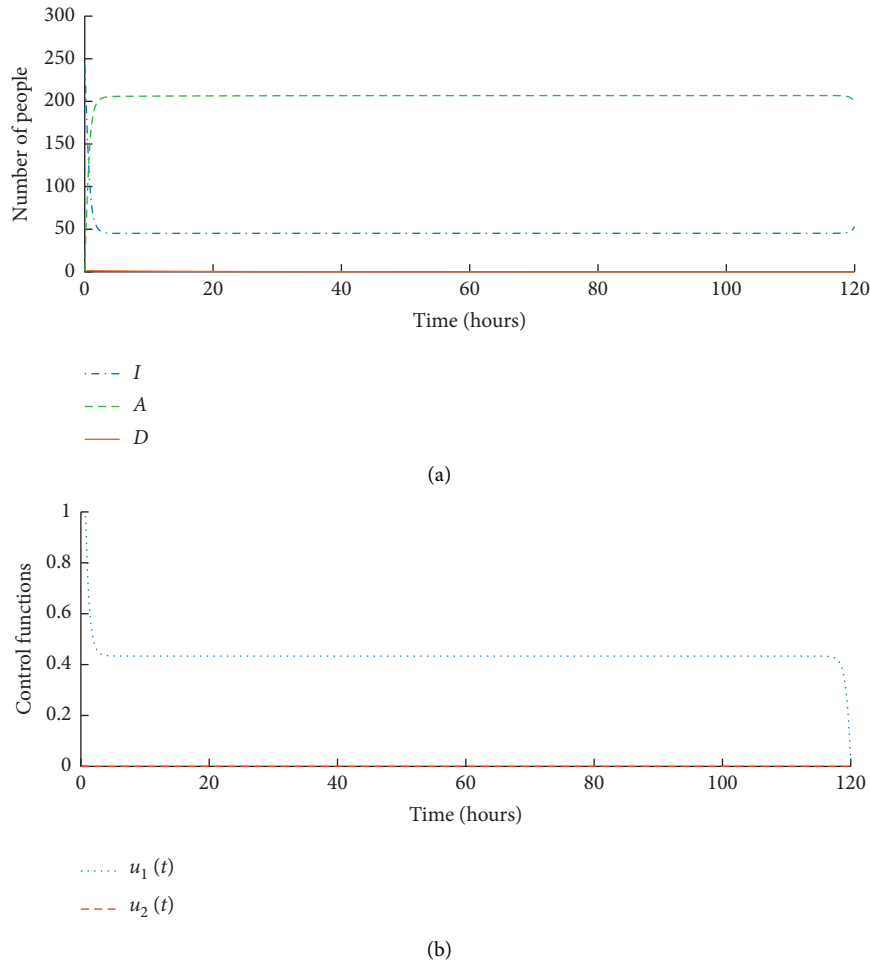In this example, we use the estimated parameters in [19] corresponding to the data of the approval poll done by the

Figure 5: State variables of the model (4)–(6) with controls. Only the control u_1 is used.

Reuters polling system on how the US president fulfills his role [28]. The data of this survey are collected for approximately 115 weeks, from January 29, 2017, to April 7, 2019, and they are given in the form of proportions. This poll is answered with the following three questions: approve, disapprove, and mixed feelings.

Figure 7 depicts the state variables I, A, and D of the model (1)–(3) when there is no control intervention. It can be seen that, from the beginning of the poll until about 25 weeks, the number of people disapproving increases significantly and stabilizes at about 55% until the end of the survey, while the number of people who agree decreases by about 25 weeks and stabilizes at about 41% until the end of the survey. The number of indifferent people is almost fixed at about 8% from the beginning to the end of that poll.

Therefore, around 30 weeks, we can predict the result of this survey. When the survey is dominated by disagreeing opinion, i.e. when the number of people who disagree becomes greater than the number of people who agree. Thus, it is recommended to introduce some controls to bring the situation under control.

Figure 8(a) depicts the state variables of the model (4)–(6) when controls are applied from the beginning of the survey. It can be seen that the number of people who

approved increased and stabilized at around 80% of the population until the end of the survey. And the number of indifferent people increased and stabilized at around 16% until the end of the survey, while the number of disapproving people takes small values about 4%. This control strategy gives satisfactory results within less than 10 weeks.

Note that the control functions take decreasing values until the end of the poll, see Figure 8(b). The u_2 control takes small values compared to values taken by u_1, which means that putting more effort into investing in positive media and publicizing the electoral platform, combined with a little effort to upset competitors with negative media, is an important control strategy to help the candidate bring in more people by his side. This simulation shows the effectiveness of optimal controls to reduce the number of people who disapprove and increase the number of people who approve.

Here, we also consider a different scenario in which the control strategy is implemented after detecting a decrease in the number of people approving, that is, after 40 weeks from the start of the poll. Figure 9 depicts the state variables of the model (4)–(6) in the case when the control u_1 is introduced at instant $t = 40$. We can see that, after
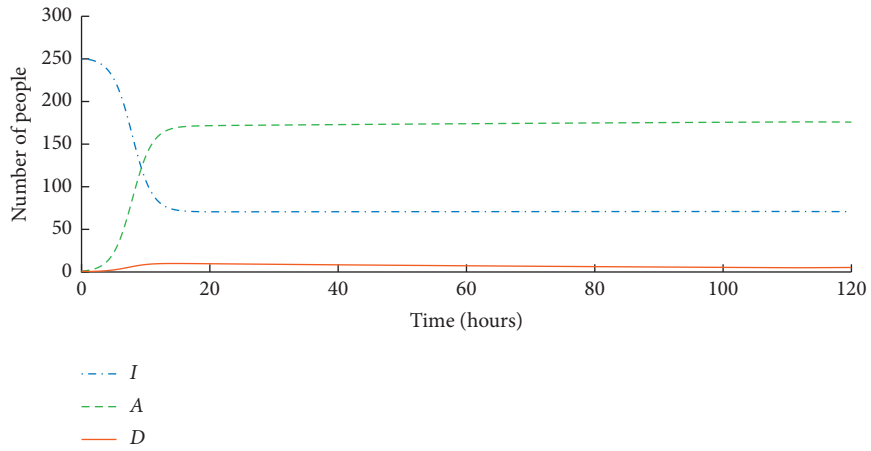
(a)
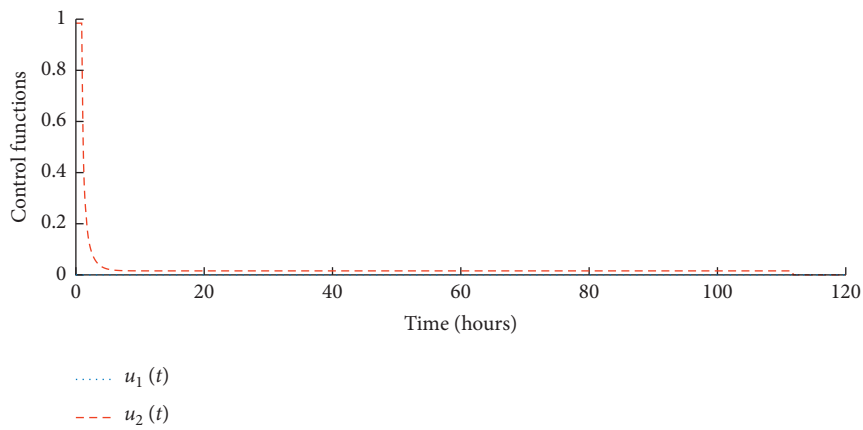
(b)

FIGURE 6: State variables of the model (4)–(6) with controls. Only the control u_2 is used.
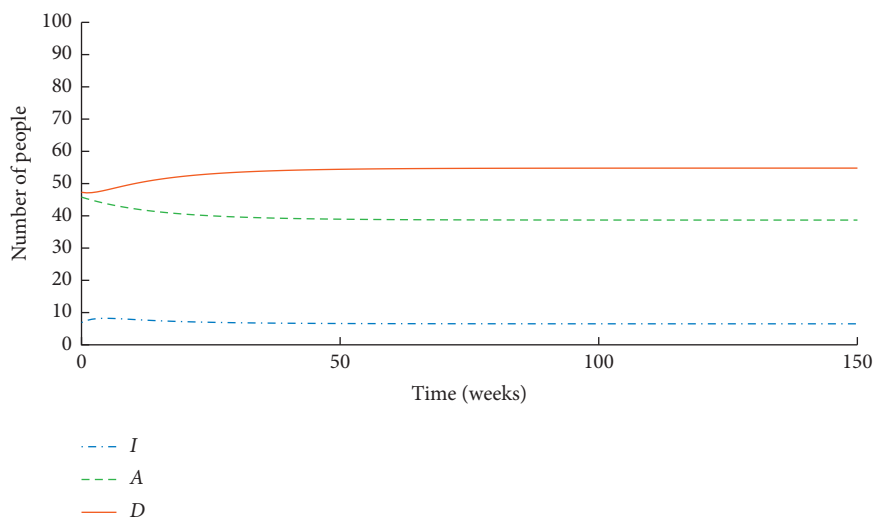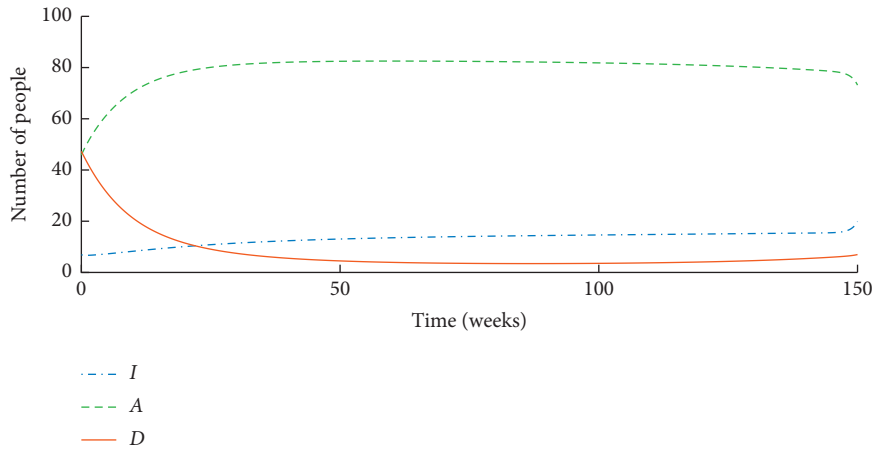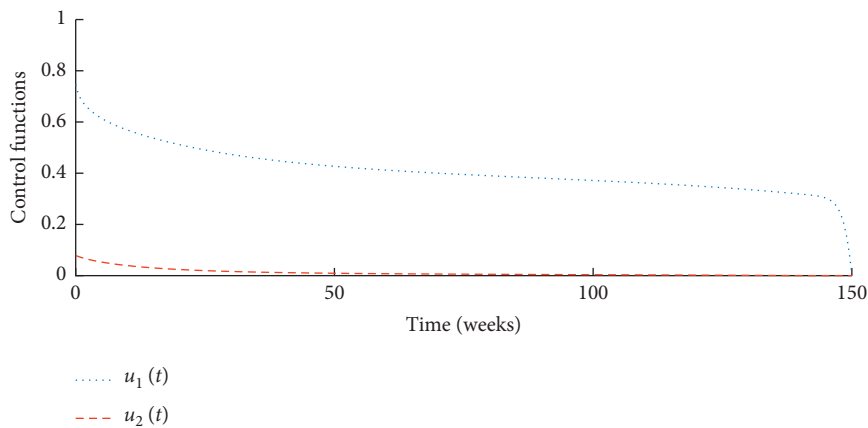

FIGURE 7: State variables of the model (1)–(3) without controls using the data of example 2.

(a)

(b)

FIGURE 8: State variables of the model (4)–(6) with controls in example 2.

discovering the decrease in the number of people in agreement, between $t = 1$ and $t = 40$, the control is established to increase the number of individuals in agreement and thus changing the increase in the number of disagreeing people to a decrease towards small values. The simulation of this scenario shows that it is possible to effectively control the survey results even if the controls are not implemented at the start of the survey and with only one control u_1.

Figure 10 depicts the state variables of the model (4)–(6) in the case when only the control u_2 is introduced at instant $t = 40$. It can be seen that the control u_2 helps to reduce the number of disagreeing people and increase the number of indifferent people, but the number of agreeing individuals is still less than the number of people who disagree, which means that only the control u_2 is not sufficient to change the poll outcome. Obviously, using one control is less expensive than using both, and using the controls from the beginning of the survey requires more effort and cost.

5. Global Sensitivity Analysis

Sensitivity analysis is a method of quantifying uncertainty in any type of model. The objective of the sensitivity analysis is to identify the critical inputs of a model and to quantify the impact of the uncertainty of the inputs on the outputs of the model. When the input factors are known with little uncertainty, we can examine the partial derivative of the output function with respect to the input factors. This technique is called local sensitivity analysis (LSA) because it studies the impact on the output of the model based on changes in factors very close to the nominal values. This can easily be calculated numerically by performing several simulations and varying the input factors around the nominal value [16].

In order to search for the parameters that are key factors to opinion control, we perform a global sensitivity analysis (GSA). The approach of the global sensitivity analysis (GSA) makes it possible to identify the efficiency of the parameters or inputs of the model and thus provides essential
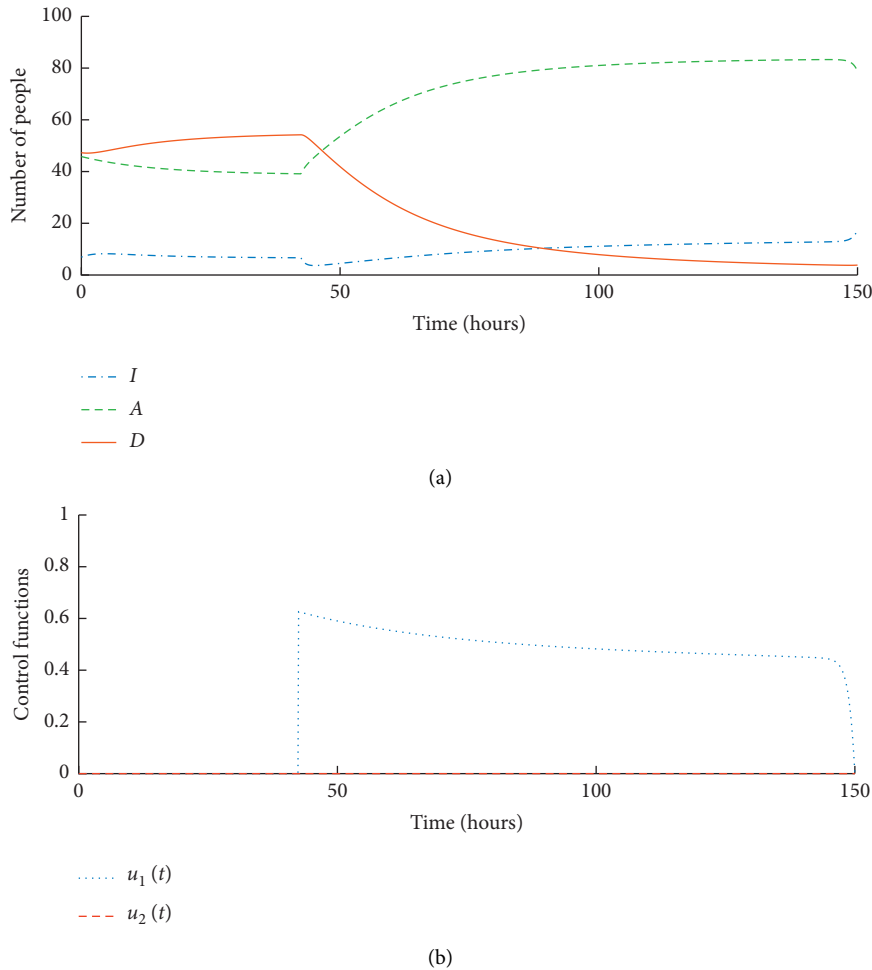
(a)

(b)

FIGURE 9: State variables of the model (4)–(6) with controls using the data of example 2. Only the control u_1 is used.

information on the performances of the model. Among the many methods of the global sensitivity analysis, there is the method of partial rank correlation coefficient (PRCC). These coefficients are similar to the common Pearson correlation coefficient but also deal with nonlinearities, as long as the output is monotonic in the parameters [16]. We assessed the impact of variations in the model's parameters on the outputs $J(x, u)$ using the partial rank correlation coefficient (PRCC).

The PRCC is a method based on sampling. One of the most efficient methods used for sampling is the Latin hypercube sampling (LHS). The LHS belongs to the Monte Carlo sampling methods' category that provides an unbiased estimate of the model's average output, with the advantage that it requires fewer samples than simple random sampling to reach the same accuracy [29]. The LHS is so-called stratified sampling without replacement technique, where the random distributions of the parameters are divided into 400 equal probability intervals, which are

then sampled, assuming that the sampling is carried out independently for each parameter, where this sampling is done by randomly selecting values from each pdf given in Table 2. Each interval for each parameter is sampled exactly once so that the full range of each parameter is explored. This produces 400 sets of values of $J(x, u)$, from 400 sets of different parameter values mixed randomly, calculated by using equation (11).

The parameters β_1 and β_2 are considered following normal distribution with mean and standard deviation 0.5 and 0.01, respectively. The other parameters are considered following triangular distribution with minimum, maximum, and mode as 0.0001, 0.8, and baseline value from Table 1 (data 1), respectively. A summary of probability distribution functions is given in Table 2.

We consider a significance level of 0.05, meaning that PRCCs with p values greater than 0.05 are not significant. We found that the interest loss factor of agree individuals γ_1 and the interest loss factor of disagree individuals γ_2 are
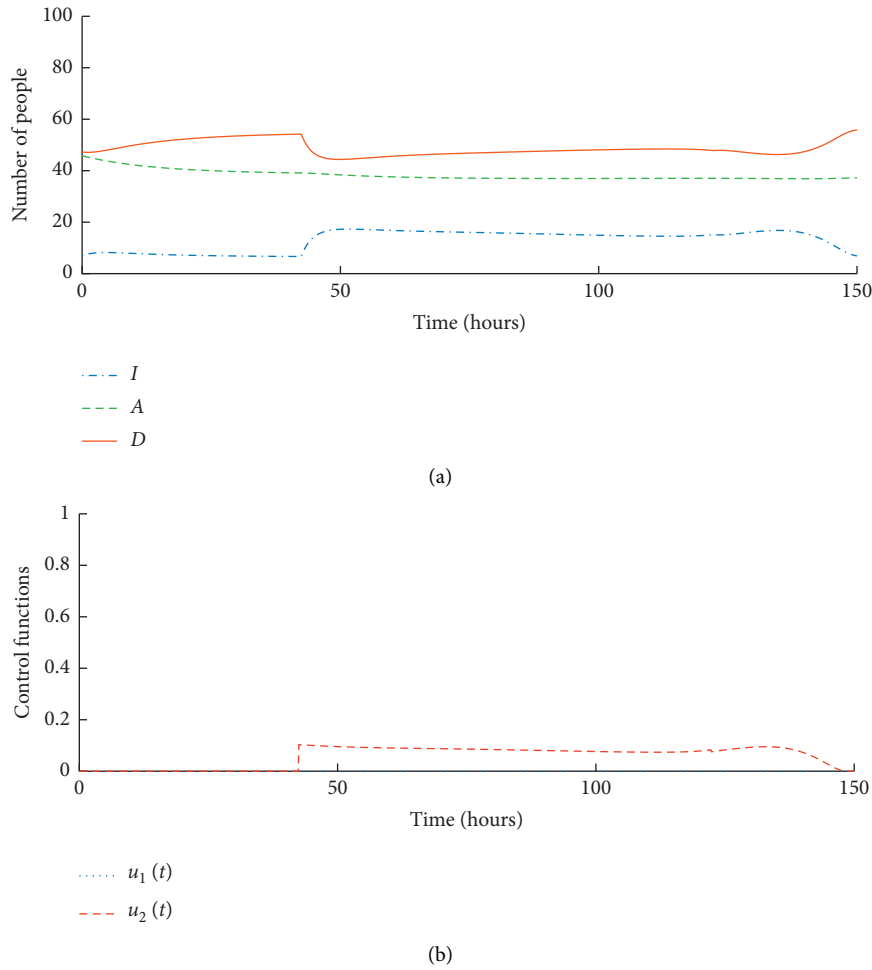
(a)

(b)

FIGURE 10: State variables of the model (4)–(6) with controls using the data of example 2. Only the control u_2 is used.

TABLE 2: Probability distributions.

Parameter	Normal distribution		
	Mean		Variance
β_1	0.5		0.01
β_2	0.5		0.01
	Triangular distribution		
	Lower bound	Nominal value	Upper bound
α_1	0.0001	0.3346	0.8
α_2	0.0001	0.2459	0.8
γ_1	0.0001	0.2579	0.8
γ_2	0.0001	0.0938	0.8

positively correlated with $J(x, u)$, while the indifferent to disagree transmission rate β_2 and the disagree to agree transmission rate α_1 are negatively correlated with $J(x, u)$. The indifferent to agree transmission rate β_1 and the agree to disagree transmission rate α_2 have insignificant PRCCs.

As shown in Table 3, the parameters γ_1, γ_2, and β_2 are highly correlated with $J(x, u)$ with corresponding values 0.3070, 0.2623, and −0.2183, respectively. Moderate correlation exists between transmission rate α_1 and $J(x, u)$ with corresponding value −0.1668. Scatter plots comparing $J(x, u)$ against each of the six parameters, β_1, β_2, α_1, α_2, γ_1, and γ_2, are shown in Figure 11 based on Latin hypercube

TABLE 3: PRCCs for $J(u_1, u_2)$ and six input parameters.

Parameters	Sampling	PRCCs	p value
β_1	LHS	0.0460	0.3585
β_2	LHS	−0.2183	$1.0532e - 05$
α_1	LHS	−0.1668	0.00081
α_2	LHS	0.0959	0.0552
γ_1	LHS	0.3070	$3.4993e - 10$
γ_2	LHS	0.2623	$1.0194e - 07$

sampling with a sample size of 400. These scatter plots clearly show the linear relationships between the outcome of $J(x, u)$ and input parameters.
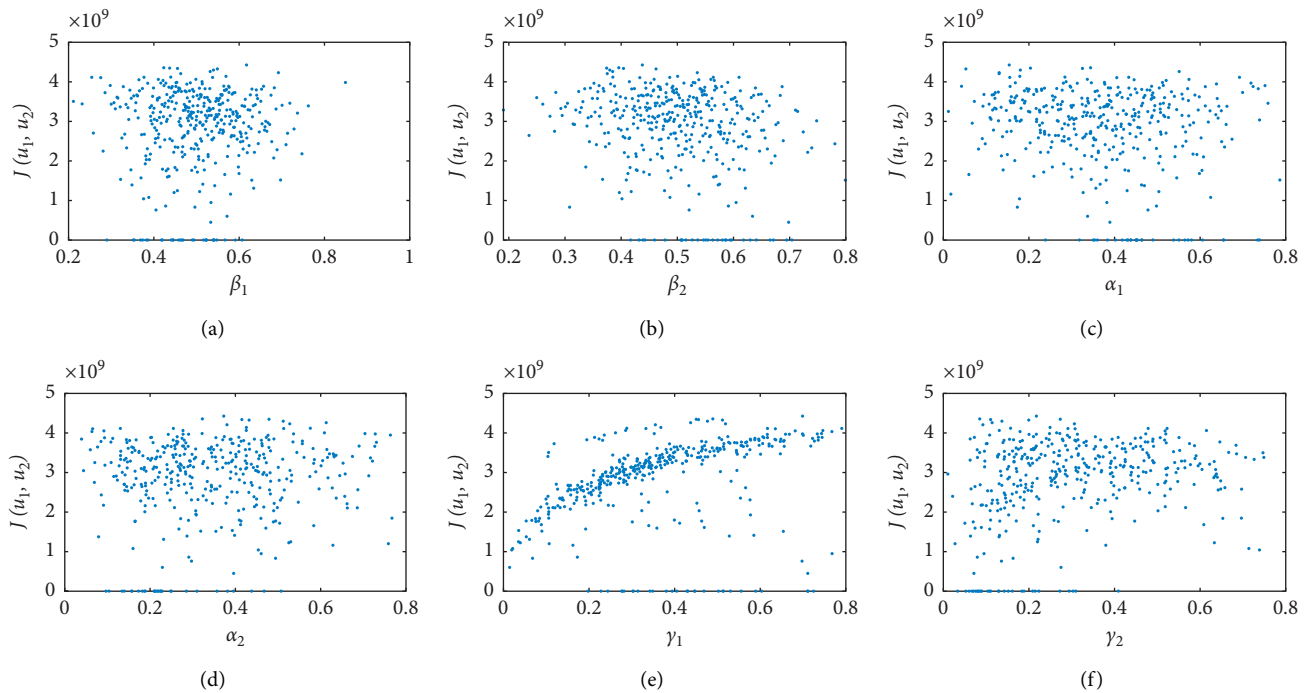
Figure 11: Scatter plots for the basic reproduction number $J(u_1, u_2)$ and six sampled input parameters' values. These results have been obtained from Latin hypercube sampling using a sample size of 400.

6. Conclusion

In this paper, we applied optimal control to a new mathematical model that we presented and analyzed in [13] which describes the evolution of opinions during polls. We incorporated two control variables. The first presents the effects of the media and publicity to convince people to change their mind and then bring them to the agreeing group. The second control is the effects of negative media against competitors by providing people with negative information about the competitor or information clarifying certain ambiguities to at least motivate them not to vote. We proved the existence of optimal controls that ensure the minimization of indifferent and disagreeing individuals by using possible minimal costs of control application. We characterized optimal controls by using Pontryagin's maximum principle, and we developed the optimality system and solved it using an iterative numerical method in order to simulate several possible scenarios, with and without optimal controls.

A partial rank correlation coefficient (PRCC) sensitivity analysis on parameters with the objective functional as the output is carried out. This global sensitivity analysis shows that the cost function is most sensitive to the interest loss factor of agree individuals γ_1, the interest loss factor of disagree individuals γ_2, and the indifferent to disagree transmission rate β_2.

Data Availability

The parameter values are from [19].

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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