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Research Article

On One Evolution Equation of Parabolic Type with Fractional Differentiation Operator in S Spaces

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In the paper, we investigate a nonlocal multipoint by a time problem for the evolution equation with the operator $A = (I - \Delta)^{\omega/2}$, $\Delta = (\mathrm{d}^2/\mathrm{d}x^2)$, and $\omega \in [1; -2)$ is a fixed parameter. The operator A is treated as a pseudodifferential operator in a certain space of type S. The solvability of this problem is proved. The representation of the solution is given in the form of a convolution of the fundamental solution with the initial function which is an element of the space of generalized functions of ultradistribution type. The properties of the fundamental solution are investigated. The behavior of the solution at $t \longrightarrow +\infty$ (solution stabilization) in the spaces of generalized functions of type S' and the uniform stabilization of the solution to zero on $\mathbb{R}$ are studied.

1. Introduction

A rather wide class of differential equations with partial derivatives forms linear parabolic and B-parabolic equations, and the theory of which originates from the study of the heat conduction equation. The classical theory of the Cauchy problem and boundary-value problems for such equations and systems of equations is constructed in the works of I.G. Petrovskiy, S.D. Eidelman, S.D. Ivasyshen, M.I. Matiychuk, M.V. Zhitarashu, A. Friedman, S. Teklind, V.O. Solonnikov, and V.V. Krekhivskiy et al. The Cauchy problem with initial data from the spaces of generalized functions of the type of distributions and ultradistributions was studied by G.E. Shilov, B.L. Gurevich, M.L. Gorbachuk, V.I. Gorbachuk, O.I. Kashpirovskiy, Ya.I. Zhytomyrskiy, S.D. Ivasyshen, V.V. Gorodetskiy, and V.A. Litovchenko et al.

A formal extension of the class of parabolic type equations is the set of evolution equations with the pseudodifferential operator (PDO), which can be represented as $A = J^{-1}_{\sigma \longrightarrow x}[a(t, x; \sigma) J_{x \longrightarrow \sigma}]$, $\{x, \sigma\} \subset \mathbb{R}^n$, $t > 0$, where a is a function (symbol) that satisfies certain conditions and $J\,(J^{-1})$ is the direct (inverse) Fourier or Bessel transform. The PDO includes differential operators, fractional differentiation and integration operators, convolution operators, and the Bessel operator $B_\nu = (\mathrm{d}^2/\mathrm{d}x^2) + (2\nu + 1)x^{-1}(\mathrm{d}/\mathrm{d}x)$, $\nu > -(1/2)$, which contains the expression $(1/x)$ in its structure and is formally represented as $B_\nu = F^{-1}_{B_\nu}[-\sigma^2 F_{B_\nu}]$, where F_{B_ν} is the Bessel integral transformation.

If A is a nonnegative self-adjoint operator in a Hilbert space H, then it is known [1] that a continuous on $[0, T)$ function $u(t)$ is continuously differentiable solution of the operator differential equation $u'(t) + Au(t) = 0$, $t \in (0, T)$, which refers to abstract equations of parabolic type, if and only if it is given as $u(t) = e^{-tA} f$, $f = u(0) \in H$. It turns out [1] that all continuously differentiable functions within the interval $(0, T)$ solutions of this equation are described by the same formula where f is an element of the wider than H space H'_a conjugate to the space H_a of analytic vectors of the operator A; the role of A is played by the extension $\widehat{A}$ of the operator A to the space H'_a, and the boundary value of $u(t)$ at the point 0 exists in the space H'_a.

If $A = (I - \Delta)^{1/2}$, $\Delta = (\mathrm{d}^2/\mathrm{d}x^2)$, then A is a nonnegative self-adjoint operator in $H = L_2(\mathbb{R})$, since $(i\mathrm{d}/\mathrm{d}x)$ is a self-adjoint in $L_2(\mathbb{R})$ operator with the domain $\mathscr{D}(i\mathrm{d}/\mathrm{d}x) = \{\varphi \in L_2(\mathbb{R})\colon \exists \varphi' \in L_2(\mathbb{R})\}$. If E_λ, $\lambda \in \mathbb{R}$, is the spectral function of the operator $(i\mathrm{d}/\mathrm{d}x)$, then, due to the basic spectral theorem for self-adjoint operators,

$$A\varphi = \left(I + \left(i\frac{\mathrm{d}}{\mathrm{d}x} \right)^2 \right)^{1/2} \varphi = \int_{-\infty}^{+\infty} \left(1+\lambda^2\right)^{1/2} \mathrm{d}E_\lambda \varphi. \quad (1)$$

It is known (see, for example, [2]) that

$$E_\lambda \varphi(t) = \frac{1}{2\pi} \int_{-\infty}^{\lambda} \int_{-\infty}^{+\infty} \left\{ \varphi(\tau) e^{i\sigma\tau} \mathrm{d}\tau \right\} e^{-it\sigma} \mathrm{d}\sigma. \quad (2)$$

It follows from this that $\mathrm{d}E_\lambda \varphi(t) = (1/2\pi) F[\varphi](\lambda) e^{-it\lambda} \mathrm{d}\lambda$. So,

$$A\varphi = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(1+\lambda^2\right)^{1/2} F[\varphi](\lambda) e^{-it\lambda} \mathrm{d}\lambda = F^{-1}\left[\left(1+\lambda^2\right)^{1/2} F[\varphi] \right]. \quad (3)$$

Following [3], we call the operator A the Bessel operator of fractional differentiation. Therefore, A can be understood as a pseudodifferential operator constructed on the function symbol $(1+\lambda^2)^{1/2}$, $\lambda \in \mathbb{R}$. This allows us to interpret the function $e^{-tA} f$ that is a solution of the corresponding Cauchy problem as a convolution of the form $G(t,\cdot) * f$ [4, 5], where $G(t,\cdot) = F^{-1}[e^{-(1+\lambda^2)^{1/2}}]$.

In this paper, we give a similar depiction of the solution of a nonlocal multipoint time problem for the equation $(\partial u(t,x)/\partial t) + Au(t,x) = 0$, $(t,x) \in (0,+\infty) \times \mathbb{R}$ when the initial condition $u|_{t=0} = f$ is replaced by the condition $\sum_{k=0}^{m} \alpha_k B_k u(t,\cdot)|_{t=t_k}$, where $t_0 = 0$, $\{t_1,\ldots,t_m\} \subset (0,+\infty)$, $\{\alpha_0,\alpha_1,\ldots,\alpha_m\} \subset \mathbb{R}$, and $m \in \mathbb{N}$ are fixed and $B_1,\ldots,B_m$ are pseudodifferential operators constructed of smooth symbols (if $\alpha_0 = 1$, $\alpha_1 = \cdots = \alpha_m = 0$, $B_0 = I$, then obviously we have a Cauchy problem). This condition is interpreted in the classical or weak sense as if f is a generalized function (generalized element of the operator A) of ultradistribution type. Properties of the fundamental solution of the specified multipoint problem are investigated. The behavior of the solution at $t \longrightarrow +\infty$ (solution stabilization) in the spaces of generalized functions of type S' and the uniform stabilization of the solution to zero on $\mathbb{R}$ are studied.

Note that the nonlocal multipoint time problem relates to nonlocal problems for operator differential equations and partial differential equations. Such problems arise when modeling many processes and problems of practice with boundary-value problems for partial differential equations, when describing correct problems for a particular operator and constructing a general theory of boundary-value problems (see, for example, [6–11]).

2. Spaces of Test and Generalized Functions

Gelfand and Shilov introduced in [12] a series of spaces, which they called the spaces S. They consist of infinitely differentiable on $\mathbb{R}$ functions, which satisfy certain conditions on the decrease at infinity and the growth of derivatives. These conditions are given by the inequalities $|x^k \varphi^{(m)}(x)| \le c_{km}$, $x \in \mathbb{R}$, $\{k,m\} \subset \mathbb{Z}_+$, where $\{c_{km}\}$ is some double sequence of positive numbers. If there are no restrictions on elements of the sequence $\{c_{km}\}$, then obviously we have L. Schwartz's space $S \equiv S(\mathbb{R})$ of quickly descending at infinity functions. However, if the numbers c_{km} satisfy certain conditions, then the corresponding specific spaces are contained in S and they are called the spaces of S type. Let us define some of them.

For any α, $\beta > 0$, let us put

$$S_\alpha^\beta(\mathbb{R}) \equiv S_\alpha^\beta := \{\varphi \in S \mid \exists c > 0 \exists A > 0 \exists B > 0 \forall \{m,n\} \subset \mathbb{Z}_+ \; \forall x \in \mathbb{R}: \left| x^m \varphi^{(n)}(x) \right| \le c A^m B^n m^{m\alpha} n^{n\beta} \right\}. \quad (4)$$

The introduced S spaces can also be described as in [12]. S_α^β consists of those infinitely differentiable on $\mathbb{R}$ functions $\varphi(x)$ that satisfy the following inequalities:

$$\left| \varphi^{(n)}(x) \right| \le c B^n n^{n\beta} \exp\left(-a|x|^{1/\alpha}\right), \quad n \in \mathbb{Z}_+, \; x \in \mathbb{R}, \quad (5)$$

with some positive constants c, B, and a dependent only on the function φ.

If $0 < \beta < 1$ and $\alpha \ge 1 - \beta$, then S_α^β consists of only those $\varphi \in C^\infty(\mathbb{R})$, which admit an analytic extension into the complex plane $\mathbb{C}$ such that

$$|\varphi(x+iy)| \le c \exp\left(-a|x|^{1/\alpha} + b|y|^{1/(1-\beta)}\right), \quad c > 0, \; a > 0, \; b > 0. \quad (6)$$

The space S_α^1 consists of functions $\varphi \in C^\infty(\mathbb{R})$, which can be analytically extended into some band $|\mathrm{Im} z| < \delta$ (dependent on φ) of the complex plane, so that the estimate

$$|\varphi(x+iy)| \le c \exp\left(-a|x|^{1/\alpha}\right), \quad c, a > 0, \; \forall y \in \mathbb{R}, \quad (7)$$

is carried out.

The spaces S_α^β, $\alpha, \beta > 0$, are nontrivial if $\alpha + \beta \ge 1$, and they form dense sets in $L_2(\mathbb{R})$.

The topological structure in S_α^β is defined as follows. The symbol $S_{\alpha,A}^{\beta,B}$, $A, B > 0$, denotes the set of functions $\varphi \in S_\alpha^\beta$ that satisfy the condition

$$\forall \overline{A} > A \forall \overline{B} > B: \left| x^k \varphi^{(m)}(x) \right| \le c \overline{A}^k \overline{B}^m k^{k\alpha} m^{m\beta}, \quad \{k,m\} \subset \mathbb{Z}_+, \; x \in \mathbb{R}. \quad (8)$$

This set is transformed into a complete, countably normed space, if the norms in it are defined by means of relations:

$$\|\varphi\|_{\delta\rho} = \sup_{x,k,m} \frac{\left| x^k \varphi^{(m)}(x) \right|}{(A+\delta)^k (B+\rho)^m k^{k\alpha} m^{m\beta}}, \quad \{\delta,\rho\} \subset \left\{ 1, \frac{1}{2}, \frac{1}{3}, \ldots \right\}. \quad (9)$$

The specified norm system is sometimes replaced by an equivalent norm system:

$$\|\varphi\|'_{\delta\rho} = \sup_{x,m} \frac{\exp\left\{ a(1-\delta)|x|^{1/\alpha} \right\} \cdot \left| \varphi^{(m)}(x) \right|}{(B+\rho)^m m^{m\beta}}, \quad \{\delta,\rho\} \subset \left\{ \frac{1}{2}, \frac{1}{3}, \ldots \right\}, \quad a = \frac{\alpha}{eA^{1/\alpha}}. \quad (10)$$

If $A_1 < A_2$, $B_1 < B_2$, then $S_{\alpha,A_1}^{\beta,B_1}$ is continuously embedded into $S_{\alpha,A_2}^{\beta,B_2}$ and $S_\alpha^\beta = \cup_{A,B>0} S_{\alpha,A}^{\beta,B}$, that is, S_α^β is endowed by the inductive limit topology of the spaces $S_{\alpha,A}^{\beta,B}$ [12]. Therefore, the convergence of a sequence $\{\varphi_\nu, \nu \ge 1\} \subset S_\alpha^\beta$ to zero in the space S_α^β is the convergence in the topology of some space $S_{\alpha,A}^{\beta,B}$, to which all the functions φ_ν belong. In other words (see [12]), $\varphi_\nu \longrightarrow 0$ in S_α^β as $\nu \longrightarrow \infty$, if and only if for every $n \in \mathbb{Z}_+$, the sequence $\varphi_\nu^{(n)}$, $\nu \ge 1$, converges to zero uniformly on an arbitrary segment $[a,b] \subset \mathbb{R}$ and for some $c, a, B > 0$ independent of ν, and the inequality

$$\left|\varphi_\nu^{(n)}(x)\right| \le cB^n n^{n\beta} \exp\left\{-a|x|^{1/\alpha}\right\}, \quad x \in \mathbb{R}, n \in \mathbb{Z}_+. \quad (11)$$

holds.

In S_α^β, the continuous translation operation T_x: $\varphi(\xi) \longrightarrow \varphi(\xi + x)$ is defined. This operation is also differentiable (even infinitely differentiable [12]) in the sense that the limit relation $(\varphi(x+h) - \varphi(x))h^{-1} \longrightarrow \varphi'(x)$, $h \longrightarrow 0$, is true for every function $\varphi \in S_\alpha^\beta$ in the sense of convergence in the S_α^β-topology. In S_α^β, the continuous differentiation operator is also defined. The spaces of type S are perfect [12] (that is, the spaces and all bounded sets of which are compact) and closely related to the Fourier transform, namely, the formula $F[S_\alpha^\beta] = S_\beta^\alpha$, $\alpha, \beta > 0$, is correct, where

$$F\left[S_\alpha^\beta\right] := \left\{\psi : \psi(\sigma) = \int_{\mathbb{R}} \varphi(x) e^{i\sigma x} \mathrm{d}x, \ \varphi \in S_\alpha^\beta\right\}. \quad (12)$$

Moreover, the operator $F : S_\alpha^\beta \longrightarrow S_\beta^\alpha$ is continuous.

Let $(S_\alpha^\beta)'$ denote the space of all linear continuous functionals on S_α^β with weak convergence. Since the translation operator T_x is defined in the space S_α^β of test functions, the convolution of a generalized function $f \in (S_\alpha^\beta)'$ with the test function $\varphi \in S_\alpha^\beta$ is given by the following formula:

$$(f * \varphi) = \langle f, T_{-x}\breve{\varphi}(\cdot)\rangle \equiv \langle f, \varphi(x - \cdot)\rangle, \quad \breve{\varphi}(\xi) = \varphi(-\xi). \quad (13)$$

It follows from the infinite differentiability property of the argument translation operation in S_α^β that the convolution $f * \varphi$ is a usual infinitely differentiable function on $\mathbb{R}$.

We define the Fourier transform of a generalized function $f \in (S_\alpha^\beta)'$ by the relation

$$\langle F[f], \varphi\rangle = \langle f, F[\varphi]\rangle, \quad \forall \varphi \in S_\beta^\alpha, \quad (14)$$

where the operator F: $(S_\alpha^\beta)' \longrightarrow (S_\beta^\alpha)'$ is continuous.

Let $f \in (S_\alpha^\beta)'$. If $f * \varphi \in S_\alpha^\beta$, $\forall \varphi \in S_\alpha^\beta$, and the convergence $\varphi_\nu \longrightarrow 0$ in the S_α^β-topology as $\nu \longrightarrow \infty$ implies that $f * \varphi_\nu \longrightarrow 0$ in the S_α^β-topology as $\nu \longrightarrow \infty$, then the function f is called a convolutor in S_α^β. If $f \in (S_\alpha^\beta)'$ is a convolutor in S_α^β, then for an arbitrary function $\varphi \in S_\alpha^\beta$, the formula $F[f * \varphi] = F[f]F[\varphi]$ is valid, where $F[f]$ is a multiplier in S_α^β [12].

Recall that the function $g \in C^\infty(\mathbb{R})$ is called a multiplier in S_α^β if $g\varphi \in S_\alpha^\beta$ for an arbitrary function $\varphi \in S_\alpha^\beta$, and the mapping $\varphi \longrightarrow g\varphi$ is continuous in the space S_α^β.

3. Nonlocal Multipoint by Time Problem

Consider the function $a(\sigma) = (1 + \sigma^2)^{\omega/2}$, $\sigma \in \mathbb{R}$, where $\omega \in [1, 2)$ is a fixed parameter. Obviously, the function $a(\sigma)$ satisfies the inequality

$$a(\sigma) \le c_\varepsilon \exp\{\varepsilon|\sigma|^\omega\}, \quad \sigma \in \mathbb{R}, \quad (15)$$

where $c_\varepsilon = 2^{\omega/2}\max\{1, (1/\varepsilon)\}$ for an arbitrary $\varepsilon > 0$. Using direct calculations and the Stirling formula, we find that

$$\left|D_\sigma^m a(\sigma)\right| \le c_0 B_0^m m! \le c_1 B_1^m m^m, \quad m \in \mathbb{N}, \sigma \in \mathbb{R}, \quad (16)$$

where $B_0 = B_0(\omega) > 0$, $B_1 = B_1(\omega) > 0$, and $c_0, c_1 > 0$. It follows from (15) and (16) that $a(\sigma)$ is a multiplier in $S_{1/\omega}^1$. Indeed, let $\varphi \in S_{1/\omega}^1$, that is, the function φ and its derivatives satisfy the inequality

$$\left|D_\sigma^k \varphi(\sigma)\right| \le cA^k k^k \exp\{-a|\sigma|^\omega\}, \quad k \in \mathbb{Z}_+, \sigma \in \mathbb{R}, \quad (17)$$

with some positive constants c, A, and a. Then, using the Leibniz formula for differentiating the product of two functions, as well as inequalities (15)–(17), we find that

$$\begin{aligned}\left|(a(\sigma)\varphi(\sigma))^{(k)}\right| &\le \sum_{l=0}^{k} C_k^l \left|a^{(l)}(\sigma)\right| \left|\varphi^{(k-l)}(\sigma)\right| = \left|a(\sigma)\varphi^{(k)}(\sigma)\right| + \\ &+ \sum_{l=1}^{k} C_k^l \left|a^{(l)}(\sigma)\right| \cdot \left|\varphi^{(k-l)}(\sigma)\right| \le c_\varepsilon c A^k k^k \exp\{-a|\sigma|^\omega\} \\ &+ c_1 c \sum_{l=1}^{k} C_k^l B_1^l l^l A^{k-l} (k-l)^{k-l} \exp\{-a|\sigma|^\omega\}.\end{aligned} \quad (18)$$

Since $\varepsilon > 0$ is arbitrary, we can put $\varepsilon = (a/2)$. Then,

$$\begin{aligned}\left|(a(\sigma)\varphi(\sigma))^{(k)}\right| &\le c_\varepsilon c A^k k^k \exp\left\{-\frac{a}{2}|\sigma|^\omega\right\} \\ &+ c_1 c \sum_{l=1}^{k} C_k^l B_1^l l^l A^{k-l} (k-l)^{(k-l)} \\ &\times \exp\left\{-\frac{a}{2}|\sigma|^\omega\right\} \le c_2 B_2^k k^k \exp\left\{-\frac{a}{2}|\sigma|^\omega\right\}, \\ &\sigma \in \mathbb{R}, k \in \mathbb{Z}_+,\end{aligned} \quad (19)$$

where $c_2 = c(c_\varepsilon + c_1)$ and $B_2 = 2\max\{A, B_1\}$. It follows from (19) that $a \cdot \varphi$ is an element of $S_{1/\omega}^1$.

The operation of multiplying by $a(\sigma)$ is continuous in the space $S_{1/\omega}^1$. In fact, let $\{\varphi_n, n \ge 1\}$ be a sequence of functions from $S_{1/\omega}^1$ convergent to 0 in this space. This means that $\{\varphi_n, n \ge 1\} \subset S_{1/\omega,A_0}^{1,B_0}$ with some constants A_0, $B_0 > 0$ and

$$\begin{aligned}&\|\varphi_n\|_{\delta\rho} = \sup_{k,\sigma} \frac{\exp\{a(1-\delta)|\sigma|^\omega\} \cdot \left|\varphi_n^{(k)}(\sigma)\right|}{(B_0 + \rho)^k k^k} \longrightarrow 0, \\ &n \longrightarrow \infty, \\ &\{\delta, \rho\} \subset \left\{\frac{1}{2}, \frac{1}{3}, \ldots\right\}, \\ &a = \frac{1}{\omega e A_0^\omega}.\end{aligned} \quad (20)$$

In other words, for an arbitrary $\tilde{\varepsilon} > 0$, there exists a number $n_0 = n_0(\tilde{\varepsilon})$ such that, for $n \ge n_0$,

$$\left|\varphi_n^{(k)}(\sigma)\right| \le \tilde{\varepsilon}(B_0 + \rho)^k k^k \exp\{-a(1-\delta)|\sigma|^\omega\}, \quad \sigma \in \mathbb{R}, k \in \mathbb{Z}_+. \quad (21)$$

Using inequalities (15) and (16) (when putting in (15), $\varepsilon = (a/2)(1-\delta)$), we get

$$\left|(a(\sigma)\varphi_n(\sigma))^{(k)}\right| \leq c_3\widetilde{\varepsilon}(\widetilde{B}+\rho)^k \exp\left\{-\frac{a}{2}(1-\delta)|\sigma|^{\omega}\right\},$$
$$k \in \mathbb{Z}_+,\ n \geq n_0, \tag{22}$$

where $c_3 = c_\varepsilon + c_1$ and $\widetilde{B} = 2\max\{B_0, B_1\}$. It follows from the last inequality that $\|a\varphi_n\|_{\delta\rho} \leq c_3\widetilde{\varepsilon}$, $\forall n \geq n_0$, that is, the sequence $\{a\varphi_n, n \geq 1\}$ converges to zero in the space $S^{1,\widetilde{B}}_{1/\omega,\widetilde{A}}$ where $\widetilde{A} = 2^{1/\omega}A_0$ and $\widetilde{B} = 2\max\{B_0, B_1\}$. This means that the sequence $\{a\varphi_n, n \geq 1\}$ converges to zero in the space $S^1_{1/\omega}$, which is what we needed to prove.

Remark 1. It follows from the proven property that the function $a(\sigma) = (1+\sigma^2)^{\omega/2}$, $\sigma \in \mathbb{R}$, is also a multiplier in every space $S^{\beta}_{1/\omega}$, where $\beta > 1$. Therefore, in the space $S^{1/\omega}_{\beta}$, $\beta \geq 1$, defined as a continuous linear pseudodifferential operator A, constructed by the function $a(\sigma)$:

$$A\varphi = F^{-1}[a(\sigma)F[\varphi]], \quad \forall \varphi \in S^{1/\omega}_{\beta}, \tag{23}$$

where $A\varphi = (I-\Delta)^{\omega/2}\varphi$, $\Delta = (\mathrm{d}^2/\mathrm{d}x^2)$.

Let us consider the evolution equation with the operator A (the Bessel fractional differentiation operator):

$$\frac{\partial u}{\partial t} + Au = 0, \quad (t,x) \in (0,+\infty)\times\mathbb{R} \equiv \Omega. \tag{24}$$

By a solution of equation (24), we mean the function $u(t,x)$, $(t,x) \in \Omega$, such that (1) $u(t,\cdot) \in C^1(0,+\infty)$ for every $x \in \mathbb{R}$; (2) $u(\cdot,x) \in S^{1/\omega}_{\beta}$ for every $t \in (0,+\infty)$; (3) $u(t,x)$ is continuous at every point $(0,x)$ of the boundary $\Gamma_\Omega = \{0\}\times\mathbb{R}$ of the region Ω; (4) $\exists M$: $\mathbb{R} \longrightarrow [0,\infty)$, $\forall t \in (0,+\infty) : |u_t(t,x)| \leq M(x)$, $\mathrm{int}_{\mathbb{R}} M(x)\mathrm{d}x < \infty$; and (5) $u(t,x)$, $(t,x) \in \Omega$ satisfies equation (24).

For equation (24), we consider the nonlocal multipoint by time problem of finding a solution of equation (24) that satisfies the condition

$$\mu u(0,x) - \mu_1 B_1 u(t_1,x) - \cdots - \mu_m B_m u(t_m,x) = f(x),$$
$$x \in \mathbb{R},\ f \in S^{1/\omega}_{\beta}, \tag{25}$$

where $u(0,x) := \lim_{t\longrightarrow +0} u(t,x)$, $x \in \mathbb{R}$, $m \in \mathbb{N}$, $\{\mu, \mu_1, \ldots, \mu_m\} \subset (0,\infty)$, and $\{t_1, \ldots, t_m\} \subset (0,\infty)$ are fixed numbers, $0 < t_1 < t_2 < \cdots < t_m < +\infty$, $\mu > \sum_{k=1}^m \mu_k$, and $B_1, \ldots, B_m$ are pseudodifferential operators in $S^{1/\omega}_{\beta}$ constructed by functions (characters) $g_k : \mathbb{R} \longrightarrow (0,+\infty)$, respectively, $B_k = F^{-1}[g_k(\sigma)F]$, $k \in \{1,\ldots,m\}$. The functions $g_k \in C^\infty(\mathbb{R})$ and $k \in \{1,\ldots,m\}$ satisfy the conditions

$$\begin{gathered}\forall \varepsilon > 0, \forall \sigma \in \mathbb{R} : g_k(\sigma) \leq \exp\{\varepsilon|\sigma|^{\omega}\},\\ \exists L_k > 0, \forall s \in \mathbb{N} : \left|D^s_\sigma g_k(\sigma)\right| \leq L^s_k s^s.\end{gathered} \tag{26}$$

Note that the above properties of the functions g_k imply that g_k, $k \in \{1,\ldots,m\}$, is a multiplier in $S^{1/\omega}_{\beta}$.

We are looking for the solution of problems (24) and (25) via the Fourier transform. Due to condition (19),

$$F[u_t] = \int_{\mathbb{R}} u_t(t,x)e^{i\sigma x}\mathrm{d}x = \frac{\partial}{\partial t}\int_{\mathbb{R}} u(t,x)e^{i\sigma x}\mathrm{d}x = \frac{\partial}{\partial t}F[u]. \tag{27}$$

We introduce the notation $F[u(t,x)] = v(t,\sigma)$. Given the form of the operators A, B_1, …, B_m, we get

$$\begin{gathered}F[Au(t,x)] = F\left[F^{-1}[a(\sigma)F[u]]\right] = a(\sigma)F[u(t,x)] = a(\sigma)v(t,\sigma),\\ F[B_k u(t,x)] = g_k(\sigma)F[u(t,x)] = g_k(\sigma)v(t,\sigma), \quad k \in \{1,\ldots,m\}.\end{gathered} \tag{28}$$

So, for the function v: $\Omega \longrightarrow \mathbb{R}$, we arrive at a problem with parameter σ:

$$\frac{\mathrm{d}v(t,\sigma)}{\mathrm{d}t} + a(\sigma)v(t,\sigma) = 0, \quad (t,\sigma) \in \Omega, \tag{29}$$

$$\mu v(0,\sigma) - \sum_{k=1}^m \mu_k g_k(\sigma)v(t_k,\sigma) = \widetilde{f}(\sigma), \quad \sigma \in \mathbb{R}, \tag{30}$$

where $\widetilde{f}(\sigma) = F[f](\sigma)$. The general solution of equation (29) has the form

$$v(t,\sigma) = c\exp\{-ta(\sigma)\}, \quad (t,\sigma) \in \Omega, \tag{31}$$

where $c = c(\sigma)$ is determined by condition (30). Substituting (30) into (31), we find that

$$c = \widetilde{f}(\sigma)\left(\mu - \sum_{k=1}^m \mu_k g_k(\sigma)\exp\{-t_k a(\sigma)\}\right)^{-1}, \quad \sigma \in \mathbb{R}. \tag{32}$$

Now, put $G(t,x) := F^{-1}[Q(t,\sigma)]$ and $Q(t,\sigma) = Q_1(t,\sigma)Q_2(\sigma)$, and thus,

$$\begin{aligned}Q_1(t,\sigma) &= \exp\{-ta(\sigma)\},\\ Q_2(\sigma) &= \left(\mu - \sum_{k=1}^m \mu_k g_k(\sigma)Q_1(t_k,\sigma)\right)^{-1}.\end{aligned} \tag{33}$$

Then, thinking formally, we come to the relation

$$u(t,x) = \int_{\mathbb{R}} G(t,x-\xi)f(\xi)\mathrm{d}\xi = G(t,x)*f(x). \tag{34}$$

Indeed,

$$\begin{aligned}u(t,x) &= (2\pi)^{-1}\int_{\mathbb{R}} Q(t,\sigma)\left(\int_{\mathbb{R}} f(\xi)e^{i\sigma\xi}\mathrm{d}\xi\right)e^{-i\sigma x}\mathrm{d}\sigma\\ &= \int_{\mathbb{R}}\left((2\pi)^{-1}\int_{\mathbb{R}} Q(t,\sigma)e^{-i\sigma(x-\xi)}\mathrm{d}\sigma\right)f(\xi)\mathrm{d}\xi\\ &= \int_{\mathbb{R}} G(t,x-\xi)f(\xi)\mathrm{d}\xi = G(t,x)*f(x), \quad (t,x) \in \Omega.\end{aligned} \tag{35}$$

The correctness of the transformations performed, the convergence of the corresponding integrals and, consequently, the correctness of formula (35) follow from the properties of the function G, which are given below. The properties of G are determined by the properties of Q

because $G = F^{-1}[Q]$. So, let us first examine the properties of the function $Q(t,\sigma)$ as a function of the variable σ.

Lemma 1. *For derivatives of $Q_1(t,\sigma)$, $(t,\sigma)\in\Omega$, the estimates*

$$\left|D_\sigma^s Q_1(t,\sigma)\right| \le cA^s t^{\gamma s} s^s \exp\{-t|\sigma|^\omega\}, \tag{36}$$

are valid, where $\gamma = 0$ if $0 < t \le 1$ and $\gamma = 1$ if $t > 1$, and the constants $c > 1$ and $A > 0$ do not depend on t.

Proof. To prove the statement, we use the Faa di Bruno formula for differentiation of a complex function:

$$D_\sigma^s F(g(\sigma)) = \sum_{p=1}^{s} \frac{d^p}{dg^p} F(g) \sum \frac{s!}{p_1!\dots p_l!}\left(\frac{d}{d\sigma}g(\sigma)\right)^{p_1} \cdots \left(\frac{1}{l!}\frac{d^l}{d\sigma^l}g(\sigma)\right)^{p_l}, \quad s\in\mathbb{N}, \tag{37}$$

where the sum sign is applied to all the solutions in positive integers of the equation $p_1 + 2p_2 + \cdots + lp_l = s$, $p_1 + \cdots + p_l = m$. Put $F = e^g$ and $g = -ta(\sigma)$. Then,

$$D_\sigma^s e^{-ta(\sigma)} = e^{-ta(\sigma)} \sum_{p=1}^{s}\sum \frac{s!}{p_1!\dots p_l!}\Lambda, \tag{38}$$

where the symbol Λ denotes the expression, and

$$\Lambda := \left(\frac{d}{d\sigma}(-ta(\sigma))\right)^{p_1}\cdot\left(\frac{1}{2!}\frac{d^2}{d\sigma^2}(-ta(\sigma))\right)^{p_2}\cdots\left(\frac{1}{l!}\frac{d^l}{d\sigma^l}(-ta(\sigma))\right)^{p_l}. \tag{39}$$

Given estimate (16) is fulfilled, we find that

$$|\Lambda| \le c_0^{p_1+\cdots+p_l} B_0^{p_1+2p_2+\cdots+lp_l} t^{p_1+\cdots+p_l} \le \widetilde{c}_0^{\,s} t^p B_0^s, \quad \widetilde{c}_0 = \max\{1, c_0\}. \tag{40}$$

Using (40) and the Stirling formula, we arrive at the inequalities

$$\begin{aligned}\left|D_\sigma^s Q_1(t,\sigma)\right| &\le \widetilde{c}_0^{\,s} B_0^s t^{\gamma s} s! \exp\{-ta(\sigma)\}\\ &\le cA^s t^{\gamma s} s^s \exp\{-t|\sigma|^\omega\}, \quad \sigma\in\mathbb{R},\end{aligned} \tag{41}$$

where $\gamma = 0$ if $0 < t \le 1$ and $\gamma = 1$ if $t > 1$, and the values $c > 1$ and $A > 0$ do not depend on t.

Lemma is proved. □

Remark 2. It follows from estimate (41) that $Q_1(t,\cdot)\in S_{1/\omega}^1$ for every $t > 0$.

Lemma 2. *The function Q_2 is a multiplier in $S_{1/\omega}^2$.*

Proof. To prove the assertion, let us estimate the derivatives of Q_2. For this we use formula (37) in which we put $F = \varphi^{-1}$ and $\varphi = R$, where

$$R(\sigma) = \mu - \sum_{k=1}^{m}\mu_k g_k(\sigma)\exp\{-t_k a(\sigma)\}. \tag{42}$$

Then, $Q_2 = F(\varphi) = R^{-1}$ and

$$\left|D_\sigma^s Q_2(\sigma)\right| = \left|\sum_{p=1}^{s}\frac{d^p}{dR^p}R^{-1}\sum\frac{s!}{p_1!\dots p_l!}\left(\frac{d}{d\sigma}R(\sigma)\right)^{p_1} \cdots\left(\frac{1}{l!}\frac{d^l}{d\sigma^l}R(\sigma)\right)^{p_l}\right|, \quad s\in\mathbb{N}. \tag{43}$$

Given the properties of $g_1,\dots,g_m$ and inequality (41), we find that

$$\begin{aligned}\left|\frac{1}{l!}\frac{d^j}{d\sigma^j}R(\sigma)\right| &\le \frac{1}{j!}\sum_{k=1}^{m}\mu_k\sum_{i=0}^{j}C_j^i\left|D_\sigma^i g_k(\sigma)\right|\cdot\left|D_\sigma^{j-i}e^{-t_k a(\sigma)}\right|\\ &\le \frac{1}{j!}\sum_{k=1}^{m}\mu_k\sum_{j=0}^{i}C_j^i L_k^i i!\widetilde{c}_0^{\,j-i} t_k^{\gamma(j-i)}(j-i)!e^{-t_k a(\sigma)}\\ &\le \sum_{k=1}^{m}\mu_k\sum_{i=0}^{j}C_j^i L_k^i \widetilde{c}_0^{\,j-i} t^{\gamma(j-i)},\end{aligned} \tag{44}$$

where we took into account that $i!(j-i)! \le j!$).

Let

$$\begin{aligned}\widetilde{L} &= \max\{L_1,\dots,L_m\},\\ L_0 &= 2\max\{\widetilde{L}, \widetilde{c}_0 T\},\end{aligned} \tag{45}$$

then

$$\begin{aligned}&\left|\frac{1}{j!}\frac{d^j}{d\sigma^j}R(\sigma)\right| \le \alpha_0 L_0^j, \quad \alpha_0 = \sum_{k=1}^{m}\mu_k,\ j\in\{1,\dots,l\},\\ &\left|\left(\frac{d}{d\sigma}R(\sigma)\right)^{p_1}\right|\cdots\left|\left(\frac{1}{l!}\frac{d^l}{d\sigma^l}R(\sigma)\right)^{p_l}\right| \le (\alpha_0 L_0)^p\left(\alpha_0 L_0^2\right)^{p_2}\cdots\left(\alpha_0 L_0^l\right)^{p_l}\\ &\qquad = \alpha_0^{p_1+\cdots+p_l} L_0^{p_1+2p_2+\cdots+lp_l} = \alpha_0^p L_0^s.\end{aligned} \tag{46}$$

In addition, $(d^p/dR^p)R^{-1} = (-1)^p p! R^{-(p+1)}$ and

$$\begin{aligned}R^{-1}(\sigma) = Q_2(\sigma) &= \left(\mu - \sum_{k=1}^{m}\mu_k g_k(\sigma)\exp\{-t_k a(\sigma)\}\right)^{-1}\\ &\le \left(\mu - \sum_{k=1}^{m}\mu_k\exp\{-t_1|\sigma|^\omega + \varepsilon|\sigma|^\omega\} \le \left(\mu - \sum_{k=1}^{m}\mu_k\right)^{-1}\right.\\ &\equiv \beta_0 > 0, \quad \varepsilon = t_1.\end{aligned} \tag{47}$$

Since, by assumption, $\mu > \sum_{k=1}^{m}\mu_k$ (assuming the properties of $g_1,\dots,g_m$, and $0 < t_1 < t_2 < \cdots < t_m$). So,

$$\left|\frac{\mathrm{d}^p}{\mathrm{d}R^p}R^{-1}\right| \leq \beta_0^{p+1}p!,$$

$$\begin{aligned}\left|D_\sigma^s Q_2(\sigma)\right| &\leq s!\sum_{p=1}^{s}\beta_0^{p+1}p!\alpha_0^p L_0^s \\ &\leq \beta_0 L_0^s (s!)^2\sum_{p=1}^{s}\beta_0^p \leq \beta_0 L_0^s s\cdot(s!)^2\beta_1^s \leq \beta_2\beta_3^s s^{2s}, \quad s\in\mathbb{N}.\end{aligned} \tag{48}$$

It follows from the last inequality and boundedness of the function $Q_2(\sigma)$ on $\mathbb{R}$ that Q_2 is a multiplier in $S_{1/\omega}^2$. □

Corollary 1. *For every $t>0$, the function $Q(t,\sigma)=Q_1(t,\sigma)Q_2(\sigma)$, $\sigma\in\mathbb{R}$, is an element of the space $S_{1/\omega}^2$, and the estimates*

$$\left|D_\sigma^s Q(t,\sigma)\right| \leq \tilde{c}\tilde{A}^s t^{\gamma s}s^{2s}\exp\{-t|\sigma|^\omega\}, \quad s\in\mathbb{Z}_+,\ (t,\sigma)\in\Omega, \tag{49}$$

are valid, where the constants $\tilde{c}$ and $\tilde{A}>0$ do not depend on t.

Taking into account the properties of the Fourier transform (direct and inverse) and the formula $F^{-1}[S_{1/\omega}^2]=S_2^{1/\omega}$, we get that $G(t,\cdot)\in S_2^{1/\omega}$ for every $t>0$. We remove in the estimates of derivatives of the function G (in the variable x), the dependence on t, assuming $t>1$. To do this, we use the relations

$$\begin{aligned}x^k D_x^s F[\varphi](x) &= i^{k+s}F\left[\left(\sigma^s\varphi(\sigma)\right)^{(k)}\right] \\ &= i^{k+s}\int_{\mathbb{R}}\left(\sigma^s\varphi(\sigma)\right)^{(k)}e^{i\sigma x}\mathrm{d}\sigma, \quad \{k,s\}\subset\mathbb{Z}_+,\ \varphi\in S_{1/\omega}^2.\end{aligned} \tag{50}$$

So,

$$x^k D_x^s G(t,x) = (2\pi)^{-1}i^{k+s}(-1)^s\int_{\mathbb{R}}\left(\sigma^s Q(t,-\sigma)\right)^{(k)}e^{i\sigma x}\mathrm{d}\sigma. \tag{51}$$

Applying the Leibniz formula for differentiating the product of two functions and estimating the derivatives of the function $Q(t,\sigma)$, we find that

$$\begin{aligned}\left|\left(\sigma^s Q(t,-\sigma)\right)^{(k)}\right| &= \left|\sum_{p=0}^{k}C_k^p\left(\sigma^s\right)^{(p)}Q^{(k-p)}(t,-\sigma)\right| \\ &\leq \left|\sigma^s Q^{(k)}(t,-\sigma)\right| + ks\left|\sigma^{s-1}Q^{(k-1)}(t,-\sigma)\right| \\ &\quad + \frac{k(k-1)}{2!}s(s-1)\left|\sigma^{s-2}Q^{(k-2)}(t,-\sigma)\right| \\ &\quad + \cdots \leq \tilde{c}\Big[\tilde{A}^k t^{\gamma k}\tilde{B}^s t^{-(s/\omega)}m_{ks} + ks\tilde{A}^{k-1}t^{\gamma(k-1)} \\ &\quad \cdot \tilde{B}^{s-1}t^{-((s-1)/\omega)}m_{k-1,s-1} \\ &\quad + \frac{1}{2!}k(k-1)s(s-1)\tilde{A}^{k-2}t^{\gamma(k-2)}\tilde{B}^{s-2}t^{-((s-2)/\omega)} \\ &\quad \cdot m_{k-2,s-2} + \ldots\Big]e^{-(t/2)|\sigma|^\omega},\end{aligned} \tag{52}$$

where $m_{ks}=k^{2k}s^{s/\omega}$. Taking into account the results in (see [12], p. 236–243), we find that this double sequence satisfies the inequality

$$ks\frac{m_{k-1,s-1}}{m_{ks}} \leq \tilde{\gamma}(k+s), \quad \tilde{\gamma}>0. \tag{53}$$

Bearing in mind the last inequality and also that $t>1$, we obtain

$$\begin{aligned}\left|\left(\sigma^s Q(t,-\sigma)\right)^{(k)}\right| &\leq \tilde{c}\tilde{A}^k t^k\tilde{B}^s m_{ks}\left(1+\frac{ks}{\tilde{A}\tilde{B}}\frac{m_{k-1,s-1}}{m_{ks}}+\frac{1}{2!}k(k-1)s(s-1)\frac{1}{\tilde{A}^2\tilde{B}^2}\frac{m_{k-2,s-2}}{m_{ks}}+\cdots\right)e^{-(t/2)|\sigma|^\omega} \leq \tilde{c}\tilde{A}^k t^k\tilde{B}^s m_{ks} \\ &\quad \times\left(1+\frac{ks}{\tilde{A}\tilde{B}}\frac{m_{k-1,s-1}}{m_{ks}}+\frac{1}{2!}\frac{1}{\tilde{A}^2\tilde{B}^2}ks\frac{m_{k-1,s-1}}{m_{ks}}(k-1)(s-1)\frac{m_{k-2,s-2}}{m_{k-1,s-1}}+\cdots\right)e^{-(t/2)|\sigma|^\omega} \\ &\leq \tilde{c}\tilde{A}^k t^k\tilde{B}^s m_{ks}\left(1+\frac{\tilde{\gamma}}{\tilde{A}\tilde{B}}(k+s)+\frac{1}{2!}\frac{\tilde{\gamma}^2}{(\tilde{A}\tilde{B})^2}(k+s)^2+\cdots\right)e^{-(t/2)|\sigma|^\omega} \\ &\leq c_1\overline{A}^k t^k\overline{B}^s m_{ks}e^{-(t/2)|\sigma|^\omega}, \\ \overline{A} &= \tilde{A}\exp\left\{\frac{\tilde{\gamma}}{\tilde{A}\tilde{B}}\right\}, \\ \overline{B} &= \tilde{B}\exp\left\{\frac{\tilde{\gamma}}{\tilde{A}\tilde{B}}\right\}.\end{aligned} \tag{54}$$

So,

$$\left|x^k D_x^s G(t,x)\right| \le (2\pi)^{-1} c_1 \overline{A}^k t^k \overline{B}^s m_{ks} \int_{\mathbb{R}} e^{-(t/2)|\sigma|^{\omega}} d\sigma = c_2 \overline{A}^k t^k \overline{B}^s t^{-(1/\omega)} k^{2k} s^{s/\omega}, \quad \{k,s\} \subset \mathbb{Z}_+. \tag{55}$$

Then,

$$\left|D_x^s G(t,x)\right| \le c_2 \overline{B}^s t^{-(1/\omega)} s^{s/\omega} \inf_k \frac{\overline{A}^k k^{2k}}{(t^{-1}|x|)^k} \le c_3 \overline{B}^s t^{-(1/\omega)} s^{s/\omega} \exp\left\{-a_0 t^{-(1/2)} |x|^{1/2}\right\}, \tag{56}$$

where the constants c_3, $\overline{B}$, and $a_0 > 0$ do not depend on t; here, we used the well-known inequality (see [12], p. 204)

$$e^{-(\alpha/e)|\xi|^{1/\alpha}} \le \inf_k \frac{L^k k^{k\alpha}}{|\xi|^k} \le c e^{-(\alpha/e)|\xi|^{1/\alpha}}, \quad c = e^{\alpha e/2}, \tag{57}$$

in which $\alpha = 2$ and $L = \widetilde{A}$. Thus, such a statement is correct.

Lemma 3. *The derivatives (by variable x) of the function $G(t,x)$ for $t > 1$ satisfy the inequality*

$$\left|D_x^s G(t,x)\right| \le c_3 \overline{B}^s s^{s/\omega} t^{-(1/\omega)} \exp\left\{-a_0 t^{-(1/2)} |x|^{1/2}\right\}, \quad s \in \mathbb{Z}_+, \tag{58}$$

where the constants c_3, $\overline{B}$, and $a_0 > 0$ do not depend on t.

Remark 3. In what follows, we assume that in condition (25), $\beta = 2$.

Here are a few more properties of the function $G(t,x)$.

Lemma 4. *The function $G(t,\cdot)$, $t \in (0,+\infty)$, as an abstract function of t with values in the space $S_2^{1/\omega}$, is differentiable in t.*

Proof. It follows from the continuity of the Fourier transform (direct and inverse) that, to prove the statement, it suffices to establish that the function $F[G(t,\cdot)] = Q(t,\cdot)$, as a function of a parameter t with values in the space $S_{1/\omega}^2$, is differentiable in t. In other words, it is necessary to prove that the boundary value relation

$$\Phi_{\Delta t}(\sigma) := \frac{1}{\Delta t}\left[Q(t+\Delta t,\cdot) - Q(t,\cdot)\right] \longrightarrow \frac{\partial}{\partial t} Q(t,\cdot), \quad \Delta t \longrightarrow 0, \tag{59}$$

is performed in the sense as follows:

(1) $D_\sigma^s \Phi_{\Delta t}(\sigma) \underset{\Delta t \to 0}{\longrightarrow} D_\sigma^s(-a(\sigma)Q(t,\sigma))$, $s \in \mathbb{Z}_+$, uniformly on each segment $[a,b] \subset \mathbb{R}$

(2) $|D_\sigma^s \Phi_{\Delta t}(\sigma)| \le \overline{c}\overline{B}^s s^{2s} \exp\{-\overline{a}|\sigma|^{\omega}\}$, $s \in \mathbb{Z}_+$, where the constants $\overline{c}$, $\overline{B}$, and $\overline{a} > 0$ do not depend on Δt if Δt is small enough

The function $Q(t,\sigma)$, $(t,\sigma) \in \Omega$, is differentiable in t in the usual sense. Due to the Lagrange theorem on finite increments,

$$\Phi_{\Delta t}(\sigma) = -a(\sigma) Q(t+\theta\Delta t,\sigma). \tag{60}$$

So,

$$D_\sigma^s \Phi_{\Delta t}(\sigma) = -\sum_{l=0}^{s} C_s^l D_\sigma^l a(\sigma) D_\sigma^{s-l} Q(t+\theta\Delta t,\sigma), \tag{61}$$

$$D_\sigma^s\left(\Phi_{\Delta t}(\sigma) - \frac{\partial}{\partial t}Q(t,\sigma)\right) = -\sum_{l=0}^{s} C_s^l D_\sigma^l a(\sigma)\left[D_\sigma^{s-l} Q(t+\theta\Delta t,\sigma) - D_\sigma^{s-l} Q(t,\sigma)\right]. \tag{62}$$

Since

$$D_\sigma^{s-l} Q(t+\theta\Delta t,\sigma) - D_\sigma^{s-l} Q(t,\sigma) = D_\sigma^{s-l+1} Q(t+\theta_1\Delta t,\sigma), \quad 0 < \theta_1 < 1, \tag{63}$$

it follows from this and estimate (49) that

$$D_\sigma^{s-l+1} Q(t+\theta_1\Delta t,\sigma)\theta\Delta t \longrightarrow 0, \quad \Delta t \longrightarrow 0, \tag{64}$$

uniformly on an arbitrary segment $[a,b] \subset \mathbb{R}$. Then,

$$D_\sigma^s \Phi_{\Delta t}(\sigma) \longrightarrow D_\sigma^s\left(\frac{\partial}{\partial t}Q(t,\sigma)\right), \quad \Delta t \longrightarrow 0, \tag{65}$$

uniformly on each segment $[a,b] \subset \mathbb{R}$, too. Therefore, condition 1 is satisfied.

Taking into account (61), estimates (15), (16), and (49) for $a(\sigma)$, $Q(t,\sigma)$, and their derivatives, we find that

$$\left|D_\sigma^s \Phi_{\Delta t}(\sigma)\right| \le c_1 c_\varepsilon \sum_{l=0}^{s} C_s^l B_1^l l^l \widetilde{A}^{s-l} t^{\gamma(s-l)} (s-l)^{2(s-l)} \exp\{-(t+\theta\Delta t)|\sigma|^{\omega} + \varepsilon|\sigma|^{\omega}\}, \tag{66}$$

where $\varepsilon > 0$ is arbitrary fixed. Take $\varepsilon = (t/2)$, and then

$$\left|D_\sigma^s \Phi_{\Delta t}(\sigma)\right| \le \overline{c}\overline{B}^s s^{2s} \exp\{-\overline{a}|\sigma|^{\omega}\}, \quad s \in \mathbb{Z}_+, \tag{67}$$

where $\overline{B} = 2\max\left\{B_1, \widetilde{A}t^{\gamma}\right\}$, $\overline{a} = (t/2)$, and all the constants do not depend on Δt. □

Corollary 2. *The formula*

$$\frac{\partial}{\partial t}(f * G(t,\cdot)) = f * \frac{\partial G(t,\cdot)}{\partial t}, \quad \forall f \in \left(S_2^{1/\omega}\right)', \ t > 0, \tag{68}$$

is correct.

Proof. According to the definition of a convolution of a generalized function with a test one, we have

$$f * G(t,\cdot) = \langle f_\xi, T_{-x}\breve{G}(t,\xi)\rangle, \quad \breve{G}(t,\xi) = G(t,-\xi). \tag{69}$$

Then,

$$\frac{\partial}{\partial t}(f * G(t,\cdot)) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[(f * G(t+\Delta t,\cdot)) - (f * G(t,\cdot))\right] = \lim_{\Delta t \to 0} \left\langle f_\xi, \frac{1}{\Delta t}\left[T_{-x}\breve{G}(t+\Delta t,\xi) - T_{-x}\breve{G}(t,\xi)\right]\right\rangle. \tag{70}$$

Due to Lemma 4, the relation

$$\frac{1}{\Delta t}\left[T_{-x}\check{G}(t+\Delta t,\xi)-T_{-x}\check{G}(t,\xi)\right]\longrightarrow\frac{\partial}{\partial t}T_{-x}\check{G}(t,\xi),\quad \Delta t\longrightarrow 0, \tag{71}$$

is performed in the sense of $S_2^{1/\omega}$ topology. So,

$$\begin{aligned}\frac{\partial}{\partial t}(f*G(t,\cdot))&=\langle f_\xi,\lim_{\Delta t\longrightarrow 0}\frac{1}{\Delta t}\left[T_{-x}\check{G}(t+\Delta t,\xi)-T_{-x}\check{G}(t,\xi)\right]\rangle\\&=\langle f_\xi,\frac{\partial}{\partial t}T_{-x}\check{G}(t,\xi)\rangle=\langle f_\xi,T_{-x}\frac{\partial}{\partial t}\check{G}(t,\xi)\rangle\\&=f*\frac{\partial}{\partial t}G(t,\cdot).\end{aligned} \tag{72}$$

The statement is proved.

Since

$$F[G(t,\cdot)]=Q(t,\sigma)=Q_1(t,\sigma)Q_2(\sigma)=\int_{\mathbb{R}}G(t,x)e^{i\sigma x}\mathrm{d}x,\quad t>0, \tag{73}$$

we get the formula

$$\begin{aligned}&\int_{\mathbb{R}}G(t,x)\mathrm{d}x=Q_1(t,0)Q_2(0)=\lambda_0e^{-t},\\&\lambda_0=\left(\mu-\sum_{k=1}^{m}\mu_kg_k(0)e^{-t_k}\right)^{-1},\end{aligned} \tag{74}$$

where $a(0)=1$ is taken into account. □

Lemma 5. *In $(S_2^{1/\omega})'$, the following relations are correct:*

$$\begin{aligned}&(1)\,G(t,\cdot)\longrightarrow F^{-1}[Q_2],\quad t\longrightarrow +0;\\&(2)\,\mu G(t,\cdot)-\sum_{k=1}^{m}\mu_kB_kG(t_k,\cdot)\longrightarrow\delta,\quad t\longrightarrow +0,\end{aligned} \tag{75}$$

where δ is the Dirac delta function.

Proof. (1) Taking into account the continuity property of the Fourier transform (direct and inverse) in spaces of S' type, it is sufficient to establish that

$$F[G(t,\cdot)]=Q_1(t,\cdot)Q_2(\cdot)\longrightarrow Q_2(\cdot),\quad t\longrightarrow +0, \tag{76}$$

in the space $(S_{1/\omega}^2)'$. To do this, we take an arbitrary function $\varphi\in S_{1/\omega}^2$, and using the fact that Q_2 is a multiplier in the space $S_{1/\omega}^2$ and the Lebesgue theorem on the limit passage under the Lebesgue integral sign, we find that

$$\begin{aligned}&\langle Q_1(t,\cdot)Q_2(\cdot),\varphi\rangle=\langle Q_1(t,\cdot),Q_2(\cdot)\varphi(\cdot)\rangle\\&=\int_{\mathbb{R}}Q_1(t,\sigma)Q_2(\sigma)\varphi(\sigma)\mathrm{d}\sigma\underset{t\longrightarrow+0}{\longrightarrow}\int_{\mathbb{R}}Q_2(\sigma)\varphi(\sigma)\mathrm{d}\sigma\\&=\langle 1,Q_2(\cdot)\varphi(\cdot)\rangle=\langle Q_2,\varphi\rangle.\end{aligned} \tag{77}$$

This leads us to statement (1) of Lemma 5.

(2) Given statement (1) and the form of the operators $B_1,\ldots,B_m$, we find that

$$\begin{aligned}\mu G(t,\cdot)-\sum_{l=1}^{m}\mu_lB_lG(t_l,\cdot)&\underset{t\longrightarrow+0}{\longrightarrow}\mu F^{-1}[Q_2]-\sum_{l=1}^{m}\mu_lB_lG(t_l,\cdot)\\&=\mu F^{-1}[Q_2]-\sum_{l=1}^{m}\mu_lF^{-1}[g_l(\cdot)Q_1(t_l,\cdot)Q_2(\cdot)]\\&=F^{-1}\left[\mu Q_2(\cdot)-\sum_{l=1}^{m}\mu_lg_l(\cdot\right)Q_1(t_l,\cdot)Q_2(\cdot)]\\&=F^{-1}\left[\mu-\sum_{l=1}^{m}\mu_lg_l(\cdot)Q_1(t_l,\cdot)Q_2(\cdot)\right]\\&=F^{-1}\left[\left(\mu-\sum_{l=1}^{m}\mu_lg_l(\cdot)Q_1(t_l,\cdot)\right)\right.\\&\quad\left.\cdot\left(\mu-\sum_{l=1}^{m}\mu_lg_l(\cdot)Q_1(t_l,\cdot)\right)^{-1}\right]=F^{-1}[1]=\delta.\end{aligned} \tag{78}$$

Therefore, relation (75) is fulfilled in the space $(S_2^{1/\omega})'$. Lemma 5 is proved. □

Remark 4. If $\mu=1$, $\mu_1=\cdots=\mu_m=0$, then problems (24) and (25) degenerate into the Cauchy problem for equation (24). In this case, $Q_2(\sigma)=1$, $\forall\sigma\in\mathbb{R}$, $G(t,x)=F^{-1}[e^{-ta(\sigma)}]$, and $G(t,\cdot)\longrightarrow F^{-1}[1]=\delta$ for $t\longrightarrow +0$ in the space $(S_1^{1/\omega})'$.

Corollary 3. *Let*

$$\omega(t,x)=f*G(t,x),\quad f\in\left(S_{2,*}^{1/\omega}\right)',\ (t,x)\in\Omega, \tag{79}$$

where $(S_{2,*}^{1/\omega})'$ is a class of convolutors in $S_2^{1/\omega}$. Then, in $(S_2^{1/\omega})'$, the following boundary relation holds:

$$\mu\omega(t,\cdot)-\sum_{k=1}^{m}\mu_kB_k\omega(t_k,\cdot)\longrightarrow f,\quad t\longrightarrow +0. \tag{80}$$

Proof. Let us prove that the relation

$$F\left[\mu\omega(t,\cdot)-\sum_{k=1}^{m}\mu_kB_k\omega(t_k,\cdot)\right]\longrightarrow F[f],\quad t\longrightarrow +0 \tag{81}$$

takes place in the space $(S_{1/\omega}^2)'$. Since $f\in(S_{2,*}^{1/\omega})'$ and $G(t,\cdot)\in S_2^{1/\omega}$ for every $t>0$, we have

$$F[\omega(t,\cdot)]=F[f*G(t,\cdot)]=F[f]F[G(t,\cdot)]=F[f]Q(t,\cdot). \tag{82}$$

So, we need to prove that

$$F[f]\left(\mu Q(t,\cdot)-\sum_{k=1}^{m}\mu_kg_k(\cdot)Q(t_k,\cdot)\right)\longrightarrow F[f], \tag{83}$$

in $(S_{1/\omega}^2)'$, when $t\longrightarrow +0$. Since $Q(t,\cdot)=Q_1(t,\cdot)Q_2(\cdot)\longrightarrow Q_2(\cdot)$ for $t\longrightarrow +0$ in the space $(S_{1/\omega}^2)'$ (see the proof of assertion 1 and Lemma 5), the correlation

$$\mu Q(t,\cdot) - \sum_{k=1}^{m} \mu_k g_k(\cdot) Q(t_k,\cdot) \underset{t \longrightarrow +0}{\longrightarrow} \mu Q_2(\cdot) - \sum_{k=1}^{m} \mu_k g_k(\cdot) Q_1(t_k,\cdot) Q_2(\cdot)$$
$$= \left(\mu - \sum_{k=1}^{m} \mu_k g_k(\cdot) Q_1(t_k,\cdot)\right) Q_2(\cdot) = 1, \tag{84}$$

is realized in the space $(S^2_{1/\omega})'$. Thus, relation (81) and (80) are fulfilled in the corresponding spaces. The statement is proved.

The function $G(t,\cdot)$ satisfies equation (24) as $t>0$. Indeed,

$$\frac{\partial}{\partial t} G(t,x) = \frac{\partial}{\partial t} F^{-1}[Q(t,\sigma)] = F^{-1}\left[\frac{\partial}{\partial t} Q(t,\sigma)\right]$$
$$= F^{-1}[a(\sigma) Q(t,\sigma)],$$
$$AG(t,x) = F^{-1}\left[a(\sigma) F^{-1}[G(t,\cdot)]\right] = F^{-1}[a(\sigma) Q(t,\sigma)]. \tag{85}$$

So,

$$\frac{\partial G(t,x)}{\partial t} + AG(t,x) = 0, \quad (t,x) \in \Omega, \tag{86}$$

which is what had to be proved.

It follows from Corollary 3 that the nonlocal multipoint by the time problem for equation (24) can be formulated in the following way: find a function $u(t,x)$, $(t,x) \in \Omega$ that satisfies equation (24) and the condition

$$\mu \lim_{t \longrightarrow +0} u(t,\cdot) - \sum_{k=1}^{m} \mu_k B_k u(t_k,\cdot) = f, \quad f \in \left(S^{1/\omega}_{2,*}\right)', \tag{87}$$

where boundary relation (87) is considered in the space $(S^{1/\omega}_2)'$, and the constraints on the parameters $\mu, \mu_1, \ldots, \mu_m, t_1, \ldots, t_m$ are the same as in case of problems (24) and (25)). □

Theorem 1. *By nonlocal multipoint by time problem (24), (87) is solvable and its solution is given by the formula*

$$u(t,x) = f * G(t,x), \quad (t,x) \in \Omega, \tag{88}$$

where $u(t,\cdot) \in S^{1/\omega}_2$ for every $t>0$.

Proof. Make sure that the function $u(t,x)$, $(t,x) \in \Omega$, satisfies equation (24). Indeed (see Corollary 2),

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial}{\partial t}(f * G(t,\cdot)) = f * \frac{\partial G(t,\cdot)}{\partial t},$$
$$Au(t,x) = F^{-1}[a(\sigma) F[f * G(t,\cdot)]]. \tag{89}$$

Since f is a convolutor in $S^{1/\omega}_2$,

$$F[f * G(t,\cdot)] = F[f] F[G(t,\cdot)] = F[f] Q(t,\cdot). \tag{90}$$

So,

$$Au(t,x) = F^{-1}[a(\sigma) Q(t,\sigma) F[f](\sigma)] = -F\left[\frac{\partial}{\partial t} Q(t,\cdot) F[f]\right]$$
$$= -F\left[F\left[\frac{\partial}{\partial t} G(t,\cdot)\right] \cdot F[f]\right]$$
$$= -F^{-1}\left[F\left[f * \frac{\partial G(t,\cdot)}{\partial t}\right]\right] = -f * \frac{\partial G(t,\cdot)}{\partial t}. \tag{91}$$

It follows from this that the function $u(t,x)$, $(t,x) \in \Omega$, satisfies equation (24). From Corollary 3, we get that $u(t,x)$ satisfies condition (87) in the defined sense. The theorem is proved. □

Remark 5. If, in condition (87), $B_1 = \cdots = B_m = I$ (I is the identity operator), then we can prove that problems (24) and (87) are well posed and its solution is given by the formula

$$u(t,x) = f * G(t,x), \quad f \in \left(S^{1/\omega}_{1,*}\right)', \ (t,x) \in \Omega, \tag{92}$$

where $G(t,\cdot) = F^{-1}[Q(t,\cdot)] \in S^{1/\omega}_1$ for every $t>0$, and

$$Q(t,\sigma) = e^{-ta(\sigma)} \left(\mu - \sum_{k=1}^{m} \mu_k e^{-t_k a(\sigma)}\right)^{-1}. \tag{93}$$

Theorem 2. *Suppose $u(t,x)$, $(t,x) \in \Omega$, is the solution of problems (24) and (87). Then, $u(t,\cdot) \longrightarrow 0$ in the space $(S^{1/\omega}_2)'$ as $t \longrightarrow +\infty$.*

Proof. Recall that the solution of problems (24) and (87) is given by the formula

$$u(t,x) = f * G(t,x) = \langle f_\xi, G(t,x-\xi) \rangle,$$
$$f \in \left(S^{1/\omega}_{2,*}\right)', \ (t,x) \in \Omega. \tag{94}$$

Let us prove that $\langle u(t,\cdot), \varphi \rangle \longrightarrow 0$ as $t \longrightarrow +\infty$ for an arbitrary function $\varphi \in S^{1/\omega}_2$. Put

$$\Psi_t(\xi) = \int_{-\infty}^{+\infty} G(t,x-\xi)\varphi(x)\mathrm{d}x,$$
$$\Psi_{t,R}(\xi) = \int_{-R}^{R} G(t,x-\xi)\varphi(x)\mathrm{d}x, \quad R>0, \ t>1. \tag{95}$$

In these notations, we prove that (a) for every $t>1$ and $R>0$, the function $\Psi_{t,R}(\xi)$ belongs to the space $S^{1/\omega}_2$ and $\Psi_{t,R}(\xi) \longrightarrow \Psi_t(\xi)$ in the space $S^{1/\omega}_2$ as $R \longrightarrow \infty$; (b) $\Psi_t(\xi) \in S^{1/\omega}_2$ for every $t>1$. From this, we get

$$\langle u(t,\cdot), \varphi \rangle = \int_{-\infty}^{+\infty} \langle f_\xi, G(t,x-\xi) \rangle \varphi(x)\mathrm{d}x$$
$$= \left\langle f_\xi, \int_{-\infty}^{+\infty} G(t,x)\varphi(x+\xi)\mathrm{d}x \right\rangle$$
$$= \left\langle f_\xi, \int_{-\infty}^{+\infty} G(t,-y)\varphi(-(y-\xi))\mathrm{d}y \right\rangle$$
$$= \left\langle f_\xi, \int_{-\infty}^{+\infty} G(t,-y)\check{\varphi}(y-\xi)\mathrm{d}y \right\rangle, \quad \check{\varphi}(z) = \varphi(-z), \tag{96}$$

where $u(t,\cdot)$ is interpreted as a regular generalized function from $(S_2^{1/\omega})'$ for every $t>0$.

So, let us set property (a). For fixed $\{k,m\}\subset\mathbb{Z}_+$, we have

$$\left|\xi^k D_\xi^m \Psi_{t,R}(\xi)\right| \le \int_{-R}^{+R}\left|\xi^k\varphi(x)D_\xi^m G(t,x-\xi)\right|\mathrm{d}x \le \int_{-\infty}^{+\infty}\left|\xi^k\varphi(\xi+\eta)D_\eta^m G(t,\eta)\right|\mathrm{d}\eta. \tag{97}$$

Since $\varphi\in S_2^{1/\omega}$, the inequality

$$\left|\xi^k D_\xi^m\varphi(\xi)\right| \le cL^kM^mk^{2k}m^{m/\omega},\quad \{k,m\}\subset\mathbb{Z}_+, \tag{98}$$

with some c, L, $M>0$, is valid. It follows from this that for every $\eta\in\mathbb{R}$,

$$\sup_{\xi\in\mathbb{R}}\left|\xi^k\varphi(\xi)\right| = \sup_{y\in\mathbb{R}}\left|(y-\eta)^k\varphi(\eta)\right| = \sup_{y\in\mathbb{R}}\left|\sum_{l=0}^{k}C_k^l y^l(-\eta)^{k-l}\varphi(y)\right| \le \sum_{l=0}^{k}C_k^l|\eta|^{k-l}\sup_{y\in\mathbb{R}}\left|y^k\varphi(y)\right| \le c\sum_{l=0}^{k}C_k^lL^ll^{2l}|\eta|^{k-l}. \tag{99}$$

Next, we will use estimate (58). Then,

$$\begin{aligned}\left|\xi^k D_\xi^m\Psi_{t,R}(\xi)\right| &\le c\sum_{l=0}^{k}C_k^lL^ll^{2l}\int_{-\infty}^{+\infty}|\eta|^{k-l}\left|D_\eta^m G(t,\eta)\right|\mathrm{d}\eta\\ &\le cc_3\overline{B}^m m^{m/\omega}t^{-(1/\omega)}\sum_{l=0}^{k}C_k^lL^ll^{2l}\\ &\cdot\int_{-\infty}^{+\infty}|\eta|^{k-l}\exp\left\{-a_0t^{-(1/2)}|\eta|^{1/2}\right\}\mathrm{d}\eta.\end{aligned} \tag{100}$$

By the direct calculations, we find that

$$\int_{-\infty}^{+\infty}|\eta|^{k-l}\exp\left\{-a_0t^{-(1/2)}|\eta|^{1/2}\right\}\mathrm{d}\eta \le c_4\tilde{L}^kk^{2(k-l)}t^{k-l+1},\quad c_4,\tilde{L}>0. \tag{101}$$

So,

$$\begin{aligned}\left|\xi^k D_\xi^m\Psi_{t,R}(\xi)\right| &\le cc_3c_4\overline{B}^mt^{1-(1/\omega)}m^{m/\omega}\sum_{l=0}^{k}C_k^lL^l\tilde{L}^{k-l}t^{k-l}l^{2l}k^{2(k-l)}\\ &\le \overline{c}\overline{B}^m\overline{L}^kk^{2k}m^{m/\omega},\end{aligned} \tag{102}$$

where $\overline{c}=cc_3c_4t^{1-(1/\omega)}$ and $\overline{L}=2\max\{L,\tilde{L}t\}$. Therefore, $\Psi_{t,R}(\xi)\in S_2^{1/\omega}$ for every $t>1$ and an arbitrary $R>0$. Furthermore, we make sure that $\Psi_{t,R}(\xi)\longrightarrow\Psi_t(\xi)$ as $R\longrightarrow\infty$, with all its derivatives uniformly in ξ on each segment $[a,b]\subset\mathbb{R}$. In addition, the set of functions $\left\{\xi^kD_\xi^m\Psi_{t,R}(\xi)\right\}$, $\{k,m\}\subset\mathbb{Z}_+$, is uniformly bounded (by R) in the space $S_2^{1/\omega}$ (this property follows from estimate (102) in which $\overline{c}$, $\overline{B}$, and $\overline{L}>0$ do not depend on R). This means that condition (a) is fulfilled.

Condition (b) follows from (a) because, in a perfect space, every bounded set is compact.

Using properties (a) and (b), we obtain the equality

$$\langle u(t,\cdot),\varphi\rangle = \int_{-\infty}^{+\infty}G(t,-y)(f*\check{\varphi})(y)\mathrm{d}y. \tag{103}$$

Since $f\in(S_2^{1/\omega})'$ is a convolutor in $S_2^{1/\omega}$, we have $f*\check{\varphi}\in S_2^{1/\omega}$. Then, given estimate (58) (at $s=0$), we find that

$$\begin{aligned}|\langle u(t,\cdot),\varphi\rangle| &\le \int_{-\infty}^{+\infty}|G(t,-y)||(f*\check{\varphi})(y)|\mathrm{d}y\\ &\le \frac{c}{t^{1/\omega}}\int_{-\infty}^{+\infty}|(f*\check{\varphi})(y)|\mathrm{d}y \le \frac{c_0}{t^{1/\omega}}\longrightarrow 0,\quad t\longrightarrow+\infty,\end{aligned} \tag{104}$$

for an arbitrary function $\varphi\in S_2^{1/\omega}$, i.e., $u(t,\cdot)\longrightarrow 0$ in the space $(S_2^{1/\omega})'$ as $t\longrightarrow+\infty$. The theorem is proved.

If the generalized function f in (87) is finite (i.e., its support $\operatorname{supp} f$ is a finite set in $\mathbb{R}$), then we can say on uniform tending a solution $u(t,x)$ of problems (24) and (87) to zero on $\mathbb{R}$ as $t\longrightarrow+\infty$. Note that every finite generalized function is a convolutor in S spaces. This property follows from the general result concerning the theory of perfect spaces (see [12], p. 137): if Φ is a perfect space with differential translation operation, then every finite functional is a convolutor in Φ. Finite functionals form a fairly wide class. In particular, every bounded set $F\subset\mathbb{R}$ is a support of some generalized function [12]. □

Theorem 3. *Let $u(t,x)$ be a solution of problems (24) and (87) with the boundary function f in condition (87), which is an element of $(S_2^\beta)'\subset(S_2^{1/\omega})'$, $\beta>1$, with finite support in $\mathbb{R}$. Then, $u(t,x)\longrightarrow 0$ uniformly on $\mathbb{R}$ as $t\longrightarrow+\infty$.*

Here is a scheme for proving this statement. Let $\operatorname{supp} f\subset[a_1,b_1]\subset[a_2,b_2]\subset\mathbb{R}$. Consider the function $\varphi\in S_2^\beta$, $\beta>1$, such that $\varphi(x)=1$, $x\in[a_1,b_1]$, and $\operatorname{supp}\varphi\subset[a_2,b_2]$. This function exists when $\beta>1$ because S_2^β contains finite functions [12]. Consider the function

$$u(t,x)=\langle f_\xi,\varphi(\xi)G(t,x-\xi)\rangle+\langle f_\xi,\gamma(\xi)G(t,x-\xi)\rangle, \tag{105}$$

where $\gamma=1-\varphi$. Since $\operatorname{supp}(\gamma(\xi)G(t,x-\xi))\cap\operatorname{supp}\varphi=\varnothing$, we have

$$u(t,x)=t^{-(1/\omega)}\langle f_\xi,t^{1/\omega}\varphi(\xi)G(t,x-\xi)\rangle. \tag{106}$$

To prove the formulated above statement, it remains to establish that the set of functions $\Phi_{t,x}(\xi)=t^{1/\omega}\varphi(\xi)G(t,x-\xi)$ is bounded in the space S_2^β, $\beta>1$, for large t and $x\in\mathbb{R}$.

For example, if, in condition (87), $f=\delta$, then δ is a convolutor in the space S, $\operatorname{supp}\delta=\{0\}$, and $u(t,x)=\delta*G(t,x)=G(t,x)$. It follows directly from estimate (58) that $G(t,x)\longrightarrow 0$ uniformly on $\mathbb{R}$ as $t\longrightarrow+\infty$.

4. Conclusion

In this paper, the solvability of a nonlocal multipoint by the time problem for the evolutionary equation $(\partial u/\partial t)+(I-(\mathrm{d}^2/\mathrm{d}x^2))^{\omega/2}u=0$, $(t,x)\in(0,+\infty)\times\mathbb{R}$ is proved, herewith the operator $(I-(\mathrm{d}^2/\mathrm{d}x^2))^{\omega/2}$ is interpreted as a pseudodifferential operator in the space $S_2^{1/\omega}$

constructed by function $(1+\sigma^2)^{\omega/2}$, $\sigma \in \mathbb{R}$, while the nonlocal multipoint by time condition also contains pseudodifferential operators constructed on smooth symbols. The representation of the solution is given in the form of a convolution of the fundamental solution with the initial function which is an element of the space of generalized functions of the ultradistribution type (the coagulator in space $S_2^{1/\omega}$). The behavior of the solution $u(t,x)$, $t \longrightarrow +\infty$ in the space of generalized functions $(S_2^{1/\omega})'$, is investigated. The conditions for the initial generalized function are found under which the solution is uniformly stabilized to zero on $\mathbb{R}$. The method of research of a nonlocal multipoint by time problem offered in this paper allows to interpret differential-operator equations of a form $(\partial u/\partial t) + \varphi(i\,\partial/\partial x)u = 0$ as an evolutionary equation with a pseudodifferential operator $\varphi(i\,\partial/\partial x) = F^{-1}[\varphi \cdot F]$ constructed by the function φ acting in certain countable-normalized space of infinite-differential functions (the choice of space depends on the properties of the function which is the symbol of the operator $\varphi(i\,\partial/\partial x)$).

Data Availability

No data were used to support the findings of this study.

Disclosure

This study was conducted in the framework of scientific activity at Yuriy Fedkovych Chernivtsi National University.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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