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Research Article

On Some Sufficiency-Type Global Stability Results for Time-Varying Dynamic Systems with State-Dependent Parameterizations

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This paper formulates sufficiency-type global stability and asymptotic stability results for, in general, nonlinear time-varying dynamic systems with state-trajectory solution-dependent parameterizations. The stability proofs are based on obtaining sufficiency-type conditions which guarantee that either the norms of the solution trajectory or alternative interval-type integrals of the matrix of dynamics of the higher-order than linear terms do not grow faster than their available supremum on the preceding time intervals. Some extensions are also given based on the use of a truncated Taylor series expansion of chosen truncation order with multiargument integral remainder for the dynamics of the differential system.

1. Introduction

It is of interest to investigate explicit solution forms, if possible, and the stability and asymptotic stability properties of ordinary differential equations whose coefficients eventually depend on the solution trajectory and on its relevant derivatives with respect to time. In particular, differential equations whose coefficients depend on the solution and derivatives up till a certain order have been formally investigated, for instance, in [1, 2] and references therein. On the other hand, the stability properties of time-varying dynamic linear and nonlinear systems have been also investigated, for instance, in [3–12] and some of the references therein. In particular, Lyapunov second method for stability theory has been successfully used to address and discuss the boundedness and stability properties of the solutions of two class of third-order differential equations whose coefficients are time-varying functions which might depend on the solution and their two first-order time-derivatives.

The duality which exists between ordinary and functional differential equations of orders higher than one is well known. This duality is also reflected in their alternative equivalent descriptions in terms of sets of systems of first-order differential equations. On the other hand, it is also well known the usefulness of such descriptions to find explicit closed forms of solutions and to perform the analysis of their stability properties. Note that the descriptions of ordinary and functional differential equations of order n through sets of n first-order differential equations, eventually coupled, is suitable to formalize both the analytic solutions and, in particular, the stability properties of real-world dynamic systems of order n. This procedure might allow, to some extent, to get standard solution trajectory analytic expressions in a closed form and to obtain conditions of stability for the original differential equation.

This paper states and proves sufficiency-type global Lyapunov's stability and Lyapunov's asymptotic stability results of a differential system of an arbitrary n-th order decomposed as a set of n first-order ordinary differential equations whose parameterization is, in general, time-varying and depends on the solution trajectory. The differential system is, in fact, the description via first-order differential equations of an ordinary, or functional, differential equation of the same order whose coefficients depend on the solution, its relevant time-derivatives, and, eventually, explicitly with time. The technical mathematical proofs are based on getting "ad hoc" mathematical expressions which are obtained analytically for upper-bounds depending on time of the supremum of the solution norm. It is assumed that such a supremum norm evolves with time at a sufficiently slow rate. It is proved that either the global stability or the global asymptotic stability holds under the stability of a matrix function which generates the fundamental matrix of a certain reference unforced system and the sufficiently smallness of the error matrix function between the dynamics of the whole differential system and the above-mentioned stability matrix.

Alternative sufficiency-type global asymptotic stability conditions are got under the time integrability of the norm, or a power of the norm, of the error matrix function with sufficiently small values of the time-interval integral. Taking advantage of the fact that the decomposition of a matrix in a sum of matrices is not unique, there is a freedom in the choice of the matrix which is requested to be stable, the so-called nominal matrix of dynamics, but then the error matrix related to that of the system has to have a sufficiently small norm to achieve the stability properties.

Some extended results are also given if a truncated Taylor series expansion with an integral remainder around the equilibrium point is developed. In particular, the remainder expression is obtained by respecting the compatibility of the regularity conditions, with respect to the state-trajectory solution and time, of the matrix defining the whole dynamics of the differential system and the truncation order in the series expansion chosen to define the error matrix. The first one of those matrices defines the nominal matrix of dynamics, which exhibits stability properties, while the second one defines the error matrix related to the whole matrix function associated to the differential system at hand.

On the other hand, it can be pointed out that there are certain epidemic models where some of the coefficients describing the differential equations are time-varying and depending on the state defined by the subpopulations which take part of the model. In particular, a normalized SIR epidemic model (that is, with susceptible, infectious, and recovered integrated subpopulations) whose recovery rate is time-varying and state dependent is described and analyzed in [13]. In [14], an epidemic model with random screening (that is, the detected infectious are removed into a special class) is proposed with a nonlinear incidence rate which depends on the susceptible and the infectious, that is on the model state. On the other hand, in [15], a true-mass action type SEIR epidemic model (that is, with susceptible, exposed, infectious, and recovered integrated subpopulations) such that the coefficient disease transmission rate is normalized with the total population is analyzed. In such a way, such a normalized parameter becomes state-dependent, and thus time-varying, in the model. The above three types of state-dependent parameterizations are very common to some commonly used epidemic models. This feature results in their describing differential equations, or their state equations, to have state-dependent and time-varying parameterizations.

The following basic notation is used through the manuscript:

$$\begin{aligned} \mathbf{R}_{+} &= \{z \in \mathbf{R}: z > 0\}, \\ \mathbf{R}_{0+} &= \{z \in \mathbf{R}: z \geq 0\}, \\ \mathbf{Z}_{+} &= \{z \in \mathbf{Z}: z > 0\}, \\ \mathbf{Z}_{0+} &= \{z \in \mathbf{Z}: z \geq 0\}. \end{aligned} \tag{1}$$

The disjunction and conjunction logic propositions are, respectively, denoted by the symbols "$\vee$" and "$\wedge$":

$$\overline{n} = \{1, 2, \ldots, n\}. \tag{2}$$

$L_{\infty}^{n}[0,t]$ and, respectively, $L_{p}^{n}[0,t]$ are the sets of real n-vector functions which are bounded or, respectively, p-integrable on $[0,t]$ for any $p \in \mathbf{Z}_{+}$. In particular, $L_{\infty}^{n} = L_{\infty}^{n}[0,\infty]$ and $L_{p}^{n} = L_{p}^{n}[0,\infty]$ for any $p \in \mathbf{Z}_{+}$; cl S denotes the closure of the set S.

2. Problem Statement

Consider the n-th differential system of first-order equations.

$$\begin{aligned} \dot{z}(t) &= B(z(t), t) z(t), \\ z(0) &= z_0, \end{aligned} \tag{3}$$

where the matrix function B: $(\mathbf{R}_{0+} \times \mathbf{R}^{n}) \times \mathbf{R}_{0+} \longrightarrow \mathbf{R}^{n\times n}$ has piecewise continuous entries for each pair $(z(t), t) \in \mathbf{R}^{n} \times \mathbf{R}_{0+}$ such that $z: \mathbf{R}_{0+} \longrightarrow \mathbf{R}^{n}$ is a solution of (3). Such a matrix function can be, in general nonuniquely, decomposed as

$$B(z(t), t) = A(z(t), t) + \widetilde{A}(z(t), t), \quad \forall t \in \mathbf{R}_{0+}. \tag{4}$$

It is assumed thoroughly in the paper that the only (nonnecessarily stable) equilibrium point of (3) for the given $\widetilde{A}(z(t), t)$ and for the case when $\widetilde{A}(z(t), t) \equiv 0$ is $z_e = 0 \in \mathbf{R}^{n}$; $\forall t \in \mathbf{R}_{0+}$. In mathematical terms,

$$(z_e \in \mathbf{R}^{n} : ([z_e \in \operatorname{Ker}(B(z_e, t))] \vee [z_e \in \operatorname{Ker}(A(z_e, t))]; \forall t \in \mathbf{R}_{0+})) \Longleftrightarrow (z_e = 0 \in R^{n}), \tag{5}$$

what implies that $\operatorname{Ker} B(0,t) = \operatorname{Ker} A(0,t) = \{0\} \subset \mathbf{R}^n$; $\forall t \in \mathbf{R}_{0+}$, as a result. This feature introduces a further constraint on the nonunique additive decomposition (4).

It is, furthermore, assumed that the matrix functions $A, \widetilde{A} : (\mathbf{R}_{0+} \times \mathbf{R}^n) \times \mathbf{R}_{0+} \longrightarrow \mathbf{R}^{n\times n}$ have piecewise continuous entries for each pair $(z(t),t) \in \mathbf{R}^n \times \mathbf{R}_{0+}$ such that $z : \mathbf{R}_{0+} \longrightarrow \mathbf{R}^n$ is a solution of (3). To fix ideas, it is interpreted that $A(.,.)$ defines the nominal dynamics and $\widetilde{A}(.,.)$ defines the error dynamics of (3), subject to (4).

Example 1. The differential system (3) can compactly describe a time-varying n-th ordinary differential system whose coefficients depend also on the derivatives up till n-th order and which are linear in the solution and its first $(n-1)$-th derivatives. For instance, a differential equation subject to a more general forcing term than equation (3) of [2] which, together with its second order analogous, plays an important role in the phase locked loop model realized by a T.V. system (see [2, 16, 17]), may be (in general, nonuniquely) described in the form (3) and (4), as follows:

$$z(t) = (x(t), \dot{x}(t), \ddot{x}(t))^{\mathrm{T}},$$
$$e(z(t),t) = c(z(t),t) + \widetilde{A}(z(t),t)z(t), \quad \forall t \in \mathbf{R}_{0+}, \tag{6}$$

according, for instance, to

(a)

$$A_a = A_a(z(t),t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_{10} & -a_{20} & -a_{30} \end{bmatrix},$$
$$\widetilde{A}_a(z(t),t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_{10} & a_{20} - b(t)g(x(t)) & a_{30} - a(t)f(\dot{x}(t)) \end{bmatrix}, \tag{7}$$

which is clearly equivalent to the third-order differential equation:

$$\dddot{x}(t) + a(t)f(\dot{x}(t))\ddot{x}(t) + b(t)g(x(t))\dot{x}(t) = e(x(t),\dot{x}(t),\ddot{x}(t),t) - c(x(t),\dot{x}(t),\ddot{x}(t),t), \tag{8}$$

independent of any real constants a_{i0} for $i = 1, 2, 3$. If $c(t, z(t)) = a(t)x(t)$ and $\widetilde{A}_b = \widetilde{A}_b(z(t),t) = 0$; $\forall t \in \mathbf{R}_{0+}$, it is possible to describe the differential system, again in alternative ways, for instance, as follows:

(b)

$$A_b(t) = A_b(z(t),t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a(t) & -b(t)g(x(t)) & -a(t)f(\dot{x}(t)) \end{bmatrix},$$
$$\widetilde{A}_b = \widetilde{A}_b(z(t),t) = 0; \quad \forall t \in \mathbf{R}_{0+}, \tag{9}$$

(c)

$$A_c(t) = A_c(z(t),t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -a_1(t) & -b(t)g(x(t)) & -a(t)f(\dot{x}(t)) \end{bmatrix},$$
$$\widetilde{A}_c(t) = \widetilde{A}_c(z(t),t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -a_2(t) & 0 & 0 \end{bmatrix}, \quad \forall t \in \mathbf{R}_{0+}, \tag{10}$$

with $a_2(t) = a(t) - a_1(t)$; $\forall t \in \mathbf{R}_{0+}$,

(d)

$$A_d = A_d(z(t),t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -a_{10} & -(a_{20} + b(0)g(x(0))) & -(a_{30} + a(0)f(\dot{x}(0))) \end{bmatrix}, \quad \forall t \in \mathbf{R}_{0+},$$
$$\widetilde{A}_d(z(t),t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ a_{10} - a(t) & a_{20} + b(0)g(x(0)) - b(t)g(x(t)) & a_{30} + a(0)f(\dot{x}(0)) - a(t)f(\dot{x}(t)) \end{bmatrix}, \quad \forall t \in \mathbf{R}_{0+}. \tag{11}$$

The following further assumptions are made to be used in some of the main subsequent results.

Assumption 1. $A(z(t),t)$ commutes with $\int_0^t A(z(\tau),\tau)d\tau$; $\forall t \in \mathbf{R}_{0+}$.

Assumption 2. The matrix function $A : (\mathbf{R}_{0+} \times \mathbf{R}^n) \times \mathbf{R}_{0+} \longrightarrow \mathbf{R}^{n\times n}$ satisfies $\|A(z(t),t)\| \le a$, and it is a stability matrix for each pair $(z(t),t) \in \mathbf{R}^n \times \mathbf{R}_{0+}$ such that $z : \mathbf{R}_{0+} \longrightarrow \mathbf{R}^n$ is a solution of (3).

Assumption 3. $\|\widetilde{A}(z(t),t)\| \le \widetilde{a}$; $\forall (z(t),t) \in \mathbf{R}^n \times \mathbf{R}_{0+}$ such that $z : \mathbf{R}_{0+} \longrightarrow \mathbf{R}^n$ is a solution of (3).

Remark 1. Note that the decomposition (4) is always possible and nonunique if $B : (\mathbf{R}_{0+} \times \mathbf{R}^n) \times \mathbf{R}_{0+} \longrightarrow \mathbf{R}^{n\times n}$ has piecewise continuous entries for each pair $(z(t),t) \in \mathbf{R}^n \times \mathbf{R}_{0+}$ such that $z : \mathbf{R}_{0+} \longrightarrow \mathbf{R}^n$ is a solution of (3). On the other hand, taking advantage that the decomposition (4) is not unique, one concludes that Assumption 1 is not restrictive at all, in practice. It would suffice, for instance, to take a diagonal $A(z(t),t)$ and $\widetilde{A}(z(t),t) = B(z(t),t) - A(z(t),t)$; $\forall t \in \mathbf{R}_{0+}$.

The main objective of the paper is to derive and prove sufficiency-type global stability and global asymptotic stability results of the differential system (3), subject to (4) by examining the growing rules of its norm through time through "ad hoc" derived integral inequalities.

3. Some Stability Results

The subsequent result relies on the existence and uniqueness of the solution of (3).

Theorem 1. *If Assumption 1 holds, then the n-th differential system (3) has a unique solution for each initial condition $z(0) = z_0 \in \mathbf{R}^n$ which is given by*

$$z(t) = e^{\int_0^t A(z(\tau),\tau)d\tau} z_0 + \int_0^t e^{\int_\tau^t A(z(\sigma),\sigma)d\sigma} \widetilde{A}(z(\tau),\tau) z(\tau) d\tau, \quad \forall t \in \mathbf{R}_{0+}, \tag{12}$$

where $\Psi(z_0,t,0) = e^{\int_0^t A(z(\tau),\tau)d\tau}$; $\forall t \in \mathbf{R}_{0+}$ is the fundamental matrix function of the n-th differential system differential system $\dot{y}(t) = A(z(t),t)y(t)$; $y(0) = y_0$, which satisfies $\Psi(z_0,0,0) = I_n$; $\forall z_0 \in \mathbf{R}^n$:

$$\begin{aligned} \dot{\Psi}(z_0,t,0) &= A(z(t),t) e^{\int_0^t A(z(\tau),\tau)d\tau} \\ &= A(z(t),t)\Psi(t,0) = \Psi(t,0)A(z(t),t), \quad \forall (t,\tau) \in \mathbf{R}_{0+} \times \mathbf{R}_{0+}. \end{aligned} \tag{13}$$

Proof. Note from (12) that $z(0) = z_0$ and that (12) is everywhere time-differentiable on $\mathbf{R}_{0+}$ whose derivative is by using Leibnitz's rule:

$$\begin{aligned} \dot{z}(t) &= A(z(t),t) e^{\int_0^t A(z(\tau),\tau)d\tau} z_0 + e^{\int_t^t A(z(\sigma),\sigma)d\sigma} \widetilde{A}(z(t),t) z(t) \\ &\quad + \int_0^t A(z(t),t) e^{\int_\tau^t A(z(\sigma),\sigma)d\sigma} \widetilde{A}(z(\tau),\tau) z(\tau) d\tau \\ &= A(z(t),t) \left[e^{\int_0^t A(z(\tau),\tau)d\tau} z_0 + \int_0^t e^{\int_\tau^t A(z(\sigma),\sigma)d\sigma} \widetilde{A}(z(\tau),\tau) z(\tau) d\tau \right] \\ &\quad + e^{\int_t^t A(z(\sigma),\sigma)d\sigma} \widetilde{A}(z(t),t) z(t) = A(z(t),t)z(t) + \widetilde{A}(z(t),t)z(t), \quad \forall t \in \mathbf{R}_{0+}, \end{aligned} \tag{14}$$

after replacing (3) in the second identity of (14). Note that (12) is a unique solution for each $z(0) = z_0 \in \mathbf{R}^n$ since it is a closed formula and that it is calculated by taking the auxiliary system $\dot{y}(t) = A(z(t),t)y(t)$ as the unforced system with $y(0) = y_0$ and whose solution is $y(t) = \Psi(z_0,t,0)y(0) = e^{\int_0^t A(z(\tau),\tau)d\tau} y_0$; $\forall t \in \mathbf{R}_{0+}$. □

It can be pointed out that time-varying matrices, contrarily to constant matrices, do not preserve the stability under arbitrary transformations. There are specific transformations, as, for instance, the so-called Bohl transformations [5], which keep the stability properties from the original representation.

Example 2. Retaking Example 1 with $e(t) \equiv 0$ and the particular parameterization of (a) of the matrix $A_a = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_{10} & -a_{20} & -a_{30} \end{bmatrix}$ yields that the fundamental matrix is $\Psi(t,0) = e^{At}$; $\forall t \in \mathbf{R}_{0+}$. Assumption 1 holds trivially and the boundedness part of Assumption 2 also holds.

The subsequent result is related to the boundedness of the solution and the stability properties of (3).

Theorem 2. *If Assumptions 1–3 hold, then the n-th differential system (3), subject to (4), has a uniformly bounded solution for any finite initial conditions and it is globally asymptotically stable at large (i.e., in $\mathbf{R}^n$) if $\widetilde{a}$ is sufficiently small satisfying a maximum measurable guaranteed upper-bound amount which is given explicitly in the proof.*

Proof. Note that, from Assumption 2, $\|e^{A(z}\ (t),t)\| \leq K_A(z(t),t)e^{-\int_0^t \rho_A(z(\tau),\tau)d\tau}$ for some bounded piecewise-continuous functions $K_A : (\mathbf{R}_{0+} \times \mathbf{R}^n) \times \mathbf{R}_{0+} \longrightarrow [1,\infty)$ and $\rho_A : (\mathbf{R}_{0+} \times \mathbf{R}^n) \times \mathbf{R}_{0+} \longrightarrow (0,+\infty)$.

Then,

$$\|\Psi(z_0,t,0)\| = \left\| e^{\int_0^t A(z(\tau),\tau)d\tau} \right\| \leq Ke^{-\int_0^t \rho_A(z(\tau),\tau)d\tau} \leq Ke^{-\rho_0(z_0,t)t} \leq Ke^{-\rho t}, \quad \forall t \in \mathbf{R}_{0+}, \tag{15}$$

where

$$K = \sup_{t\in\mathbf{R}_{0+}} \sup_{z_0\in\mathbf{R}^n} K_A(z_0,t),$$

$$0 < \rho_0(z_0,t) \leq \frac{\int_0^t \rho_A(z(\tau),\tau)d\tau}{t}, \tag{16}$$

$$\rho = \inf_{t\in\mathbf{R}_{0+}} \inf_{z_0\in\mathbf{R}^n} \rho_0(z_0,t).$$

Note that $\rho > 0$ since $\rho_0(z_0,t) > 0$ for $t \in \mathbf{R}_+$ and, from L'Hopital rule,

$$\lim_{t\longrightarrow 0^+} \rho_0(z_0,t) = \lim_{t\longrightarrow 0^+} \frac{\int_0^t \rho_A(z(\tau),\tau)d\tau}{t} = \rho_A(z_0,0) > 0. \tag{17}$$

Then,

$$\|z(t)\| \leq Ke^{-\rho t}\left(\|z_0\| + \int_0^t e^{\rho\tau}\|\widetilde{A}(z(\tau),\tau)\|\|z(\tau)\|d\tau\right); \leq Ke^{-\rho t}\left(\|z_0\| + \frac{e^{\rho t}-1}{\rho}\widetilde{a}\sup_{0\leq\tau\leq t}\|z(\tau)\|\right); \quad \forall t \in \mathbf{R}_{0+}. \tag{18}$$

Define

$$t_t = \left(\max \tau \in [0,t] : \|z(\tau)\| = \sup_{0\leq\sigma\leq\tau}\|z(\sigma)\|\right), \quad \forall t \in \mathbf{R}_{0+}, \tag{19}$$

$$\sup_{0\leq\tau\leq t}\|z(\tau)\| = \left\|z(t_t)\right\| \leq Ke^{-\rho t_t}\left\|z_0\right\| + K\frac{1-e^{-\rho t_t}}{\rho}\widetilde{a}\sup_{0\leq\tau\leq t}\|z(\tau)\|, \quad \forall t \in \mathbf{R}_{0+}. \tag{20}$$

Now, assume that $\widetilde{a} < (\rho/K)\inf_{t\geq 0} 1/(1-e^{-\rho t_t}) \leq (\rho/K)$, then $\|z(t)\| \leq \sup_{0\leq\tau<\infty}\|z(\tau)\| \leq \rho K\|z_0\|/(\rho - K\widetilde{a})$; $\forall t \in \mathbf{R}_{0+}$ since $\|z(t)\| \leq \sup_{0\leq\tau\leq t}\|z(\tau)\| \leq \rho Ke^{-\rho t_t}\|z_0\|/(\rho + K(e^{-\rho t_t}-1))\widetilde{a} \leq \rho K\|z_0\|/(\rho - K\widetilde{a}) < +\infty$; $\forall t \in \mathbf{R}_{0+}$.

Then, the solution (12) is uniformly bounded for all time for any given finite initial conditions, and the differential system (3) is globally uniformly stable at large in Lyapunov's sense as a result. It is now proved by contradiction argument that the stability is asymptotic. Assume on the contrary that there is a sequence $\{t_j\}_{j=0}^{\infty} \subset \mathbf{R}_{0+}$ such that $\sup_{t_{k+1}\leq t<t_{k+2}}\|z(t)\| \geq \sup_{t_k\leq t<t_{k+1}}\|z(t)\|$; $\forall t_k \in \{t_j\}_{j=0}^{\infty}$. Assume that the following cases can occur.

Case a. There is a subsequence $\{t_{j_i}\}_{i=0}^{\infty} \subseteq \{t_j\}_{j=0}^{\infty}$ such that $\sup_{t_{k+1}\leq t<t_{k+2}}\|z(t)\| > \sup_{t_k\leq t<t_{k+1}}\|z(t)\|$; $\forall t_k \in \{t_{j_i}\}_{i=0}^{\infty}$. This case is not possible since then the solution is not uniformly bounded for all $t \in \mathbf{R}_{0+}$.

Case b. There is a subsequence $\{t_{j_i}\}_{i=0}^{\infty} \subseteq \{t_j\}_{j=0}^{\infty}$ such that $\sup_{t_{k+1}\leq t<t_{k+2}}\|z(t)\| = \sup_{t_k\leq t<t_{k+1}}\|z(t)\| > 0$; $\forall t_k \in \{t_{j_i}\}_{i=0}^{\infty}$. Then,

$$\sup_{t_{k+j+1}\leq\tau<t_{k+j+2}}\|z(\tau)\| = \left\|z\left(t_{k+j+1}+\sigma_{k+j+1}\right)\right\| \leq Ke^{-\rho\left(t_{k+j+1}-t_{k+1}+\sigma_{k+j+1}-\sigma_{k+1}\right)} \cdot \sup_{t_{k+1}\leq\tau<t_{k+2}}\|z(\tau)\| + K\frac{1-e^{-\rho\left(t_{k+j+1}-t_{k+1}+\sigma_{k+j+1}-\sigma_{k+1}\right)}}{\rho}\widetilde{a} \cdot \sup_{0\leq\tau<t_{k+j+2}}\|z(\tau)\|, \quad \forall t_k \in \{t_{j_i}\}_{i=0}^{\infty},\ \forall t \in \mathbf{R}_{0+}, \tag{21}$$

where $\sigma_k \in [0, t_{k+1}-t_k)$, $\forall t_k \in \{t_{j_i}\}_{i=0}^{\infty}$, such that $\sup_{t_k\leq\tau<t_{k+1}}\|z(\tau)\| = \|z(t_k+\sigma_k)\|$ if and only if $\tau = 0$; i.e., $t_k + \sigma_k$ is the largest time instant in $[t_k, t_{k+1})$ where the supremum is reached within such an interval for all $t_k \in \{t_{j_i}\}_{i=0}^{\infty}$. Now, if the subsequence $\{t_{j_i}\}_{i=0}^{\infty}$ is not unique then, with no loss in generality, take the one such that there is some finite $t_{j_0} \in \{t_{j_i}\}_{i=0}^{\infty}$ and there is no $t(\geq t_{j_0}) \in \mathbf{R}_{0+}$ such that $\|z(t)\| > \sup_{t_k\leq\tau<t_{k+1}}\|z(\tau)\|$; $\forall t_k \in \{t_{j_i}\}_{i=0}^{\infty}$ so that $\sup_{\tau\geq t_{j_0}}\|z(\tau)\| = \sup_{t_{j_0}\leq\tau<t_k}\|z(\tau)\|$; $\forall t_k(\geq t_{j_0}) \in \{t_j\}_{j=0}^{\infty}$ and $t_{j_0} \in \{t_{j_i}\}_{i=0}^{\infty}$. Thus, one gets from (21) that

$$\sup_{t_{j_0}\le\tau<t_{k+j+2}}\|z(\tau)\|\le Ke^{-\rho\left(t_{k+j+1}-t_{k+1}+\sigma_{k+j+1}-\sigma_{k+1}\right)}\sup_{t_{j_0}\le\tau<t_{k+j+2}}\|z(\tau)\|+K\frac{1-e^{-\rho\left(t_{k+j+1}-t_{k+1}+\sigma_{k+j+1}-\sigma_{k+1}\right)}}{\rho}\tilde{a}\sup_{t_{j_0}\le\tau<t_{k+j+2}}\|z(\tau)\|;\quad\forall t_k\in\left\{t_j\right\}_{j=0}^{\infty},\quad\forall t\in\mathbf{R}_{0+},\tag{22}$$

$$\left(1-Ke^{-\rho\left(t_{k+j+1}-t_{k+1}+\sigma_{k+j+1}-\sigma_{k+1}\right)}-K\frac{1-e^{-\rho\left(t_{k+j+1}-t_{k+1}+\sigma_{k+j+1}-\sigma_{k+1}\right)}}{\rho}\tilde{a}\right)\sup_{t_{j_0}\le\tau<t_{k+j+2}}\|z(\tau)\|\le 0;\quad\forall t_k\in\left\{t_j\right\}_{j=0}^{\infty},\quad\forall t\in\mathbf{R}_{0+},\tag{23}$$

so that

$$\limsup_{j\longrightarrow\infty}\left[\left(1-Ke^{-\rho\left(t_{k+j+1}-t_{k+1}+\sigma_{k+j+1}-\sigma_{k+1}\right)}-K\frac{1-e^{-\rho\left(t_{k+j+1}-t_{k+1}+\sigma_{k+j+1}-\sigma_{k+1}\right)}}{\rho}\tilde{a}\right)\sup_{t_{j_0}\le\tau<t_{k+j+2}}\|z(\tau)\|\right]\le 0,\tag{24}$$

for some $t_{j_0}\in\left\{t_{j_i}\right\}_{i=0}^{\infty}$; $\forall t_k\in\left\{t_j\right\}_{j=0}^{\infty}$, $\forall t\in\mathbf{R}_{0+}$. Then, $(1-(K\tilde{a}/\rho))\sup_{t_{j_0}\le\tau<\infty}\|z(\tau)\|\le 0$ which implies that for any subsequence $\left\{t_{j_i}\right\}_{i=0}^{\infty}\subseteq\left\{t_j\right\}_{j=0}^{\infty}$ such that $\overline{z}=\limsup_{t_k\longrightarrow\infty}\sup_{t_k\le t<t_{k+1}}\|z(t)\|$; $\forall t_k\in\left\{t_{j_i}\right\}_{i=0}^{\infty}$ it holds that $\overline{z}=0$, a contradiction to the assumption of Case b. Then, Case b is not possible either. As a result, the proved global stability is also asymptotic at large. □

Some necessary, but not sufficient, conditions for global stability or global asymptotic stability can be got by simple direct inspection of the various parameterizations of the differential system (3), subject to (4), given in the particular Example 1. Such conditions are basically addressed by inspecting the (3, 3)-entry of the corresponding matrices $A(z(t),t)$ as discussed in the next result.

Proposition 1. *Consider the third-order Example 1 of the differential system (3) subject to (4). Then, the following properties hold:*

(i) If $a_{10}=0$, $b(t)=(a_{20}/g(x(t)))$ and $a(t)=(a_{30}/f(\dot{x}(t)))$ with $a_{20}>0$ and $a_{30}>0$, then the system is not globally asymptotically stable at large, but it can be globally stable at large. The property can also hold if $a_{10}=0$, $b(t)=(a_{20}/g(x(t)))$ and $a(t)=(a_{30}/f(\dot{x}(t)))$ do not hold for all time while $\|\tilde{A}(z(t),t)\|\le\tilde{a}$; $\forall(z(t),t)\in\mathbf{R}^n\times\mathbf{R}_{0+}$ (Assumption 3) with sufficiently small $\tilde{a}$.

(ii) If $A_b(t)$ satisfies Assumption 1 and its third row has no entry either being identically zero or unbounded or negative for all time, then the system (3), subject to (4), can be globally asymptotically stable only if

$$\lim_{t\longrightarrow\infty}\int_0^t a(\tau)f(\dot{x}(\tau))d\tau=-\infty.\tag{25}$$

(iii) If $A_c(t)$ satisfies Assumption 1 and its third row has no entry either being identically zero or negative for all time, and if $\|\tilde{A}_c(t)\|$ satisfies Assumption 3 with $|a_2(t)|$ being sufficiently small for all $t\in\mathbf{R}_{0+}$, then the system (3), subject to (4), can be globally asymptotically stable only if

$$\lim_{t\longrightarrow\infty}\int_0^t a(\tau)f(\dot{x}(\tau))d\tau=-\infty.\tag{26}$$

(iv) If the entries of the third row of A_d satisfy $a_{10}>0$ and, for each given initial conditions, $a_{20}>-b(0)g(x(0))$ and $a_{30}>-a(0)f(\dot{x}(0))$ while $\|\tilde{A}(z(t),t)\|$ satisfies either

$$a(t)=a_{10},\qquad b(t)=\frac{a_{20}+b(0)g(x(0))}{g(x(t))},\qquad a(t)=\frac{a_{30}+a(0)f(\dot{x}(0))}{f(\dot{x}(t))},\tag{27}$$

or if some of the above equalities fail but Assumption 3 holds, with $|a_2(t)|$ being sufficiently small for all $t\in\mathbf{R}_{0+}$, then the system (3), subject to (4), can be globally asymptotically stable.

Proof. Note that Assumption 1 and the first part of Assumption 2 on boundedness hold trivially since A_a is constant. On the other hand, since $\widetilde{A}_a(z(t),t)=0; \forall t \in \mathbf{R}_{0+}$, Assumption 3 holds trivially and the stability of the differential system reduces to the stability of the constant matrix A_a. Note that the characteristic equation of A_a is $p(s)=s^3+a_{20}s+a_{30}=0$, and since the coefficient of s^2 in the characteristic polynomial is zero, then the matrix A_a is not a stability matrix and then the differential system cannot be globally asymptotically stable but can be globally stable depending on the values of a_{20} and a_{30} since they are positive. Furthermore, since $\det e^{A_a t}=\det e^{t\,trA_a}=\det e^{-a_{30}t} \longrightarrow 0$ as $t \longrightarrow +\infty$, (at least) an eigenvalue $\Lambda_1(t)$ of the fundamental matrix of multiplicity $p_\alpha(<n) \in \mathbf{Z}_+$ vanishes exponentially with time as time tends to infinity (note that p_α cannot equalize n since then A_a would be a stability matrix). Thus, the remaining eigenvalues of $e^{A_a t}$ of spectrum $sp(e^{A_a t})=\{\Lambda_i(t); i \in \overline{n}\}$ fulfill that $\prod_{i=1}^{n-1}[\Lambda_i(t)]=o(t^{p_\alpha-1}e^{-\alpha t})$. If the given equalities fail but Assumption 3 holds, then the nonasymptotic property can still holds under extra sufficiency-type conditions from Theorem 2. Proposition 1 (i) has been proved. Propositions 1 (ii) and 1 (iii) follow in the same way provided that $\lim_{t\longrightarrow\infty}\int_0^t a(\tau)f(\dot{x}(\tau))d\tau=-\infty$ which guarantees that $\lim_{t\longrightarrow\infty}\det e^{\int_0^t A_b(\tau)d\tau}=0$, respectively, $\lim_{t\longrightarrow\infty}\det e^{\int_0^t A_c(\tau)d\tau}=0$ which is a necessary condition for global asymptotic stability. Proposition 1 (iv) follows from close arguments as those invoked for proving Proposition 1 (i) if A_d is a stability matrix (note that this is possible depending on its eigenvalues while contrarily A_a can never be a stability matrix) and $\widetilde{A}_d(z(t),t)$ is either zero or with sufficiently norm for all time.

Example 3. Assume that the system of Example 1 with $e(t)\equiv 0$ is parameterized with the parameterization of (a). The constant matrix A_a is a stability matrix if its characteristic equation $s^3+a_{10}s^2+a_{20}s+a_{30}=0$ has all its roots in the open complex left-hand-side. This holds from the Routh-Hurwitz criterion if and only if $a_{10}>0$, $a_{20}>0$, and $0<a_{30}<a_{10}a_{20}$. If the parameterization of (d) is used, then the modified matrix of dynamics A_d is a stability matrix if and only if

$$\begin{gathered} a_{10}>0, \\ a_{20}+b(0)g(x(0))>0, \\ a_{10}(a_{20}+b(0)g(x(0)))>a(0)f(\dot{x}(0)). \end{gathered} \tag{28}$$

(i) The parameterization of (a) is useful to guarantee the global asymptotic stability of the differential system (3), subject to (4), that is if A_a is stable and, furthermore, $\|\widetilde{A}_a(t)\|$ is sufficiently small for all time in the sense that if $(-\rho_{Ma})$ is the stability abscissa of A_a, i.e., the absolute value of its (stable) eigenvalue being closer to the complex imaginary axis satisfying:

$$\begin{aligned} |\widetilde{a}_1(t)|^2+|\widetilde{a}_2(t)|^2+|\widetilde{a}_3(t)|^2 &= |a_{10}-a(t)|^2 \\ &\quad + |a_{20}-b(t)g(x(t))|^2 \\ &\quad + |a_{30}-a(t)f(\dot{x}(t))|^2 \\ &< \rho_{Ma}^2. \end{aligned} \tag{29}$$

(ii) The parameterization of (d) is useful if A_d is a stability matrix of stability abscissa $(-\rho_{Md})$, and

$$\begin{aligned} &|a_{10}-a(t)|^2+|a_{20}+b(0)g(x(0))-b(t)g(x(t))|^2 \\ &\quad + |a_{30}+a(0)f(\dot{x}(0))-a(t)f(\dot{x}(t))|^2<\rho_{Md}^2. \end{aligned} \tag{30}$$

Remark 2. Theorem 2 proves that $\sup_{0\le t<\infty}\|z(t)\| \le \rho K\|z_0\|/\rho-K\widetilde{a}$ under Assumption 3 which requests the uniform boundedness of $\|A(z(t),t)\|$ and $\|\widetilde{A}(z(t),t)\|$ for any pair $(z(t),t)\in \mathbf{R}^n\times\mathbf{R}_{0+}$ such that $z:\mathbf{R}_{0+}\longrightarrow\mathbf{R}^n$ is a solution of (3). The theorem concludes the global asymptotic stability at large of (3). This suggests that the global asymptotic stability in a closed ball (rather than in the large) can be formulated for a certain closed ball of $\mathbf{R}^n$ containing the solution trajectory of (3) for any initial conditions in a given closed ball without invoking Assumption 3. Thus, define the closed balls $\mathbf{B}_0(0,r_0)=\{z_0\in\mathbf{R}^n : \|z_0\|\le r_0\}$ and

$$\mathbf{B}\left(0,\frac{\rho K r_0}{\rho-K\widetilde{a}}\right)=\left\{z\in\mathbf{R}^n:\left(\|z\|\le\frac{\rho K r_0}{\rho-K\widetilde{a}}\wedge z_0\in\mathbf{B}_0(0,r_0)\right)\right\}. \tag{31}$$

Thus, Theorem 2 has the following useful corollary which does need "a priori" boundedness conditions on the norm of the supremum of the solution as invoked in Assumption 3.

Corollary 1. *If Assumptions 1 and 2 hold, then the n-th differential system (3), subject to (4), has a uniformly bounded solution for any initial conditions in* $\mathbf{B}_0(0,r_0)=\{z_0\in\mathbf{R}^n : \|z_0\|\le r_0\}$ *and it is globally asymptotically stable with the trajectory solution contained in*

$$\mathbf{B}\left(0,\frac{\rho K r_0}{\rho-K\widetilde{a}}\right)=\left\{z\in\mathbf{R}^n:\left(\|z\|\le\frac{\rho K r_0}{\rho-K\widetilde{a}}\wedge z_0\in\mathbf{B}_0(0,r_0)\right)\right\}, \tag{32}$$

if $\widetilde{a}$ *is small enough such that* $\widetilde{a}<(\rho/K)$ *(see Theorem 2).*□

The subsequent result is similar to Theorem 2 without assuming the boundedness of $\widetilde{A}(z(t),t)$ which can be expanded in series for all values of the trajectory solution at any time according to a small time-varying parameter. Also, it can grow unboundedly with time if $\sup_{0\le\tau\le t}\|z(\tau)\|$ is strictly upper-bounded by the inverse of a small time-varying function which can vanish asymptotically.

The bounded closed domain which contains the state trajectory solution for all time may be defined depending on the relevant parameters of the differential system.

Example 4. Let the initial conditions of (3), subject to (4), satisfy $\|z_0\| \le r_0$. Then, sufficient conditions for $\|z(t)\| \le r < +\infty$; $\forall t \in \mathbf{R}_{0+}$ for some $r = \mu r_0 > r_0$ can be got from Theorem 2, equation (18), as follows:

$$\|z(t)\| \le K\left(e^{-\rho t} + \frac{1}{\rho}\sup_{0\le t<+\infty}\|\widetilde{A}(t)\|\mu\right) r_0 \le \mu r_0 = r, \quad \forall t \in \mathbf{R}_{0+},$$
$$\mu = \frac{r}{r_0} \ge K\left(1 + \frac{1}{\rho}\sup_{0\le t<+\infty}\|\widetilde{A}(t)\|\mu\right) \ge K\left(e^{-\rho t} + \frac{1}{\rho}\sup_{0\le t<+\infty}\|\widetilde{A}(t)\|\mu\right), \quad \forall t \in \mathbf{R}_{0+}. \tag{33}$$

So, $r \ge K(1 + (1/\rho)\sup_{0\le t<+\infty}\|\widetilde{A}(t)\|\mu)r_0 = Kr_0 + (K/\rho)\sup_{0\le t<+\infty}\|\widetilde{A}(t)\|\ r$, or $r \ge K\rho r_0/(\rho - K\sup_{0\le t<+\infty}\|\widetilde{A}(t)\|)$ if $\sup_{0\le t<+\infty}\|\widetilde{A}(t)\| < \rho/K$. Note that $\mu \ge K\rho/(\rho - K\sup_{0\le t<+\infty}\|\widetilde{A}(t)\|) > K \ge 1$ as a result.

Now, assume that r_0 is given as the radius of the closed ball around zero which fixes the domain for initial conditions and r is prefixed as the radius of the suitable closed ball which guarantees that the solution remains within it for all time. Thus, $\rho(r - Kr_0) \ge rK\sup_{0\le\tau<+\infty}\|\widetilde{A}(t)\|$ so that the constraint $\sup_{0\le\tau<+\infty}\|\widetilde{A}(t)\| \le \rho(\mu - K)r_0/rK$ guarantees the respective radii r_0 and $r = \mu r_0$ for a given $\rho > 0$ provided that $\sup_{0\le t<+\infty}\|\widetilde{A}(t)\| < \rho/K$. The combination of both norm constraints leads that the first one is stronger since $\sup_{0\le\tau<+\infty}\|\widetilde{A}(t)\| \le (\rho/K)\min(1, (\mu - K/\mu)) = (\rho/K)(1 - (K/\mu))$.

Theorem 3. *Let Assumptions 1–2 hold and assume also that*

(1) *The pair of real constants (ρ, K) defined in the proof of Theorem 2 satisfies the constraint $\rho > (K/1 - \overline{\varepsilon}_0)$*

(2) *There exists a function $\varepsilon : [0, t] \longrightarrow \mathbf{R}_+$; $\forall t \in \mathbf{R}_{0+}$ such that $\varepsilon_0(t) = \varepsilon(t)\sup_{0\le\tau\le t}\|z(\tau)\| \le \overline{\varepsilon}_0 < 1$ (note that $\varepsilon(t)$ is allowed to be asymptotically vanishing) and such that $\|\widetilde{A}(z(t), t)\| \le \sum_{i=0}^{\infty}\varepsilon^i(t)\sup^i_{0\le\tau\le t}\|z(\tau)\|$*

Then, the n-th differential system (3), subject to (4), has a uniformly bounded solution for any finite initial conditions, and it is globally asymptotically stable at large (i.e., in $\mathbf{R}^n$).

Proof. Note that $\varepsilon(t) = (\varepsilon_0(t)/\sup_{0\le\tau<\infty}\|z(\tau)\|)$; $\forall t \in \mathbf{R}_{0+}$ for some $\varepsilon_0(t) \in [0, \overline{\varepsilon}_0)$ and some $0 \le \overline{\varepsilon}_0 \le 1$; $\forall t \in \mathbf{R}_{0+}$ implies that

$$\|\widetilde{A}(z(t), t)\| \le \sum_{i=0}^{\infty}\varepsilon^i(t)\sup_{0\le\tau\le t}^{i}\|z(\tau)\| = \sum_{i=0}^{\infty}\varepsilon_0^i(t) = \frac{1}{1-\varepsilon_0} = \frac{1}{1 - \varepsilon(t)\sup_{0\le\tau\le t}\|z(\tau)\|}; \quad \forall t \in \mathbf{R}_{0+}, \tag{34}$$

with

$$\sup_{0\le\tau\le t}\|z(\tau)\| \le \frac{\overline{\varepsilon}_0}{\varepsilon(t)} < \frac{1}{\varepsilon(t)}. \tag{35}$$

Equation (20) is modified as follows:

$$\sup_{0\le\tau\le t}\|z(\tau)\| = \|z(t_t)\| \le Ke^{-\rho t_t}\|z_0\| + K\frac{1 - e^{-\rho t_t}}{\rho}\frac{1}{1 - \varepsilon(t)\sup_{0\le\tau<\infty}\|z(\tau)\|}\sup_{0\le\tau\le t}\|z(\tau)\|, \quad \forall t \in \mathbf{R}_{0+}, \tag{36}$$

leading to

$$\begin{aligned}(1 - \overline{\varepsilon}_0)\sup_{0\le\tau\le t}\|z(\tau)\| &\le \left(1 - \varepsilon(t)\sup_{0\le\tau<\infty}\|z(\tau)\|\right)\sup_{0\le\tau\le t}\|z(\tau)\| \\ &= \left(1 - \varepsilon(t)\sup_{0\le\tau<\infty}\|z(\tau)\|\right)\|z(t_t)\| \\ &\le K\left(1 - \varepsilon(t)\sup_{0\le\tau<\infty}\|z(\tau)\|\right)e^{-\rho t_t}\|z_0\| + K\frac{1 - e^{-\rho t_t}}{\rho}\sup_{0\le\tau\le t}\|z(\tau)\|, \quad \forall t \in \mathbf{R}_{0+},\end{aligned} \tag{37}$$

or

$$\begin{aligned}\left(1 - \overline{\varepsilon}_0 - \frac{K}{\rho}\right)\sup_{0\le\tau\le t}\|z(\tau)\| &\le \left(1 - \overline{\varepsilon}_0 - K\frac{1 - e^{-\rho t_t}}{\rho}\right)\sup_{0\le\tau\le t}\|z(\tau)\| \\ &\le K\left(1 - \varepsilon(t)\sup_{0\le\tau<\infty}\|z(\tau)\|\right)e^{-\rho t_t}\|z_0\|, \quad \forall t \in \mathbf{R}_{0+},\end{aligned} \tag{38}$$

so that

$$\left(1-\bar{\varepsilon}_0-\frac{K}{\rho}\right)\sup_{0\le\tau\le t}\|z(\tau)\| \le \frac{\rho K}{\rho(1-\bar{\varepsilon}_0)-K} \cdot \left(1-\varepsilon(t)\sup_{0\le\tau<\infty}\|z(\tau)\|\right)e^{-\rho t_t}\|z_0\|, \quad \forall t\in\mathbf{R}_{0+}, \tag{39}$$

and since $\rho > K/(1-\bar{\varepsilon}_0) \ge K/(1-\bar{\varepsilon}_0+K\varepsilon(t)e^{-\rho t_t}\|z_0\|)$ and $\sup_{0\le\tau\le t}\|z(\tau)\| = (\varepsilon_0(t)/\varepsilon(t)) \le (\bar{\varepsilon}_0/\varepsilon(t)) < (1/\varepsilon(t))$; $\forall t\in\mathbf{R}_{0+}$, one has

$$\sup_{0\le\tau\le t}\|z(\tau)\| \le \frac{\rho K}{\rho(1-\bar{\varepsilon}_0)-K} \cdot \left(1-\varepsilon(t)\sup_{0\le\tau\le t}\|z(\tau)\|\right)e^{-\rho t_t}\|z_0\|, \quad \forall t\in\mathbf{R}_{0+}, \tag{40}$$

$$\sup_{0\le\tau\le t}\|z(\tau)\| \le \frac{\rho K e^{-\rho t_t}\|z_0\|}{\rho\left(1-\bar{\varepsilon}_0+K\varepsilon(t)e^{-\rho t_t}\|z_0\|\right)-K} < +\infty, \quad \forall t\in\mathbf{R}_{0+}, \tag{41}$$

which proves the global uniform stability at large. The asymptotic stability at large is proved by contradiction arguments by the construction of the appropriate solution sequences as in Theorem 2.

Corollary 2. *Theorem 2 still holds if* $\|\widetilde{A}(z(t),t)\| \le \sum_{i=0}^{k}\varepsilon^i(t)\sup_{0\le\tau\le t}{}^i\|z(\tau)\|$; $\forall t\in\mathbf{R}_{0+}$ *for some given positive integer k with* $\varepsilon_0(t) = \varepsilon(t)\sup_{0\le\tau\le t}\|z(\tau)\| \le \bar{\varepsilon}_0 < 1$; $\forall t\in\mathbf{R}_{0+}$.

Proof. In this case, one has

$$\|\widetilde{A}(z(t),t)\| \le \sum_{i=0}^{k}\varepsilon^i(t)\sup_{0\le\tau\le t}^{i}\|z(\tau)\| = \frac{1-\varepsilon^k(t)\sup_{0\le\tau\le t}\|z(\tau)\|}{1-\varepsilon(t)\sup_{0\le\tau\le t}\|z(\tau)\|} \le \frac{1}{1-\varepsilon(t)\sup_{0\le\tau\le t}\|z(\tau)\|}; \quad \forall t\in\mathbf{R}_{0+}. \tag{42}$$

The proof follows by using the above constraint. □

The subsequent result does not invoke Assumption 3. Instead, an upper-bound maximum growing time-interval condition on the time integral of the norm of $\widetilde{A}(z(t),t)$ is used to address sufficiency-type conditions for the global stability and asymptotic stability of (3), subject to (4).

Theorem 4. *If Assumptions 1 and 2 hold and, furthermore,*

$$\int_t^{t+T}\|\widetilde{A}(z(\tau),\tau)\|^4 d\tau \le c_1 T + c_2, \quad \forall t\in\mathbf{R}_{0+}, \forall T\in\mathbf{R}_+, \tag{43}$$

for some sufficiently small $c_1, c_2 \in \mathbf{R}_{0+}$, *then the differential system (3), subject to (4), is globally asymptotically Lyapunov's stable at large.*

Proof. One gets from (18) in the proof of Theorem 1 and Hölder's inequality that

$$\begin{aligned}
\|z(t+T)\| &\le Ke^{-\rho T}\|z(t)\| + \int_t^{t+T} Ke^{-\rho(t+T-\tau)}\|\widetilde{A}(z(\tau),\tau)\|\|z(\tau)\|d\tau \\
&\le Ke^{-\rho T}\|z(t)\| + K\left(\int_t^{t+T} e^{-2\rho(t+T-\tau)}d\tau\right)^{1/2}\left(\int_t^{t+T}(\|\widetilde{A}(z(\tau),\tau)\|\|z(\tau)\|)^2 d\tau\right)^{1/2} \\
&\le Ke^{-\rho T}\|z(t)\| + K\left(\int_t^{t+T} e^{-2\rho(t-\tau)}d\tau\right)^{1/2}\left(\int_t^{t+T}\|\widetilde{A}(z(\tau),\tau)\|^4 d\tau\right)^{1/4}\left(\int_t^{t+T}\|z(\tau)\|^4 d\tau\right)^{1/4} \\
&\le Ke^{-\rho T}\|z(t)\| + K\sqrt{\frac{1-e^{-2\rho T}}{2\rho}}\left(\int_t^{t+T}\|\widetilde{A}(z(\tau),\tau)\|^4 d\tau\right)^{1/4}\left(\int_t^{t+T}\|z(\tau)\|^4 d\tau\right)^{1/4} \\
&\le Ke^{-\rho T}\|z(t)\| + K\sqrt{\frac{1-e^{-2\rho T}}{2\rho}}(c_1T+c_2)^{1/4}\sup_{t\le\tau\le t+T}\|z(\tau)\|T^{1/4} \\
&\le Ke^{-\rho T}\|z(t)\| + K\sqrt{\frac{1-e^{-2\rho T}}{2\rho}}(c_1^{1/4}T^{1/4}+c_2^{1/4})T^{1/4}\sup_{t\le\tau\le t+T}\|z(\tau)\|, \quad \forall t\in\mathbf{R}_{0+},
\end{aligned} \tag{44}$$

and for some $t_t \in [t, t+T]$ defined as $t_t = (\max \tau \in [t, t+T] : \|z(\tau)\| = \sup_{0\le\sigma\le T}\|z(t+\sigma)\|)$; $\forall t \in \mathbf{R}_{0+}$,

$$\sup_{t\le\tau\le t+T} \|z(\tau)\| = \|z(t_t)\| \le Ke^{-\rho(t_t-t)}\|z(t)\| + K\sqrt{\frac{1-e^{-2\rho T}}{2\rho}}\left(c_1^{1/4}T^{1/4} + c_2^{1/4}\right)T^{1/4} \sup_{t\le\tau\le t+T} \|z(\tau)\|, \quad \forall t \in \mathbf{R}_{0+}. \tag{45}$$

Since c_1 and c_2 are small enough such that $1 > K\sqrt{(1-e^{-2\rho T}/2\rho)}(c_1^{1/4}T^{1/4} + c_2^{1/4})T^{1/4}$, one has

$$\sup_{t\le\tau\le t+T} \|z(\tau)\| \le \frac{Ke^{-\rho(t_t-t)}\|z(t)\|}{1-K\sqrt{1-e^{-2\rho T}/2\rho}\left(c_1^{1/4}T^{1/4}+c_2^{1/4}\right)T^{1/4}}, \quad \forall t \in \mathbf{R}_{0+}. \tag{46}$$

Assume that $\lim_{t\to\infty}\|z(t)\| = +\infty$. Since T is arbitrary in (43), there are nonunique strictly increasing real sequences $\{kT\}_0^\infty \subset \mathbf{R}_{0+}$ such that $\|z((k+j_k)T)\| > \|z(kT)\|$; for some sufficiently large finite $j_k = j_k(T) \in \mathbf{Z}_+$, $\forall k \in \mathbf{Z}_{0+}$. Then, for any given finite $T > 0$ and $k \in \mathbf{Z}_{0+}$, sufficiently large values of $j_k \in \mathbf{Z}_{0+}$ and sufficiently small related constants $c_1 = c_1(T), c_2(T)$, one has

$$\begin{aligned}\|z(kT)\| &< \left\|z\left(\left(k+j_j\right)T\right)\right\| \\ &\le \frac{Ke^{-\rho j_k T}\|z(kT)\|}{1-K\sqrt{1-e^{-2\rho j_k T}/2\rho}\left(c_1^{1/4}\left(j_kT\right)^{1/4}+c_2^{1/4}\right)\left(j_kT\right)^{1/4}} \\ &< \|z(kT)\|, \quad \forall t \in \mathbf{R}_{0+},\end{aligned} \tag{47}$$

because of the dominance of the second right-hand-side numerator related to the denominator for sufficiently large j_k and sufficiently small c_1, c_2 since T is finite. Hence, a contradiction for some sufficiently large finite $T > 0$. Thus, the differential system (3), subject to (4), is globally stable at large as a result. On the other hand, consider three particular cases concerned with the differential system (3), subject to (4).

Case a. It is globally stable at large for sufficiently small constants c_1, c_2. Thus, no further proof is needed.

Case b. Its solution is unbounded for sufficiently small constants c_1 and c_2. In this case, it is unstable what contradicts its already proved global Lyapunov's stability at large. So, this case is impossible.

Case c. Its solution is bounded but oscillatory. Thus, there is a time interval $[0, t_a]$, of finite or zero measure, with $t_a = t_a(z_0) \ge 0$ such that $\sup_{0\le t\le t_a}\|z(t)\| = \sup_{0\le t<+\infty}\|z(t)\|$ so that there is a subsequence of time instants $\{t_{k_j}\}_0^\infty \subset \mathbf{R}_{0+}$ defined by $t_{k_j} = t_a + k_jT + \tau_{k_j}$, with $k_j \in Z_{0+}$ and $\tau_{k_j} \in [0, T)$; $\forall j \in \mathbf{Z}_{0+}$, such that $\|z(t_{k_j})\| = \sup_{0\le t\le t_a}\|z(t)\|$ so that one gets from the first inequality of (44) that

$$\begin{aligned}\sup_{0\le t\le t_a} \|z(t)\| &= \left\|z\left(t_{k_j}\right)\right\| \le Ke^{-\rho\left(k_jT+\tau_{k_j}\right)}\|z(t_a)\| + \sup_{0\le t\le t_a}\|z(t)\| \\ &\quad\cdot \left(\int_0^{k_jT+\tau_{k_j}} Ke^{-\rho\left(k_jT+\tau_{k_j}-\tau\right)}\left\|\widetilde{A}\left(z(\tau+t_a), \tau+t_a\right)\right\| d\tau\right) \\ &\le Ke^{-\rho\left(k_jT+\tau_{k_j}\right)}\|z(t_a)\| + \frac{K}{\rho}\left(1-e^{-\rho\left(k_jT+\tau_{k_j}\right)}\right) \\ &\quad\cdot \left(c_1\left(k_jT+\tau_{k_j}\right)+c_2\right)\sup_{0\le t\le t_a}\|z(t)\|,\end{aligned} \tag{48}$$

and then one gets the following contradiction for some sufficiently large finite $T > 0$:

$$\begin{aligned}\sup_{0\le t\le t_a} \|z(t)\| &\le \frac{Ke^{-\rho\left(k_jT+\tau_{k_j}\right)}\|z(t_a)\|}{1-(K/\rho)\left(1-e^{-\rho\left(k_jT+\tau_{k_j}\right)}\right)\left(c_1\left(k_jT+\tau_{k_j}\right)+c_2\right)} \\ &< \|z(t_a)\|, \quad \forall t \in \mathbf{R}_{0+},\end{aligned} \tag{49}$$

by defining the subsequences $\{k_j\}_{j=0}^\infty \subset \mathbf{Z}_{0+}$, $\{\varepsilon_{k_j}\}_{j=0}^\infty \subset \mathbf{R}_+$ and $\{m_{k_j}\}_{j=0}^\infty \subset \mathbf{R}_+$ with $\varepsilon_{k_j} = e^{-\rho(k_jT+\tau_{k_j})}$ (note that $\{\varepsilon_{k_j}\} \to 0$) such that

$$\begin{aligned}&0 < \varepsilon_{k_j} < \min\left(1, \frac{1}{K\|z(t_a)\|}\right), \\ &k_j = -\frac{1}{T}\left(\ln\frac{\varepsilon_{k_j}}{\rho} + \tau_{k_j}\right) \in \mathbf{Z}_+, \\ &m_{k_j} = c_1k_jT + c_2 < \left(\frac{1}{K} - \varepsilon_{k_j}\|z(t_a)\|\right)\frac{\rho}{1-\varepsilon_{k_j}},\end{aligned} \tag{50}$$

and note also that it suffices to see that the contradiction holds for the finite first element $k_0 \in \{k_j\}_{j=0}^\infty$ and sufficiently small c_1, c_2 related to ρ/K. Thus, the solution to (3), subject to (4), for any finite initial conditions cannot be bounded and oscillatory. Thus, one concludes that the differential system is globally asymptotically Lyapunov's stable at large if c_1 and c_2 are sufficiently small.

The particular 4-th power in $\|\widetilde{A}(z(\tau), \tau)\|$ in the integral (43) assumed in Theorem 4 is not crucial to the proof except for the "amount of smallness" needed for the constants c_1 and c_2 to guarantee the theorem. In this context, note the subsequent result.

Corollary 3. *Assume that the inequality (43) in Theorem 4 is replaced with*

$$\int_t^{t+T} \left\| \widetilde{A}(z(\tau),\tau) \right\|^2 d\tau \le d_1 T + d_2, \quad \forall t \in \mathbf{R}_{0+},\ \forall T \in \mathbf{R}, \tag{51}$$

for some sufficiently small $d_1, d_2 \in \mathbf{R}_{0+}$, or with

$$\int_t^{t+T} \left\| \widetilde{A}(z(\tau),\tau) \right\| d\tau \le f_1 T + f_2, \quad \forall t \in \mathbf{R}_{0+},\ \forall T \in \mathbf{R}_+, \tag{52}$$

for some sufficiently small $f_1, f_2 \in \mathbf{R}_{0+}$. Then, Theorem 4 still holds under the same given remaining assumptions.

Outline of Proof. The relevant inequalities of (44) are modified as follows if (43) is replaced with (51):

$$\begin{aligned} \|z(t+T)\| &\le K e^{-\rho T}\|z(t)\| + \int_t^{t+T} K e^{-\rho(t+T-\tau)} \left\|\widetilde{A}(z(\tau),\tau)\right\| \|z(\tau)\| d\tau \le K e^{-\rho T}\|z(t)\| + K\left(\int_t^{t+T} \left\|\widetilde{A}(z(\tau),\tau)\right\|^2 d\tau\right)^{1/2} \left(\int_t^{t+T} \|z(\tau)\|^2 d\tau\right)^{1/2} \\ &\le K e^{-\rho T}\|z(t)\| + K\left(d_1 T + d_2\right)^{1/2} \sup_{t\le\tau\le t+T} \|z(\tau)\| T^{1/2} \le K e^{-\rho T}\|z(t)\| + K\left(\sqrt{d_1}\,T + \sqrt{d_2 T}\right) \sup_{t\le\tau\le t+T} \|z(\tau)\|, \quad \forall t \in \mathbf{R}_{0+}, \end{aligned} \tag{53}$$

and the rest of the proof follows "mutatis-mutandis" to that of Theorem 4. In the same way, if (43) is replaced with (52), then the modified set of inequalities holds:

$$\begin{aligned} \|z(t+T)\| &\le K e^{-\rho T}\|z(t)\| + K\left(\int_t^{t+T} \|\widetilde{A}(z(\tau),\tau)\|^2 d\tau\right)^{1/2} \left(\int_t^{t+T} \|z(\tau)\|^2 d\tau\right)^{1/2} \\ &\le K e^{-\rho T}\|z(t)\| + K\left(f_1 T + f_2\right) \sup_{t\le\tau\le t+T} \|z(\tau)\| T^{1/2} \le K e^{-\rho T}\|z(t)\| + K\left(f_1 T^{3/2} + f_2\sqrt{T}\right) \sup_{t\le\tau\le t+T} \|z(\tau)\|; \quad \forall t \in \mathbf{R}_{0+}, \end{aligned} \tag{54}$$

and the stability result is again proved. □

Remark 3. Note that the global stability of the various proved results guarantees that the solution of the differential system and all its time-derivatives up till n-th order are bounded for all time for any given finite initial conditions. On the other hand, the global asymptotic stability guarantees in addition the asymptotic convergence of the above functions to zero.

Theorem 5. *Assume the following particular differential system (3) and (4) which is also forced with an external function*

$$\dot{z}(t) = B(t)z(t) + \omega(t),\ z(0) = z_0, \quad \forall t \in \mathbf{R}_{0+}, \tag{55}$$

where the matrix function $B : \times\mathbf{R}_{0+} \longrightarrow \mathbf{R}^{n\times n}$ has piecewise continuous entries and it is nonuniquely decomposed as $B(t) = A(t) + \widetilde{A}(t)$; $\forall t \in \mathbf{R}_{0+}$, and assume also that the amended Assumptions 1–3 hold referred just to time, instead to the pair $(z(t), t)$, in view of (55). Then, the following properties hold:

(i) *ie differential system is globally stable if the unforced system is globally asymptotically stable and $\omega \in L_\infty^n$ or $\omega \in L_p^n$ for any $p \in \mathbf{Z}_+$ if $\tilde{a}$ is small enough.*

(ii) *Assume, in addition, that ω is of exponential order $(-\rho_\omega) < 0$ with ρ_ω being larger than the average stability abscissa of $A(t)$. Then, the differential system is globally asymptotically stable at large.*

Proof. In this case, (12) is replaced as follows:

$$z(t) = z_{\text{unf}}(t) + z_f(t), \quad \forall t \in \mathbf{R}_{0+}, \tag{56}$$

where the unforced and forced solutions are

$$\begin{aligned} z_{\text{unf}}(t) &= e^{\int_0^t A(\tau)d\tau} z_0 + \int_0^t e^{\int_\tau^t A(\sigma)d\sigma} \widetilde{A}(\tau) z(\tau) d\tau, \\ z_f(t) &= \int_0^t e^{\int_\tau^t A(\sigma)d\sigma} \omega(\tau) d\tau, \quad \forall t \in \mathbf{R}_{0+}. \end{aligned} \tag{57}$$

Note that closely to the arguments used in the proof of Theorem 2 that $\left\|e^{\int_0^t A(\tau)d\tau}\right\| \le K e^{-\int_0^t \rho_A(\tau)d\tau} \le K e^{-\rho_0(z_0,t)t} \le K e^{-\rho t}$; $\forall t \in \mathbf{R}_{0+}$, where $K = \sup_{t\in\mathbf{R}_{0+}} \sup_{z_0\in\mathbf{R}^n} K_A(z_0, t)$; $0 < \rho \le \inf_{\tau, t(>\tau)\in\mathbf{R}_{0+}} \left(\int_\tau^t \rho_A(\tau)d\tau\right)/(t-\tau)$. Thus, the following additive terms have to be added for the obtained unforced upper-bounds of the unforced $\sup_{0\le\tau\le t}\|z_{\text{unf}}(\tau)\|$; $\forall t \in \mathbf{R}_{0+}$ in the various former given results applied to this particular differential system:

(a) $K\int_0^t e^{-\rho(t-\tau)}\|\omega(\tau)\|d\tau \le K\sup_{0\le\tau<\infty} \|\omega(\tau)\|(1/\rho) = (KM_\infty/\rho)$; $\forall t \in \mathbf{R}_{0+}$ if $\omega \in L_\infty^n$ and $\|\omega\|_\infty = M_\infty$. Then, $\sup_{0\le\tau\le t}\|z(\tau)\| \le \sup_{0\le\tau\le t}\|z_{\text{unf}}(\tau)\| + (KM_\infty/\rho)$.

(b) $K\int_0^t e^{-\rho(t-\tau)}\|\omega(\tau)\|d\tau \le K\int_0^t \|\omega(\tau)\|d\tau = KM_1$; $\forall t \in \mathbf{R}_{0+}$ if $\omega \in L_1^n$ and $\|\omega\|_1 = M_1$. Then, $\sup_{0\le\tau\le t}\|z(\tau)\| \le \sup_{0\le\tau\le t}\|z_{\text{unf}}(\tau)\| + KM_1$.

(c) $K\int_0^t e^{-\rho(t-\tau)}\|\omega(\tau)\|d\tau \le K(\int_0^t e^{-\rho p(t-\tau)})^{1/p}(\int_0^t \|\omega(\tau)\|^p)^{1/p} d\tau = (K/(p\rho)^{1/p})M_p$; $\forall t \in \mathbf{R}_{0+}$ if $\omega \in L_1^n$ and $\|\omega\|_1 = M_1$. Then, $\sup_{0\le\tau\le t}\|z(\tau)\| \le \sup_{0\le\tau\le t}\|z_{\text{unf}}(\tau)\| + KM_1$; if $\omega \in L_p^n$ and $\|\omega\|_p = M_p$ for any $p \in \mathbf{Z}_+$. Then, $\sup_{0\le\tau\le t}\|z(\tau)\| \le \sup_{0\le\tau\le t}\|z_{\text{unf}}(\tau)\| + (K/(p\rho)^{1/p})M_p$; $\forall t \in \mathbf{R}_{0+}$. Note that since $\rho > K \ge 1$ for the results on asymptotic stability, the upper-bound $(K/\rho)M_1$ of (c) might improve, in general, that of (b), i.e., KM_1 if $\omega \in L_1^n$. One concludes that, under any of the conditions obtained for the unforced differential system to be globally asymptotically stable, the forced one remains globally stable with the corresponding above new given norm upper-bounds for all time. Property (i) has been proved. Property (ii) follows since if ω is of negative exponential order $(-\rho_\omega)$ with $\rho_\omega > \rho$ then $z_f(t) \longrightarrow 0$ exponentially as $t \longrightarrow \infty$ since

$$\begin{aligned}\left|z_f(t)\right| &= \left|\int_0^t e^{\int_\tau^t A(\sigma)d\sigma}\omega(\tau)d\tau\right| \le K\int_0^t e^{-\int_\tau^t \rho_A(\sigma)d\sigma}K_1 e^{-\rho_\omega\tau}d\tau \\ &\le KK_1e^{-\rho t}\int_0^t e^{(\rho-\rho_\omega)\tau} \le KK_1e^{-\rho t}\frac{e^{(\rho-\rho_\omega)t}-1}{\left|\rho-\rho_\omega\right|}, \\ &\qquad \forall t \in \mathbf{R}_{0+}.\end{aligned} \tag{58}$$

Remark 4. The upper-bounds of Theorem 5 for the norm of the whole solution of (55) may be improved if the forcing function belongs jointly to several normed function spaces on $[0,\infty]$ by using the minimum of the corresponding finite norms. For instance, one has

$$\sup_{0\le\tau\le t}\|z(\tau)\| \le \sup_{0\le\tau\le t}\left\|z_{\text{unf}}(\tau)\right\| + M_{\omega 1}, \quad \forall t \in \mathbf{R}_{0+}, \tag{59}$$

where $M_{\omega 1} = (K/\sqrt{\rho})\min((M_\infty/\sqrt{\rho}), (M_1/\sqrt{\rho}))$ if $\omega \in L_\infty^n \cap L_1^n$. Since $L_\infty^n \cap L_1^n \supset L_2^n$, then $M_{\omega 1}$ may be replaced with $M_{\omega 2} = (K/\sqrt{\rho})\min((M_\infty/\sqrt{\rho}), (M_1/\sqrt{\rho}), (M_2/\sqrt{2}))$ in (59) since $\omega \in L_\infty^n \cap L_1^n \cap L_2^n$.

The so-called property of global ultimate boundedness guarantees also the global Lyapunov's (nonasymptotic) stability property since the smooth solution of (3) and (4) has to be necessarily continuous for all time so that it cannot have finite escape times (i.e., right or left finite-time discontinuity points to $\pm\infty$), as a result. An explicit related result follows.

Theorem 6. *Under Assumptions 1–2, the following properties hold for the differential system (3), subject to (4):*

(i) Assume that $\limsup_{t\longrightarrow\infty}(\|\widetilde{A}(z(t),t)z(t)\|) \le M_\infty$ for some finite real constant M_∞ and any given finite initial conditions. Then, the system is globally stable at large with ultimate boundedness satisfying the asymptotic finiteness norm constraint $\limsup_{t\longrightarrow\infty}\|z(t)\| \le (K/\rho)M_\infty$ for any given finite initial conditions and ρ being defined in Theorem 2.

(ii) Assume that $\limsup_{t\longrightarrow\infty}(\int_0^t \|\widetilde{A}(z(\tau),\tau)z(\tau)\|^p d\tau) \le M_p$ for some real $p > 0$ and finite real constant M_p and any given finite initial conditions. Then, the system is globally stable at large with ultimate boundedness satisfying the asymptotic finiteness norm constraint $\limsup_{t\longrightarrow\infty}\|z(t)\| \le K(M_p/\rho p)^{1/p}$ for any given finite initial conditions with ρ being defined in Theorem 2.

Proof. Since the unforced part of the solution converges asymptotically to zero, Property (i) follows since

$$\begin{aligned}\limsup_{t\longrightarrow\infty}\|z(t)\| &\le \limsup_{t\longrightarrow\infty}\left(\int_0^t Ke^{-\rho(t-\tau)}(\|\widetilde{A}(z(\tau),\tau)z(\tau)\|d\tau)\right) \\ &\le \frac{K}{\rho}M_\infty,\end{aligned} \tag{60}$$

and Property (ii) follows since $\limsup_{t\longrightarrow\infty}(\int_0^t e^{-\rho(t-\tau)}d\tau)^p \le (1/\rho p)$ so that

$$\begin{aligned}\limsup_{t\longrightarrow\infty}\|z(t)\| &\le K\limsup_{t\longrightarrow\infty}\left(\int_0^t e^{-\rho(t-\tau)}\|\widetilde{A}(z(\tau),\tau)z(\tau)\|d\tau\right) \\ &\le K\left(\limsup_{t\longrightarrow\infty}\left(\int_0^t e^{-\rho(t-\tau)}d\tau\right)^p\right)^{1/p} \\ &\quad\times \limsup_{t\longrightarrow\infty}\left(\int_0^t \|\widetilde{A}(z(\tau),\tau)z(\tau)\|d\tau\right)^{1/p} \\ &\le K\left(\frac{M_p}{\rho p}\right)^{1/p}.\end{aligned} \tag{61}$$

Example 5. Assume that, for some finite and positive real constants z_m, θ, and $q \ge 1$ and some bounded function $\widetilde{K} : \mathbf{R}_{0+} \longrightarrow \mathbf{R}_+$, the solution of (3), subject to (4), satisfies

$$\begin{aligned}\|\widetilde{A}(z(t),t)\| &\le \begin{cases}\widetilde{K}(t), & \text{if } \|z(t)\| \le z_m, \\ \dfrac{\widetilde{K}(t)}{\|z(t)\|^q + \theta}, & \text{if } \|z(t)\| \le z_m,\end{cases} \\ \|\widetilde{A}(z(t),t)z(t)\| &\le \|\widetilde{A}(z(t),t)\|\|z(t)\| \\ &\le \begin{cases}\widetilde{K}(t)z_m, & \text{if } \|z(t)\| > z_m, \\ \dfrac{\widetilde{K}(t)\|z(t)\|}{\|z(t)\|^q + \theta} \le \dfrac{\widetilde{K}(t)}{\|z(t)\|^{q-1} + \theta\|z(t)\|}, & \text{if } \|z(t)\| > z_m,\end{cases}\end{aligned} \tag{62}$$

then for some positive finite real constant M_∞, $\limsup_{t\longrightarrow\infty}(\|\widetilde{A}(z(t),t)\|\|z(t)\|) \le M_\infty$ so that the condition of Property

(i) of Theorem 6 holds. Thus, Theorem 6 (i) holds under Assumptions 1–2.

4. Some Remarks on the Problem Point of View under Then Taylor's Series Remainder

A brief remind of well-known results on Taylor's series expansion follow to be then used for the problem at hand in this paper. Consider a real open interval $I=(a,b)$ and a real function $f(x)$ which is continuous on $\operatorname{cl} I=[a,b]$, m-th continuously differentiable on I, and such that its m-th derivative is absolutely continuous on $\operatorname{cl} I$ and its $(m+1)$-th derivative exists and it is absolutely continuous on $\operatorname{cl} I$. Then, the Taylor's series expansion formula with truncation and integral remainder gives for any $x_0, x \in I$ that

$$\begin{aligned} f(x) &= \sum_{j=0}^{m} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j + \frac{1}{m!}\int_{x_0}^{x} f^{(m+1)}(\xi)(x-\xi)^m d\xi \\ &= \sum_{j=0}^{m} \frac{f^{(j)}(x_0)}{j!}(x-x_0)^j + \frac{1}{m!}\int_{0}^{1}(1-\xi)^m f^{(m+1)}(\xi)d\xi, \end{aligned} \tag{63}$$

where the last right-hand-side additive term is the integral remainder. Note that if $f^{(m)}(\xi)$ is just continuous, rather than absolutely continuous, on $[x_0,x]$, then one gets from the mean value theorem,

$$\begin{aligned} R_m(x) &= \frac{1}{(m+1)!} f^{(m+1)}(\xi)(x-\xi)^{m+1} \\ &= \frac{1}{m!} f^{(m+1)}(\varsigma)(x-\varsigma)^m (x-x_0), \end{aligned} \tag{64}$$

for some $\xi, \varsigma \in (x_0, x)$, the first expression of (64) being the Lagrange remainder form while the second one is the Cauchy remainder form.

The formula (63), subject to (64), be easily extended to the differential system (3), subject to (4), by expanding $\dot{z}(t)$ in Taylor's series about the equilibrium point, that is, by expanding the real vector function $(B(z(t),t)z(t))$ for each fixed $t \in \mathbf{R}_{0+}$ with the assignations $x_0 \longrightarrow z_e = 0 \in \mathbf{R}^n$, $x \longrightarrow z(t)$ for each $t \in \mathbf{R}_{0+}$, in particular, $x \longrightarrow z_0$ for $t=0$. Then, assume that all the components of $(B(z(t),t))$, of rows $B_i(z(t),t)$ for $i=1,2,\ldots,n$, are continuously differentiable with respect to all the components of $z(t)=(z_1(t),z_2(t),\ldots,z_n(t))^T$ up till some order m and that the $(m+1)$-th higher-order differentials of all the entries of $B(z(t),t)$ exist at $z_e=0\in\mathbf{R}^n$, in some open subset of $\mathbf{R}^n$ containing z_0 and the unique equilibrium point $z_e=0\in\mathbf{R}^n$. Thus, we have

$$\begin{aligned} \dot{z}(t) &= 0 + \nabla_{z^T}\left(B_{ij}(0,t)z(t)\right)\Big]_{z_e=0} z(t) + R_1(z,t) = \underbrace{B(0,t).0}_{0} + \nabla_{z^T}\left(B_{ij}(0,t)\right)\Big]_{z_e=0} z(t) \\ &+ \sum_{j=2}^{m}\sum_{k=1}^{n} \begin{pmatrix} \dfrac{1}{j!}\left[\displaystyle\sum_{i_1=1}^{j}\cdots\sum_{i_j=1}^{j}\frac{\partial^j\left(B_{1k}(z(t),t)z_k\right)}{\partial z_{i1}\cdots\partial z_{ij}}\right]_{z_e=0}\left(\displaystyle\prod_{\ell=1}^{j} z_{i_\ell}\right) \\ \vdots \\ \vdots \\ \dfrac{1}{j!}\left[\displaystyle\sum_{i_1=1}^{j}\cdots\sum_{i_j=1}^{j}\frac{\partial^j\left(B_{nk}(z(t),t)z_k\right)}{\partial z_{i1}\cdots\partial z_{ij}}\right]_{z_e=0}\left(\displaystyle\prod_{\ell=1}^{j} z_{i_\ell}\right) \end{pmatrix} + R_m(z(t),t), \quad \forall t \in \mathbf{R}_{0+}, \end{aligned} \tag{65}$$

where the ∇ (Nabla)-operator stands for the gradient, i.e., $\nabla_{z^T}(B_{ij}(0,t))]_{z_e=0} = ([\partial B_{ij}(0,t)/\partial z_j]_{ij})$; $\forall t \in \mathbf{R}_{0+}$. A useful simplified notation for (65) is

$$\dot{z}(t) = \nabla_{z^T}\left(B_{ij}(0,t)\right)\Big]_{z_e=0} z(t) + \begin{pmatrix} \displaystyle\sum_{2\le|\alpha|\le m}\frac{1}{\alpha!}\left(D^\alpha B_{1k}(z,t)z\right)(0)\mathbf{z}^\alpha \\ \vdots \\ \vdots \\ \displaystyle\sum_{2\le|\alpha|\le m}\frac{1}{\alpha!}\left(D^\alpha B_{nk}(z,t)z\right)(0)\mathbf{z}^\alpha \end{pmatrix} \times id(m-1) + R_m(z(t),t), \quad \forall t \in \mathbf{R}_{0+}, \tag{66}$$

where the discrete binary indicator function $id : \mathbf{Z}_{0+} \longrightarrow \{0,1\}$ is defined as $id(m-1)=0$ if $m=0,1$ and $id(m-1)=1$ if $m\geq 2$, and the remainder real vector of the series expansion in differential and integral forms becomes

$$R_m(z(t),t)=\sum_{k=1}^{n}\begin{pmatrix}\left[\dfrac{1}{(m+1)!}\displaystyle\sum_{i_1=1}^{m+1}\cdots\sum_{i_1=1}^{m+1}\frac{\partial^j(B_{1k}(z(t),t)z_k(t))}{\partial z_{i1}\cdots\partial z_{ij}}\right]_{z_e=0}\left(\displaystyle\prod_{\ell=1}^{m+1}z_{i_\ell}\right)\\ \vdots\\ \vdots\\ \dfrac{1}{(m+1)!}\left[\displaystyle\sum_{i_1=1}^{m+1}\cdots\sum_{i_j=1}^{m+1}\frac{\partial^j(B_{nk}(z(t),t)z_k(t))}{\partial z_{i1}\cdots\partial z_{ij}}\right]_{z_e=0}\left(\displaystyle\prod_{\ell=1}^{m+1}z_{i_\ell}\right)\end{pmatrix}, \tag{67}$$

$$=\begin{pmatrix}\displaystyle\sum_{|\alpha|=m+1}\frac{m+1}{\alpha!}\mathbf{z}^{\alpha}\int_0^1(1-t)^m((D^{\alpha}B_{1k})(tz(t)))dt\\ \vdots\\ \vdots\\ \displaystyle\sum_{|\alpha|=m+1}\frac{m+1}{\alpha!}\mathbf{z}^{\alpha}\int_0^1(1-t)^m((D^{\alpha}B_{nk})(tz(t)))dt\end{pmatrix},\quad \forall t\in\mathbf{R}_{0+}, \tag{68}$$

where $D^{\alpha}f=(\partial^{|\alpha|}f/\partial_{x_1}^{\alpha_1}\ldots\partial_{x_n}^{\alpha_n})$, where $|\alpha|=\sum_{i=1}^{n}\alpha_i$, $\alpha!=\prod_{i=1}^{n}[\alpha_i!]$, and $\mathbf{z}^{\alpha}=\prod_{i=1}^{n}[z_i-z_{e_i}]=\prod_{i=1}^{n}[z_i]$ since $z_e=0$ so that $z_e+t(z-z_3)=tz(t)$ in (68) after using a variable change $s\longrightarrow z_e+t(z-z_e)$ to convert the integral from $z_e=0$ to z into one from $t=0$ to $t=1$. Note that, if (12), subject to (67), equivalently to (68), is used to describe (3), subject to (4), with

$$A(z(t),t)=\nabla_{z^T}\left(B_{ij}(0,t)\right)\big]_{z_e=0},$$

$$\widetilde{A}(z(t),t)z(t)=\begin{pmatrix}\displaystyle\sum_{2\le|\alpha|\le m}\frac{1}{\alpha!}\left(D^{\alpha}B_{1k}(z,t)z\right)(0)\mathbf{z}^{\alpha}\\ \vdots\\ \vdots\\ \displaystyle\sum_{2\le|\alpha|\le m}\frac{1}{\alpha!}\left(D^{\alpha}B_{nk}(z,t)z\right)(0)\mathbf{z}^{\alpha}\end{pmatrix}\times id(m-1)+R_m(z(t),t),\quad\forall t\in\mathbf{R}_{0+}, \tag{69}$$

then Assumptions 1–2 hold provided that the unique equilibrium point $z_e=0$ is stable and $A(z(t),t)=A(z_e,t)=A(0,t)=A;\forall t\in\mathbf{R}_{0+}$ is a constant (i.e., independent of time) stability matrix.

Theorem 2 holds under the following simplified "ad hoc" form.

Theorem 7. *Assume that the n-th differential system (3), subject to (4), is described by (69), subject to (68) with $A(z(t),t)=A$; $\forall t\in\mathbf{R}_{0+}$ being a stability matrix, such that the unique equilibrium point $z_e=0$ is locally asymptotically stable, and that Assumption 3 holds with $\tilde{a}$ being sufficiently small related to the absolute value of the stability abscissa of A. Then, the system is globally asymptotically stable at large.*

Data Availability

The data supporting this work are from previously reported studies and datasets, which have been cited in the list of references.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this manuscript.

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