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Research Article

Positive Solutions for a Coupled System of Nonlinear Semipositone Fractional Boundary Value Problems

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In this paper, we consider a four-point coupled boundary value problem for system of the nonlinear semipositone fractional differential equation $D_{0^+}^{\alpha}u(t) + \lambda f(t, u(t), v(t)) = 0$, $0 < t < 1$, $D_{0^+}^{\alpha}v(t) + \mu g(t, u(t), v(t)) = 0$, $0 < t < 1$, $u(0) = v(0) = 0$, $a_1 D_{0^+}^{\beta}u(1) = b_1 D_{0^+}^{\beta}v(\xi)$, $a_2 D_{0^+}^{\beta}v(1) = b_2 D_{0^+}^{\beta}u(\eta)$, $\eta, \xi \in (0,1)$, where the coefficients $a_i, b_i, i = 1, 2$ are real positive constants, $\alpha \in (1,2]$, $\beta \in (0,1]$, $D_{0^+}^{\alpha}$, $D_{0^+}^{\beta}$ are the standard Riemann-Liouville derivatives. Values of the parameters λ and μ are determined for which boundary value problem has positive solution by utilizing a fixed point theorem on cone.

1. Introduction

In recent years, fractional-order calculus has been one of the most rapidly developing areas of mathematical analysis. In fact, a natural phenomenon may depend not only on the time instant but also on the previous time history, which can be successfully modeled by fractional calculus. Fractional-order differential equations are naturally related to systems with memory, as fractional derivatives are usually nonlocal operators. Thus, fractional differential equations (FDEs) play an important role because of their applications in various fields of science, such as mathematics, physics, chemistry, optimal control theory, finance, biology, and engineering [1–6]. In particular, a great interest has been shown by many authors in the subject of fractional-order boundary value problems (BVPs), and a variety of results for BVPs equipped with different kinds of boundary conditions have been obtained; for instance, see [7–18] and the references cited therein.

We consider the four-point coupled system of nonlinear fractional differential equations:

$$\begin{aligned} &D_{0^+}^{\alpha}u(t) + \lambda f(t, u(t), v(t)) = 0, \quad 0 < t < 1, \\ &D_{0^+}^{\alpha}v(t) + \mu g(t, u(t), v(t)) = 0, \quad 0 < t < 1, \end{aligned} \tag{1}$$

with the coupled boundary conditions

$$\begin{aligned} &u(0) = v(0) = 0, \\ &a_1 D_{0^+}^{\beta}u(1) = b_1 D_{0^+}^{\beta}v(\xi), \\ &a_2 D_{0^+}^{\beta}v(1) = b_2 D_{0^+}^{\beta}u(\eta), \\ &\qquad \eta, \xi \in (0,1), \end{aligned} \tag{2}$$

where $\alpha \in (1,2]$, $\beta \in (0,1]$, $D_{0^+}^{\alpha}$ and $D_{0^+}^{\beta}$ are the standard Riemann-Liouville derivatives, $f, g \in C([0,1] \times [0,+\infty) \times [0,+\infty), [0,+\infty))$ and $a_i, b_i, i = 1, 2$ are real positive constants.

Here we emphasize that our problem is new in the sense of nonseparated coupled boundary conditions introduced here. To the best of our knowledge, fractional-order coupled system (1) has yet to be studied with the boundary conditions (2). In consequence, our findings of the present work will be a useful contribution to the existing literature on the topic. The existence of positive solution results for the given problem is new, though they are proved by applying the well-known fixed point theorem.

We present intervals for parameters λ, μ, f, and g such that the above problem (1)-(2) has at least one positive solution. By a positive solution (1)-(2), we mean a pair of functions $(u, v) \in C[0,1] \times C[0,1]$ satisfying (1) and (2) with $u(t) \geq 0, v(t) \geq 0$ for all $t \in [0,1]$ and $u(t) > 0, v(t) > 0$.

We use the following notations for our convenience:

$$K_i = \int_0^1 G_i(1,s)\,ds \text{ and } L_i = \int_0^1 H_i(1,s)\,ds \quad \text{for } i = 1,2.$$
$$A_i = \int_{s\in I} G_i(1,s)\,ds \text{ and } B_i = \int_{s\in I} H_i(1,s)\,ds \quad \text{for } i = 1,2. \tag{3}$$

Before stating our results, we make precise assumptions throughout the paper:

(H1) The functions $f, g \in C((0,1) \times [0,\infty) \times [0,\infty), (-\infty,\infty))$ and there exist functions $p_1, p_2 \in C([0,1] \times [0,\infty))$ such that $f(t,u,v) \geq -p_1(t)$ and $g(t,u,v) \geq -p_2(t)$ for any $t \in [0,1]$ and $(u,v) \in [0,\infty)$.

(H2) a_1, a_2, b_1, b_2 are positive constants such that $a_1 a_2 \geq b_1 b_2/(\xi^{1-\alpha+\beta}\eta^{1-\alpha+\beta})$.

(H3) $f(t,0,0) > 0, g(t,0,0) > 0$ for all $t \in [0,1]$.

(H4) The functions $f, g \in C((0,1) \times [0,\infty) \times [0,\infty), (-\infty,\infty))$, f, g may be singular at $t = 0$ and/or $t = 1$, and there exist functions $p_1, p_2 \in C((0,1), [0,\infty)), \alpha_1, \alpha_2 \in C((0,1),(0,\infty)), \beta_1, \beta_2 \in C([0,1] \times [0,\infty),[0,\infty))$ such that $-p_1(t) \leq f(t,u,v) \leq \alpha_1(t)\beta_1(t,u,v), -p_2(t) \leq g(t,u,v) \leq \alpha_2(t)\beta_2(t,u,v)$ for all $t \in (0,1), u, v \in [0,\infty)$, with $0 < \int_0^1 p_i(s)ds < \infty, 0 < \int_0^1 \alpha_i(s)ds < \infty,\ i = 1,2$.

(H5) There exists $t \in I = [1/4, 3/4] \subset (0,1)$ such that

$$f_\infty = \lim_{u+v\to\infty} \min_{t\in I} \frac{f(t,u,v)}{u+v} = \infty \quad \text{or } g_\infty = \lim_{u+v\to\infty} \min_{t\in I} \frac{g(t,u,v)}{u+v} = \infty. \tag{4}$$

The rest of the paper is organized as follows. In Section 2, we construct the Green functions for the associated linear fractional-order boundary value problems and estimate the bounds for these Green functions. In Section 3, we establish the existence of at least one positive solution of the boundary value problem (1)-(2) by applying fixed point theorem. Finally, as an application, we give an example to illustrate our result.

2. Green Functions and Bounds

In this section, we construct the Green functions for the associated linear fractional-order boundary value problems and estimate the bounds for these Green functions, which are needed to establish the main results.

Lemma 1. *Let $\alpha > 0$. Then, the differential equation $D_{0^+}^{\alpha} u(t) = 0$ has a solution*

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n} \tag{5}$$

for some $c_i \in \mathbb{R},\ i = 1,2,\ldots,n$, where n is the smallest integer greater than or equal to α.

Lemma 2. *Let $\alpha > 0$. Then, $I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$ for some $c_i \in \mathbb{R},\ i = 1,2,\ldots,n$, where n is the smallest integer greater than or equal to α.*

Lemma 3. *Let $\Delta = \Gamma(\alpha)\mathcal{N} \neq 0$ and $\mathcal{N} = a_1 a_2 - b_1 b_2 \xi^{\alpha-\beta-1}\eta^{\alpha-\beta-1}$. Let $x, y \in C[0,1]$ be given functions. Then, the boundary value problem,*

$$\begin{gathered} D_{0^+}^{\alpha} u(t) + x(t) = 0, \quad 0 < t < 1, \\ D_{0^+}^{\alpha} v(t) + y(t) = 0, \quad 0 < t < 1, \\ u(0) = v(0) = 0, \\ a_1 D_{0^+}^{\beta} u(1) = b_1 D_{0^+}^{\beta} v(\xi), \\ a_2 D_{0^+}^{\beta} v(1) = b_2 D_{0^+}^{\beta} u(\eta), \\ \xi, \eta \in (0,1), \end{gathered} \tag{6}$$

has an integral representation

$$\begin{aligned} u(t) &= \int_0^1 G_1(t,s)\,x(s)\,ds + \int_0^1 H_1(t,s)\,y(s)\,ds, \\ v(t) &= \int_0^1 G_2(t,s)\,y(s)\,ds + \int_0^1 H_2(t,s)\,x(s)\,ds, \end{aligned} \tag{7}$$

where

$$G_1(t,s) = \frac{1}{\Delta} \begin{cases} a_1 a_2 t^{\alpha-1}(1-s)^{\alpha-\beta-1} - \mathcal{N}(t-s)^{\alpha-1} - b_1 b_2 t^{\alpha-1}\xi^{\alpha-\beta-1}(\eta-s)^{\alpha-\beta-1}, & 0 \leq s \leq t \leq 1,\ s \leq \eta, \\ a_1 a_2 t^{\alpha-1}(1-s)^{\alpha-\beta-1} - \mathcal{N}(t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1,\ s \geq \eta, \\ a_1 a_2 t^{\alpha-1}(1-s)^{\alpha-\beta-1} - b_1 b_2 t^{\alpha-1}\xi^{\alpha-\beta-1}(\eta-s)^{\alpha-\beta-1}, & 0 \leq t \leq s \leq 1,\ s \leq \eta, \\ a_1 a_2 t^{\alpha-1}(1-s)^{\alpha-\beta-1}, & 0 \leq t \leq s \leq 1,\ s \geq \eta, \end{cases} \tag{8}$$

$$G_2(t,s)=\frac{1}{\Delta}\begin{cases}a_1a_2t^{\alpha-1}(1-s)^{\alpha-\beta-1}-\mathcal{N}(t-s)^{\alpha-1}-b_1b_2t^{\alpha-1}\eta^{\alpha-\beta-1}(\xi-s)^{\alpha-\beta-1}, & 0\le s\le t\le 1,\ s\le\xi,\\ a_1a_2t^{\alpha-1}(1-s)^{\alpha-\beta-1}-\mathcal{N}(t-s)^{\alpha-1}, & 0\le s\le t\le 1,\ s\ge\xi,\\ a_1a_2t^{\alpha-1}(1-s)^{\alpha-\beta-1}-b_1b_2t^{\alpha-1}\eta^{\alpha-\beta-1}(\xi-s)^{\alpha-\beta-1}, & 0\le t\le s\le 1,\ s\le\xi,\\ a_1a_2t^{\alpha-1}(1-s)^{\alpha-\beta-1}, & 0\le t\le s\le 1,\ s\ge\xi,\end{cases} \tag{9}$$

$$H_1(t,s)=\frac{1}{\Delta}\begin{cases}a_2b_1t^{\alpha-1}\xi^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}-a_2b_1t^{\alpha-1}(\xi-s)^{\alpha-\beta-1}, & s\le\xi,\\ a_2b_1t^{\alpha-1}\xi^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}, & s\ge\xi,\end{cases} \tag{10}$$

$$H_2(t,s)=\frac{1}{\Delta}\begin{cases}a_1b_2t^{\alpha-1}\eta^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}-a_1b_2t^{\alpha-1}(\eta-s)^{\alpha-\beta-1}, & s\le\eta,\\ a_1b_2t^{\alpha-1}\eta^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}, & s\ge\eta.\end{cases} \tag{11}$$

Lemma 4. *Assume that condition* $(H2)$ *is satisfied. Then, the Green functions* $G_1(t,s)$ *and* $H_1(t,s)$ *defined, respectively, by (8) and (10) are nonnegative, for all* $t,s\in[0,1]$.

Lemma 5. *Assume that condition* $(H2)$ *is satisfied. Then, the Green functions* $G_2(t,s)$ *and* $H_2(t,s)$ *defined, respectively, by (9) and (11) are nonnegative, for all* $t,s\in[0,1]$.

Lemma 6. *Assume that condition* $(H2)$ *is satified. Then, the Green functions* $G_1(t,s)$ *and* $H_1(t,s)$ *defined, respectively, by (8) and (10) have the following properties:*

(C1) $G_1(t,s)\le G_1(1,s)$, $H_1(t,s)\le H_1(1,s)$ *for all* $(t,s)\in[0,1]\times[0,1]$,

(C2) $G_1(t,s)\ge(1/4)^{\alpha-1}G_1(1,s)$, $H_1(t,s)\ge(1/4)^{\alpha-1}H_1(1,s)$, *for all* $(t,s)\in I\times(0,1)$, *where* $I=[1/4,3/4]$.

Lemma 7. *Assume that condition* $(H2)$ *is satified. Then, the Green functions* $G_2(t,s)$ *and* $H_2(t,s)$ *defined, respectively, by (9) and (11) have the following properties:*

(C3) $G_2(t,s)\le G_2(1,s)$ *and* $H_2(t,s)\le H_2(1,s)$ *for all* $(t,s)\in[0,1]\times[0,1]$,

(C4) $G_2(t,s)\ge(1/4)^{\alpha-1}G_2(1,s)$ *and* $H_2(t,s)\ge(1/4)^{\alpha-1}H_2(1,s)$, *for all* $(t,s)\in I\times(0,1)$, *where* $I=[1/4,3/4]$.

In the proof of our main results, we shall use the nonlinear alternative of Leray-Schauder type and the Guo-Krasnosel'skii fixed point theorem presented below [19, 20].

Theorem 8. *Let* X *be a Banach space with* $\Omega\subset X$ *closed and convex. Assume* U *is a relatively open subset of* Ω *with* $0\in U$, *and let* $S:\overline{U}\longrightarrow\Omega$ *be a completely continuous operator (continuous and compact). Then, either*

(i) S *has a fixed point in* $\overline{U}$, *or*

(ii) *there exist* $u\in\partial U$ *and* $v\in(0,1)$ *such that* $u=vSu$.

Theorem 9 (*Krasnosel′skii*). *Let* $\mathcal{B}$ *be a Banach space, and let* $\mathcal{P}\subset\mathcal{B}$ *be a cone in* $\mathcal{B}$. *Assume that* Ω_1 *and* Ω_2 *are two bounded open subsets of* $\mathcal{B}$ *with* $0\in\Omega_1\subset\overline{\Omega}_1\subset\Omega_2$, *and let* $T:\mathcal{P}\cap(\overline{\Omega}_2\setminus\Omega_1)\longrightarrow\mathcal{P}$ *be a completely continuous operator such that either*

(i) $\|Tu\|\le\|u\|, u\in\mathcal{P}\cap\partial\Omega_1$, *and* $\|Tu\|\ge\|u\|, u\in\mathcal{P}\cap\partial\Omega_2$, *or*

(ii) $\|Tu\|\ge\|u\|, u\in\mathcal{P}\cap\partial\Omega_1$, *and* $\|Tu\|\le\|u\|, u\in\mathcal{P}\cap\partial\Omega_2$.

Then, T *has a fixed point in* $\mathcal{P}\cap(\overline{\Omega}_2\setminus\Omega_1)$.

3. Main Results

In this section, we investigate the existence of positive solutions for our problem (1)-(2).

We consider the system of nonlinear fractional differential equations

$$\begin{aligned}&D^{\alpha}_{0^+}x(t)+\lambda\left(f\left(t,[x(t)-q_1(t)]^*,[y(t)-q_2(t)]^*\right)\right.\\&\quad\left.+p_1(t)\right)=0,\quad 0<t<1,\\&D^{\alpha}_{0^+}y(t)+\mu\left(g\left(t,[x(t)-q_1(t)]^*,[y(t)-q_2(t)]^*\right)\right.\\&\quad\left.+p_2(t)\right)=0,\quad 0<t<1,\end{aligned}\tag{12}$$

with the boundary conditions

$$\begin{aligned}&x(0)=y(0)=0,\\&a_1D^{\beta}_{0^+}x(1)=b_1D^{\beta}_{0^+}y(\xi),\\&a_2D^{\beta}_{0^+}y(1)=b_2D^{\beta}_{0^+}x(\eta),\\&\qquad\eta,\xi\in(0,1),\end{aligned}\tag{13}$$

where a modified function $[z(t)]^*$ for any $z\in C[0,1]$ by

$$\begin{aligned}&[z(t)]^*=z(t),\quad\text{if } z(t)\ge 0,\ \text{and}\\&[z(t)]^*=0,\quad\text{if } z(t)=0.\end{aligned}\tag{14}$$

Here (q_1,q_2) with

$$\begin{aligned}q_1(t)=\lambda\int_0^1G_1(t,s)\,p_1(s)\,ds\\+\mu\int_0^1H_1(t,s)\,p_2(s)\,ds,\quad t\in[0,1],\end{aligned}$$

$$q_2(t)=\mu\int_0^1 G_2(t,s)\,p_2(s)\,ds + \lambda\int_0^1 H_2(t,s)\,p_1(s)\,ds, \quad t\in[0,1], \tag{15}$$

is solution of the system of fractional differential equations

$$\begin{aligned} &D_{0^+}^{\alpha}q_1(t)+\lambda p_1(t)=0, \quad 0<t<1,\\ &D_{0^+}^{\alpha}q_2(t)+\mu p_2(t)=0, \quad 0<t<1, \end{aligned} \tag{16}$$

with the boundary conditions

$$\begin{aligned} &q_1(0)=q_2(0)=0,\\ &a_1D_{0^+}^{\beta}q_1(1)=b_1D_{0^+}^{\beta}q_2(\xi),\\ &a_2D_{0^+}^{\beta}q_2(1)=b_2D_{0^+}^{\beta}q_1(\eta),\\ &\eta,\xi\in(0,1). \end{aligned} \tag{17}$$

Under the assumptions $(H1)$ and $(H2)$ or $(H2)$ and $(H4)$, we have $q_1(t)\ge 0, q_2(t)\ge 0$ for all $t\in[0,1]$.

We shall prove that there exists a solution (x,y) for the boundary value problem (12)-(13) with $x(t)\ge q_1(t)$ and $y(t)\ge q_2(t)$ on $[0,1], x(t)>q_1(t), y(t)>q_2(t)$ on $(0,1)$. In this case, (u,v) with $u(t)=x(t)-q_1(t)$ and $v(t)=y(t)-q_2(t), t\in[0,1]$ represents a positive solution of boundary value problem (1)-(2).

By using Lemma 3, a solution of the system

$$\begin{aligned} x(t)&=\lambda\int_0^1 G_1(t,s)\cdot\left(f\left(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*\right)+p_1(s)\right)ds+\mu\int_0^1 H_1(t,s)\\ &\quad\cdot\left(g\left(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*\right)+p_2(s)\right)ds, \quad t\in[0,1],\\ y(t)&=\mu\int_0^1 G_2(t,s)\cdot\left(g\left(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*\right)+p_2(s)\right)ds+\lambda\int_0^1 H_2(t,s)\\ &\quad\cdot\left(f\left(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*\right)+p_1(s)\right)ds, \quad t\in[0,1], \end{aligned} \tag{18}$$

is a solution for problem (12)-(13).

We consider the Banach space $X=C[0,1]$ with supremum norm $\|\cdot\|$ and the Banach space $Y=X\times X$ with the norm $\|(u,v)\|=\|u\|+\|v\|$. We define the cone $P\subset Y$

$$P=\left\{(x,y)\in Y: x(t)\ge 0, y(t)\ge 0\ \forall t\in[0,1]\ \text{and}\ \min_{t\in I}\{x(t)+y(t)\}\ge\left(\frac{1}{4}\right)^{\alpha-1}\|(x,y)\|\right\}, \tag{19}$$

where $I=[1/4,3/4]$.

For $\lambda,\mu>0$, we define the operators $Q_1,Q_2: Y\longrightarrow X$ and $Q: Y\longrightarrow Y$ defined by $Q(x,y)=(Q_1(x,y),Q_2(x,y))$, $(x,y)\in Y$ with

$$\begin{aligned} Q_1(x,y)&=\lambda\int_0^1 G_1(t,s)\cdot\left(f\left(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*\right)+p_1(s)\right)ds+\mu\int_0^1 H_1(t,s)\\ &\quad\cdot\left(g\left(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*\right)+p_2(s)\right)ds, \quad t\in[0,1],\\ Q_2(x,y)&=\mu\int_0^1 G_2(t,s)\cdot\left(g\left(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*\right)+p_2(s)\right)ds+\lambda\int_0^1 H_2(t,s)\\ &\quad\cdot\left(f\left(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*\right)+p_1(s)\right)ds, \quad t\in[0,1]. \end{aligned} \tag{20}$$

It is clear that if (x,y) is a fixed point of operator Q, then (x,y) is a solution of problem (12)-(13).

Lemma 10. *If $(H1)$ and $(H2)$ or $(H2)$ and $(H4)$ hold, then operator $Q: P\longrightarrow P$ is a completely continuous operator.*

Proof. The operators Q_1 and Q_2 are well defined. To prove this, let $(x,y)\in P$ be fixed with $\|(x,y)\|=\widetilde{L}$. Then we have

$$\begin{aligned} &[x(s)-q_1(s)]^*\le x(s)\le\|x\|\le\|(x,y)\|=\widetilde{L}, \quad \forall s\in[0,1],\\ &[y(s)-q_2(s)]^*\le y(s)\le\|y\|\le\|(x,y)\|=\widetilde{L}, \quad \forall s\in[0,1]. \end{aligned} \tag{21}$$

If (H_1) and (H_2) hold, then we deduce easily that $Q_1(x,y)(t)<\infty$ and $Q_2(x,y)(t)<\infty$ for all $t\in[0,1]$. If (H_2) and (H_4) hold, we deduce, for all $t\in[0,1]$:

$$\begin{aligned} Q_1(x,y)&\le\lambda\int_0^1 G_1(1,s)\left[\alpha_1(s)\right.\\ &\quad\cdot\beta_1\left(s,(x(s)-q_1(s))^*,(y(s)-q_2(s))^*\right.\end{aligned}$$

$$+ p_1(s)\big] ds + \mu \int_0^1 H_1(1,s) \big[\alpha_2(s) \cdot \beta_2\left(s, (x(s) - q_1(s))^*, (y(s) - q_2(s))^*\right) + p_2(s)\big] ds, \le M\left(\lambda \int_0^1 G_1(1,s) \cdot (\alpha_1(s) + p_1(s))\, ds + \mu \int_0^1 H_1(1,s) \cdot (\alpha_2(s) + p_2(s))\, ds\right) < \infty,$$

$$Q_2(x,y) \le \mu \int_0^1 G_2(1,s) \big[\alpha_2(s) \cdot \beta_2\left(s, (x(s) - q_1(s))^*, (y(s) - q_2(s))^*\right. + p_2(s)\big] ds + \lambda \int_0^1 H_2(1,s) \big[\alpha_1(s) \cdot \beta_1\left(s, (x(s) - q_1(s))^*, (y(s) - q_2(s))^*\right) + p_1(s)\big] ds, \le M\left(\mu \int_0^1 G_2(1,s) \cdot (\alpha_2(s) + p_2(s))\, ds + \lambda \int_0^1 H_2(1,s) \cdot (\alpha_1(s) + p_1(s))\, ds\right) < \infty, \quad (22)$$

where $M = \max\{\max_{t\in[0,1],u,v\in[0,\widetilde{L}]}\beta_1(t,u,v), \max_{t\in[0,1],u,v\in[0,\widetilde{L}]}\beta_2(t,u,v), 1\}$.

Thus, $Q : P \longrightarrow Y$ is well defined.

Next, we show that $T : P \longrightarrow P$. For any fixed $(x,y) \in P$, by Lemmas 6 and 7, we have

$$\min_{t\in I} Q_1(x,y)(t) = \min_{t\in I}\Big[\lambda \int_0^1 G_1(t,s) \cdot \left(f\left(s, [x(s) - q_1(s)]^*, [y(s) - q_2(s)]^*\right) + p_1(s)\right) ds + \mu \int_0^1 H_1(t,s) \cdot \left(g\left(s, [x(s) - q_1(s)]^*, [y(s) - q_2(s)]^*\right) + p_2(s)\right) ds\Big] \ge \left(\frac{1}{4}\right)^{\alpha-1}\Big[\lambda \int_0^1 G_1(1,s) \cdot \left(f\left(s, [x(s) - q_1(s)]^*, [y(s) - q_2(s)]^*\right) + p_1(s)\right) ds + \mu \int_0^1 H_1(1,s) \left(g\left(s, [x(s) - q_1(s)]^*, [y(s) - q_2(s)]^*\right) + p_2(s)\right) ds\Big] \ge \left(\frac{1}{4}\right)^{\alpha-1} \|Q_1(x,y)\|. \quad (23)$$

Similarly, $\min_{t\in I} Q_2(x,y)(t) \ge (1/4)^{\alpha-1}\|Q_2(x,y)\|$. Therefore,

$$\min_{t\in I}\{Q_1(x,y)(t) + Q_2(x,y)(t)\} \ge \left(\frac{1}{4}\right)^{\alpha-1} \|Q_1(x,y)\| + \left(\frac{1}{4}\right)^{\alpha-1} \|Q_2(x,y)\| = \left(\frac{1}{4}\right)^{\alpha-1} \|(Q_1(x,y), Q_2(x,y))\| = \left(\frac{1}{4}\right)^{\alpha-1} \|Q(x,y)\|. \quad (24)$$

Hence, $Q(x,y) \in P$. This implies that $Q(P) \subset P$. According to the Ascoli-Arzela theorem, we can easily get that $Q : P \longrightarrow P$ is completely continuous. □

Theorem 11. *Assume that $(H1)-(H3)$ hold. Then, there exist constants $\lambda_0 > 0$ and $\mu_0 > 0$ such that, for any $\lambda \in (0,\lambda_0]$ and $\mu \in (0,\mu_0]$, the boundary value problem (1)-(2) has at least one positive solution.*

Proof. Let $\delta \in (0,1)$ be fixed. From $(H1)$ and $(H3)$, there exist $R_0 \in (0,1]$ such that

$$f(t,u,v) \ge \delta f(t,0,0), \quad g(t,u,v) \ge \delta g(t,0,0), \quad \forall t \in [0,1],\ u,v \in [0,R_0]. \quad (25)$$

We define

$$\overline{f}(R_0) = \max_{t\in[0,1],u,v\in[0,R_0]}\{f(t,u,v) + p_1(t)\} \ge \max_{t\in[0,1]}\{\delta f(t,0,0) + p_1(t)\} > 0,$$
$$\overline{g}(R_0) = \max_{t\in[0,1],u,v\in[0,R_0]}\{g(t,u,v) + p_2(t)\} \ge \max_{t\in[0,1]}\{\delta g(t,0,0) + p_2(t)\} > 0, \quad (26)$$
$$\lambda_0 = \max\left\{\frac{R_0}{8K_1\overline{f}(R_0)}, \frac{R_0}{8L_2\overline{f}(R_0)}\right\},$$
$$\mu_0 = \max\left\{\frac{R_0}{8L_1\overline{g}(R_0)}, \frac{R_0}{8K_2\overline{g}(R_0)}\right\}.$$

We will show that, for any $\lambda \in (0,\lambda_0]$ and $\mu \in (0,\mu_0]$, problem (12)-(13) has at least one positive solution.

So, let $\lambda \in (0,\lambda_0]$ and $\mu \in (0,\mu_0]$ be arbitrary but fixed for the moment. We define the set $U = \{(x,y) \in P, \|(x,y)\| < R_0\}$. We suppose that there exist $(x,y) \in \partial U (\|(x,y)\| = R_0$ or $\|x\| + \|y\| = R_0)$ and $\theta \in (0,1)$ such that $(x,y) = \theta Q(x,y)$ or $x = \theta Q_1(x,y)$, $y = \theta Q_2(x,y)$.

We deduce that

$$[x(t) - q_1(t)]^* = x(t) - q_1(t) \le x(t) \le R_0, \quad \text{if } x(t) - q_1(t) \ge 0,$$
$$[x(t) - q_1(t)]^* = 0, \quad \text{for } x(t) - q_1(t) < 0,\ \forall t \in [0,1],$$

$$[y(t)-q_2(t)]^* = y(t)-q_2(t) \le y(t) \le R_0, \quad \text{if } y(t)-q_2(t) \ge 0,$$
$$[y(t)-q_2(t)]^* = 0, \quad \text{for } y(t)-q_2(t) < 0, \ \forall t \in [0,1]. \tag{27}$$

Then by Lemma 3, for all $t \in [0,1]$, we obtain

$$\begin{aligned} x(t) &= \theta Q_1(x,y)(t) < Q_1(x,y)(t) \\ &\le \lambda \int_0^1 G_1(1,s)\overline{f}(R_0)\,ds + \mu \int_0^1 H_1(1,s)\overline{g}(R_0)\,ds \\ &\le \lambda_0 K_1 \overline{f}(R_0) + \mu_0 L_1 \overline{g}(R_0) \le \frac{R_0}{8} + \frac{R_0}{8} = \frac{R_0}{4}, \\ y(t) &= \theta Q_2(x,y)(t) < Q_2(x,y)(t) \\ &\le \mu \int_0^1 G_2(1,s)\overline{g}(R_0)\,ds + \lambda \int_0^1 H_2(1,s)\overline{f}(R_0)\,ds \\ &\le \mu_0 K_2 \overline{g}(R_0) + \lambda_0 L_2 \overline{f}(R_0) \le \frac{R_0}{8} + \frac{R_0}{8} = \frac{R_0}{4}. \end{aligned} \tag{28}$$

Hence, $\|x\| \le R_0/4$ and $\|y\| \le R_0/4$. Then, $R_0 = \|(x,y)\| = \|x\| + \|y\| \le R_0/4 + R_0/4 = R_0/2$, which is contradiction.

Therefore, by Theorem 8 (with $\Omega = P$), we deduce that Q has a fixed point $(x_0, y_0) \in \overline{U} \cap P$. That is, $(x_0, y_0) = Q(x_0, y_0)$ or $x_0 = Q_1(x_0, y_0)$, $y_0 = Q_2(x_0, y_0)$, and $\|x_0\| + \|y_0\| \le R_0$ with $x_0 \ge (1/4)^{\alpha-1}\|x_0\|$ and $y_0(t) \ge (1/4)^{\alpha-1}\|y_0\|$ for all $t \in [0,1]$. Moreover, by (25), we conclude

$$\begin{aligned} x_0(t) &= Q_1(x_0, y_0)(t) \\ &\ge \lambda \int_0^1 G_1(t,s)(\delta f(t,0,0) + p_1(s))\,ds + \mu \int_0^1 H_1(t,s)(\delta g(t,0,0) + p_2(s))\,ds \\ &\ge \lambda \int_0^1 G_1(t,s)p_1(s)\,ds + \mu \int_0^1 H_1(t,s)p_2(s)\,ds = q_1(t), \quad \forall t \in [0,1], \\ x_0(t) &> \lambda \int_0^1 G_1(t,s)p_1(s)\,ds + \mu \int_0^1 H_1(t,s)p_2(s)\,ds = q_1(t), \quad \forall t \in (0,1), \\ y_0(t) &= Q_2(x_0, y_0)(t) \\ &\ge \mu \int_0^1 H_2(t,s)(\delta g(t,0,0) + p_2(s))\,ds + \lambda \int_0^1 G_2(t,s)(\delta f(t,0,0) + p_1(s))\,ds \\ &\ge \mu \int_0^1 H_2(t,s)p_2(s)\,ds + \lambda \int_0^1 G_2(t,s)p_1(s)\,ds = q_2(t), \quad \forall t \in [0,1], \\ y_0(t) &> \mu \int_0^1 H_2(t,s)p_2(s)\,ds + \lambda \int_0^1 G_2(t,s)p_1(s)\,ds = q_2(t), \quad \forall t \in (0,1). \end{aligned} \tag{29}$$

Therefore, $x_0(t) \ge q_1(t)$, $y_0(t) \ge q_2(t)$ for all $t \in [0,1]$, and $x_0(t) > q_1(t)$, $y_0(t) > q_2(t)$ for all $t \in (0,1)$. Let $u_0(t) = x_0(t) - q_1(t)$ and $v_0(t) = y_0(t) - q_2(t)$ for all $t \in [0,1]$. Then, $u_0(t) \ge 0$, $v_0(t) \ge 0$ for all $t \in [0,1]$, $u_0(t) > 0$, $v_0(t) > 0$ for all $t \in (0,1)$. Therefore, (u_0, v_0) is a positive solution of (1)-(2). □

Theorem 12. *Assume that* $(H1)$, $(H4)$, *and* $(H5)$ *hold. Then, there exist* $\lambda^* > 0$ *and* $\mu^* > 0$ *such that, for any* $\lambda \in (0, \lambda^*]$ *and* $\mu \in (0, \mu^*]$, *the boundary value problem (1)-(2) has at least one positive solution.*

Proof. We choose a positive number

$$R_1 > \max\left\{1, 2\int_0^1 (G_1(1,s)p_1(s) + G_2(1,s)p_2(s))\,ds\right\} \tag{30}$$

and we define the set $\Omega_1 = \{(x,y) \in P, \|(x,y)\| < R_1\}$.

We introduce

$$\begin{aligned} \lambda^* &= \min\left\{1, \frac{R_1}{4M_1}\left(\int_0^1 G_1(1,s)(\alpha_1(s) + p_1(s))\,ds\right)^{-1}, \frac{R_1}{4M_1}\left(\int_0^1 H_2(1,s)(\alpha_1(s) + p_1(s))\,ds\right)^{-1}\right\}, \\ \mu^* &= \min\left\{1, \frac{R_1}{4M_2}\left(\int_0^1 H_1(1,s)(\alpha_2(s) + p_2(s))\,ds\right)^{-1}, \frac{R_1}{4M_2}\left(\int_0^1 G_2(1,s)(\alpha_2(s) + p_2(s))\,ds\right)^{-1}\right\}, \end{aligned} \tag{31}$$

with

$$M_1 = \max\left\{\max_{t\in[0,1],u,v\geq 0,u+v\leq R_1} \beta_1(t,u,v), 1\right\},$$
$$M_2 = \max\left\{\max_{t\in[0,1],u,v\geq 0,u+v\leq R_1} \beta_2(t,u,v), 1\right\}. \tag{32}$$

Let $\lambda \in (0, \lambda^*]$ and $\mu \in (0, \mu^*]$. Then, for any $(x, y) \in P \cap \partial\Omega_1$ and $s \in [0,1]$, we have

$$[x(s) - q_1(s)]^* \leq x(s) \leq \|x\| \leq R_1,$$
$$[y(s) - q_2(s)]^* \leq y(s) \leq \|y\| \leq R_1. \tag{33}$$

Then, for any $(x, y) \in P \cap \partial\Omega_1$, we obtain

$$\begin{aligned}\|Q_1(x,y)\| &\leq \lambda \int_0^1 G_1(1,s)\big[\alpha_1(s) \\ &\cdot \beta_1\big(s,(x(s)-q_1(s))^*,(y(s)-q_2(s))^* \\ &+ p_1(s)\big]\,ds + \mu \int_0^1 H_1(1,s)\big[\alpha_2(s) \\ &\cdot \beta_2\big(s,(x(s)-q_1(s))^*,(y(s)-q_2(s))^*\big) \\ &+ p_2(s)\big]\,ds, \leq \lambda^* M_1 \int_0^1 G_1(1,s)(\alpha_1(s) \\ &+ p_1(s))\,ds + \mu^* M_2 \int_0^1 H_1(1,s)(\alpha_2(s) \\ &+ p_2(s))\,ds \leq \frac{R_1}{4} + \frac{R_1}{4} = \frac{R_1}{2} = \frac{\|(x,y)\|}{2}, \\ \|Q_2(x,y)\| &\leq \mu \int_0^1 G_2(1,s)\big[\alpha_2(s) \\ &\cdot \beta_2\big(s,(x(s)-q_1(s))^*,(y(s)-q_2(s))^* \\ &+ p_2(s)\big]\,ds + \mu \int_0^1 H_2(1,s)\big[\alpha_1(s) \\ &\cdot \beta_1\big(s,(x(s)-q_1(s))^*,(y(s)-q_2(s))^*\big) \\ &+ p_1(s)\big]\,ds, \leq \mu^* M_2 \int_0^1 G_2(1,s)(\alpha_2(s) \\ &+ p_2(s))\,ds + \lambda^* M_1 \int_0^1 H_2(1,s)(\alpha_1(s) \\ &+ p_1(s))\,ds \leq \frac{R_1}{4} + \frac{R_1}{4} = \frac{R_1}{2} = \frac{\|(x,y)\|}{2}.\end{aligned} \tag{34}$$

Therefore,

$$\|Q(x,y)\| = \|Q_1(x,y)\| + \|Q_2(x,y)\| \leq \|(x,y)\|, \quad \forall (x,y) \in P \cap \partial\Omega_1. \tag{35}$$

On the other hand, we choose a constant $L > 0$ such that

$$\lambda\left(\frac{1}{4}\right)^{2(\alpha-1)} A_1 L \geq 4,$$
$$\mu\left(\frac{1}{4}\right)^{2(\alpha-1)} A_2 L \geq 4. \tag{36}$$

From $(H5)$, we deduce that there exists a constant $M_0 > 0$ such that

$$\begin{aligned}& f(t,u,v) \geq L(u+v) \\ \text{or } & g(t,u,v) \geq L(u+v), \\ & \forall t \in I,\ u,v \geq 0,\ u+v \geq M_0.\end{aligned} \tag{37}$$

Now we define

$$R_2 = \max\Big\{2R_1, 4^\alpha M_0, 4\int_0^1 (G_1(1,s)p_1(s) + H_1(1,s)p_2(s))\,ds\Big\} > 0, \tag{38}$$

and let $\Omega_2 = \{(x,y) \in P, \|(x,y)\| < R_2\}$.

We suppose that $f_\infty = \infty$, that is, $f(t,u,v) \geq L(u+v)$ for all $t \in I$ and $u, v \geq 0, u+v \geq M_0$. Then, for any $(x,y) \in P \cap \partial\Omega_2$, we have $\|(x,y)\| = R_2$ or $\|x\| + \|y\| = R_2$. We deduce that $\|x\| \geq R_2/2$ or $\|y\| \geq R_2/2$.

We suppose that $\|x\| \geq R_2/2$. Then, for any $(x,y) \in P \cap \partial\Omega_2$, we obtain

$$\begin{aligned}x(t) - q_1(t) &= x(t) - \lambda \int_0^1 G_1(t,s)p_1(s)\,ds \\ &- \mu \int_0^1 H_1(t,s)p_2(s)\,ds \geq x(t) - \left(\frac{1}{4}\right)^{\alpha-1} \\ &\cdot \left(\int_0^1 G_1(1,s)p_1(s)\,ds + \int_0^1 H_1(1,s)p_2(s)\,ds\right) \\ &\geq x(t) - \frac{x(t)}{\|x\|} \\ &\cdot \int_0^1 (G_1(1,s)p_1(s) + H_1(1,s)p_2(s))\,ds = x(t) \\ &\cdot \Big[1 \\ &- \frac{1}{\|x\|}\int_0^1 (G_1(1,s)p_1(s) + H_1(1,s)p_2(s))\,ds\Big] \\ &\geq \Big[1 \\ &- \frac{2}{R_2}\int_0^1 (G_1(1,s)p_1(s) + H_1(1,s)p_2(s))\,ds\Big] \\ &\geq \frac{1}{2}x(t) \geq 0.\end{aligned} \tag{39}$$

Therefore, we conclude

$$\begin{aligned}[x(t)-q_1(t)]^* &= x(t)-q_1(t) \ge \frac{1}{2}x(t) \\ &\ge \frac{1}{2}\left(\frac{1}{4}\right)^{\alpha-1}\|x\| \ge \frac{1}{4}\left(\frac{1}{4}\right)^{\alpha-1}R_2 \\ &= \left(\frac{1}{4}\right)^{\alpha}R_2 \ge M_0, \quad \forall t\in I.\end{aligned} \tag{40}$$

Hence,

$$\begin{aligned}&[x(t)-q_1(t)]^* + [y(t)-q_2(t)]^* \ge [x(t)-q_1(t)]^* \\ &\quad = x(t)-q_1(t) \ge M_0, \quad \forall t\in I.\end{aligned} \tag{41}$$

Then, for any $(x,y)\in P\cap\partial\Omega_2$ and $t\in I$, by (37) and (41), we deduce

$$\begin{aligned}&f\left(t,[x(t)-q_1(t)]^*,[y(t)-q_2(t)]^*\right) \\ &\quad\ge L\left([x(t)-q_1(t)]^*+[y(t)-q_2(t)]^*\right) \\ &\quad\ge L[x(t)-q_1(t)]^* \ge \frac{L}{2}x(t), \quad \forall t\in I.\end{aligned} \tag{42}$$

It follows that, for any $(x,y)\in P\cap\partial\Omega_2, t\in I$, we obtain

$$\begin{aligned}Q_1(x,y)(t) &\ge \lambda\int_0^1 G_1(t,s) \\ &\cdot\left(f\left(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*\right)\right. \\ &\left.+p_1(s)\right)ds \ge \lambda\int_{s\in I}G_1(t,s) \\ &\cdot\left(f\left(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*\right)\right. \\ &\left.+p_1(s)\right)ds \ge \left(\frac{1}{4}\right)^{\alpha-1}\lambda\int_{s\in I}G_1(1,s) \\ &\cdot L\left([x(s)-q_1(s)]^*\right)ds \ge \lambda\left(\frac{1}{4}\right)^{\alpha-1} \\ &\cdot A_1\left(\frac{1}{4}\right)^{\alpha-1}\frac{L}{4}R_2 = \lambda\frac{L}{4}\left(\frac{1}{4}\right)^{2(\alpha-1)}A_1R_2 \ge R_2.\end{aligned} \tag{43}$$

Then, $\|Q_1(x,y)\|\ge\|(x,y)\|$ and

$$\|Q(x,y)\| \ge \|(x,y)\|, \quad \forall (x,y)\in P\cap\partial\Omega_2. \tag{44}$$

If $\|y\| \ge R_2/2$, then by a similar approach, we obtain again relation (44).

We suppose that $g_\infty=\infty$, that is, $g(t,u,v)\ge L(u+v)$, for all $t\in I$ and $u,v\ge 0, u+v\ge M_0$. Then, for any $(x,y)\in P\cap\partial\Omega_2$, we have $\|(x,y)\|=R_2$. Hence, $\|x\|\ge R_2/2$ or $\|y\|\ge R_2/2$.

If $\|x\|\ge R_2/2$, then for any $(x,y)\in P\cap\partial\Omega_2$ we deduce in a similar manner as above that $x(t)-q_1(t)\ge(1/2)x(t)$ for all $t\in[0,1]$ and

$$\begin{aligned}Q_1(x,y)(t) &\ge \mu\int_0^1 G_2(t,s) \\ &\cdot\left(g\left(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*\right)\right. \\ &\left.+p_2(s)\right)ds \ge \mu\int_{s\in I}G_2(t,s) \\ &\cdot\left(g\left(s,[x(s)-q_1(s)]^*,[y(s)-q_2(s)]^*\right)\right. \\ &\left.+p_2(s)\right)ds \ge \left(\frac{1}{4}\right)^{\alpha-1}\mu\int_{s\in I}G_2(1,s) \\ &\cdot L\left([x(s)-q_1(s)]^*\right)ds \ge \mu\left(\frac{1}{4}\right)^{\alpha-1} \\ &\cdot A_2\left(\frac{1}{4}\right)^{\alpha-1}\frac{L}{4}R_2ds = \mu\frac{L}{4}\left(\frac{1}{4}\right)^{2(\alpha-1)}A_2R_2 \ge R_2, \\ &\forall t\in I.\end{aligned} \tag{45}$$

Hence, we obtain relation (44). If $\|y\| \ge R_2/2$, then in a similar way as above, we deduce again relation (44). Therefore, by Theorem 9, relation (35), and (44), we conclude that Q has a fixed point $(x,y)\in P\cap(\overline{\Omega_2}\setminus\Omega_1)$. □

4. Example

In this section, we give an example to illustrating our result. Let

$$\begin{aligned}\alpha &= \frac{3}{2}, \\ \beta &= \frac{1}{4}, \\ \eta &= \frac{2}{3}, \\ \xi &= \frac{1}{3}, \\ a_1 &= a_2 = 1, \\ b_1 &= b_2 = 1.\end{aligned} \tag{46}$$

Consider the system of fractional differential equations,

$$\begin{aligned}&D_{0^+}^{3/2}u(t)+\lambda f(t,u(t),v(t))=0, \quad t\in(0,1), \\ &D_{0^+}^{3/2}v(t)+\mu g(t,u(t),v(t))=0, \quad t\in(0,1), \\ &u(0)=v(0)=0, \\ &D_{0^+}^{1/4}u(1)=D_{0^+}^{1/4}v\left(\frac{1}{3}\right), \\ &D_{0^+}^{1/4}v(1)=D_{0^+}^{1/4}u\left(\frac{2}{3}\right),\end{aligned} \tag{47}$$

where $f(t,u,v)=(u+v)^3+\cos u, g(t,u,v)=(u+v)^{1/3}+\cos v$. We have $p_1(t)=p_2(t)=1$ for all $t\in[0,1]$, and then assumption $(H1)$ is satisfied. Besides, assumption $(H3)$

is also satisfied, because $f(t,0,0) = 1$ and $g(t,0,0) = 1$ for all $t \in [0,1]$. Let $\delta = 1/3 < 1$ and $R_0 = 1$. Then $f(t,u,v) \geq \delta f(t,0,0) = 1/3$, $g(t,u,v) \geq \delta g(t,0,0) = 1/3, \forall t \in [0,1], u, v \in [0,1]$. In addition,

$$\begin{aligned} \overline{f}(R_0) &= \overline{f}(1) = \max_{t\in[0,1],u,v\in[0,1]} \{f(t,u,v) + p_1(t)\} \\ &\approx 9.999848, \\ \overline{g}(R_0) &= \overline{g}(1) = \max_{t\in[0,1],u,v\in[0,1]} \{g(t,u,v) + p_2(t)\} \\ &\approx 3.259769. \end{aligned} \tag{48}$$

We also obtain $\Delta = (0.8865)(0.3133) \approx 0.2778 > 0$, $M_1 = 992$, $M_2 = 1280$, $K_1 = 0.1488$, $K_2 = 0.01598$, $L_1 = 0.0536$, $L_2 = 0.1268$, and then $\lambda_0 = \max\{R_0/8K_1\overline{f}(R_0), R_0/8K_2\overline{f}(R_0)\} \approx 0.782239674$, $\mu_0 = \max\{R_0/8L_1\overline{g}(R_0), R_0/8L_2\overline{g}(R_0)\} \approx 0.7154155$. We can apply Theorem 11. So we conclude that there exist $\lambda_0, \mu_0 > 0$ such that, for every $\lambda \in (0, \lambda_0]$ and $\mu \in (0, \mu_0]$, the boundary value problem (47) has at least one positive solution.

5. Conclusions

This paper studies the existence of positive solution of a four-point coupled system of nonlinear fractional differential equations. We give sufficient conditions on λ, μ, f, and g such that the system has at least one positive solution. The existence of positive solution is discussed by using Guo-Krasnosel'skii fixed point theorem. Also, an example which illustrates the obtained result is presented.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that no competing interests exist.

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