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Research Article

Investigation of a Mild Solution to Coupled Systems of Impulsive Hybrid Fractional Differential Equations

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The study of coupled systems of hybrid fractional differential equations requires the attention of scientists for the exploration of their different important aspects. Our aim in this paper is to study the existence and uniqueness of the solution for impulsive hybrid fractional differential equations. The novelty of this work is the study of a coupled system of impulsive hybrid fractional differential equations with initial and boundary hybrid conditions. We used the classical fixed-point theorems such as the Banach fixed-point theorem and Leray–Schauder alternative fixed-point theorem for existence results. We also give an example of the main results.

1. Introduction

Fractional differential equations appear naturally in a number of fields such as physics, engineering, biophysics, blood flow phenomena, aerodynamics, electron-analytical chemistry, biology, and control theory. An excellent account of fractional differential equations is given in this study. Undergoing abrupt changes at certain moments of time like earthquake, harvesting, and shock, these perturbations can be well approximated as instantaneous change of state or impulses. Furthermore, these processes are modelled by impulsive differential equations.

On the other hand, impulsive differential equations appear as a natural description of many evolutionary phenomena in the real world. The majority of processes in applied sciences are represented by differential equations. However, the situation is different in certain physical phenomena undergoing abrupt changes during their evolution as mechanical systems with impact, biological systems (heartbeat, blood flow, and so on), the dynamics of populations, natural disasters, etc. These changes are often of very short duration and are therefore produced instantly in the form of pulses. The modeling of such phenomena requires the use of forms that explicitly and simultaneously involve the continuous evolution of the phenomenon as well as instantaneous changes.

Such models are said to be "impulsive;" they are evolutionary of continuous processes governed by differential equations combined with difference equations representing the effect impulsive has undergone.

For some recent developments on the topic, see [1–5] and the references therein.

Hybrid fractional differential equations have also been studied by several researchers. This class of equations involves the fractional derivative of an unknown hybrid function with the nonlinearity depending on it. Some recent results on hybrid differential equations can be found in a series of papers [6–11].

By applying the Banach fixed-point theorem and Kransnoselskii fixed-point theorem, the existence results for the solution are obtained. Shah et al. [12] studied the coupled system of fractional impulsive boundary problems:

$$\begin{cases} D^{\alpha}x(t) = \Phi(t, x(t), y(t)) \text{ a.e.} & t \in J = [0,1],\ t \neq t_i,\ 1 < \alpha \leq 2, \\ \Delta x|_{t=t_i} = I_i(x(t_i)), \Delta' x\big|_{t=t_i} = \overline{I_i}(x(t_i)), & t_i \in (0,1),\ i = 1, 2, \ldots, m, \\ x(0) = h(x), x(1) = g(x), & \\ D^{\beta}y(t) = \Psi(t, x(t), y(t)) \text{ a.e.} & t \in j',\ 1 < \beta \leq 2, \\ \Delta x|_{t=t_j} = J_i\big(x(t_j)\big),\ \Delta' x\big|_{t=t_i} = \overline{J_j}\big(x(t_j)\big), & t_j \in (0,1),\ j = 1, 2, \ldots, m, \\ x(0) = \kappa(x), x(1) = f(x), & \end{cases} \tag{1}$$

where $1 < \alpha$, $\beta \leq 2$, Φ, Ψ, I_i, $\overline{I_i}$, J_j, and $\overline{J_j}$ are continuous functions, g, h, κ, and f are fixed continuous functionals, and $\Delta x|_{t=t_i} = x(t_i^+) - x(t_i^-)$, $\Delta' x|_{t=t_i} = x'(t_i^+) - x'(t_i^-)$ and $\Delta y|_{t=t_j} = y(t_j^+) - y(t_j^-)$, $\Delta' y|_{t=t_j} = y'(t_j^+) - y'(t_j^-)$.

Motivated by some recent studies to the boundary value problem of a class of impulsive hybrid fractional differential, we consider the problem of coupled hybrid fractional differential equations:

$$\begin{cases} D^{\alpha}\left(\dfrac{u(t)}{f_1(t, u(t), v(t))}\right) = g_1(t, u(t), v(t)), & t \in [0,1],\ t \neq t_i,\ i = 1, 2, \ldots, n,\ 0 < \alpha < 1, \\ u(t_i^+) = u(t_i^-) + I_i(u(t_i^-)), & t_i \in (0,1),\ i = 1, 2, \ldots, n, \\ D^{\beta}\left(\dfrac{u(t)}{f_2(t, u(t), v(t))}\right) = g_2(t, u(t), v(t)), & t \in [0,1],\ t \neq t_j,\ j = 1, 2, \ldots, m,\ 0 < \beta < 1, \\ v\big(t_j^+\big) = v\big(t_j^-\big) + I_j\big(v\big(t_j^-\big)\big), & t_j \in (0,1),\ j = 1, 2, \ldots, m, \\ \dfrac{u(0)}{f_1(0, u(0), v(0))} = \phi(u), \dfrac{v(0)}{f_2(0, u(0), v(0))} = \psi(v), & \end{cases} \tag{2}$$

D^{α} and D^{β} stand for Caputo fractional derivative of order α and β, respectively; $f_i \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}\backslash\{0\})$, $g_i \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $(i = 1, 2)$, and $\phi, \psi : C([0,1], \mathbb{R}) \longrightarrow \mathbb{R}$ are continuous functions defined by $\phi(u) = \sum_{i=1}^{n} \lambda_i u(\xi_i)$ and $\psi(v) = \sum_{j=1}^{n} \delta_j v(\eta_j)$, where $\xi_i, \eta_j \in (0,1)$ for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$; and $I_k : \mathbb{R} \longrightarrow \mathbb{R}$ and $u(t_k^+) = \lim_{\varepsilon \longrightarrow 0^+} u(t_k + \varepsilon)$ and $u(t_k^-) = \lim_{\varepsilon \longrightarrow 0^-} u(t_k - \varepsilon)$ represent the right and left limits of $u(t)$ at $t = t_k$, $(k = i, j)$.

We assume that $\sum_{i=1}^{n} \lambda_i u(\xi_i)^{\alpha-1} < 1$ and $\sum_{j=1}^{m} \delta_j v(\eta_j)^{\beta-1} < 1$.

This paper is arranged as follows. In Section 2, we recall some concepts and some fractional calculation laws and establish preparation results. In Section 3, we present the main results. Section 4 is devoted to an example of the main results.

2. Preliminaries

In this section, we recall some basic definitions and properties of the fractional calculus theory and preparation results.

Throughout, this paper denotes $J_0 = [0, t_1]$, $J_1 = (t_1, t_2]$, $\ldots$, $J_{p-1} = (t_{p-1}, t_p]$, $J_p = (t_p, 1]$, and we introduce the spaces: for $t_i \in (0,1)$ such that $t_1 < t_2 < \cdots < t_n$ and $I' = I\backslash\{t_1, t_2, \ldots, t_n\}$, define the space $X = \{u : [0,1] \longrightarrow \mathbb{R} : u \in C(I')$ and left $u(t_i^+)$ and right limit $u(t_i^-)$ exist and $u(t_i^-) - u(t_i), 1 \leq i \leq n\}$.

Then, clearly $(X, \|\cdot\|)$ is the Banach space under the norm $u = \max_{t \in [0,1]} |u(t)|$.

Similarly, for $t_j \in (0,1)$ such that $t_1 < t_2 < \cdots < t_m$ and $J' = J\backslash\{t_1, t_2, \ldots, t_m\}$, define the space $Y = \{v : [0,1] \longrightarrow \mathbb{R} : v \in C(J')$ and left $v(t_j^+)$ and right limit $v(t_j^-)$ exist and $v(t_j^-) - v(t_j), 1 \leq j \leq n\}$.

Then, clearly $(Y, \|\cdot\|)$ is the Banach space under the norm $v = \max_{t \in [0,1]} |v(t)|$.

Consequently, the product $X \times Y$ is a Banach space under the norms $\|(u, v)\| = \|u\| + \|v\|$ and $\|(u, v)\| = \max\{\|u\|, \|v\|\}$.

Definition 1 (see [13]). The fractional integral of the function $h \in L^1([a, b], \mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^{\alpha} h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) \mathrm{d}s, \tag{3}$$

where Γ is the gamma function.

Definition 2 (see [13]). For a function h given on the interval $[a, b]$, the Riemann–Liouville fractional-order derivative of h is defined by

$$\left({}^{C}D^{\alpha}_{a^+}h\right)(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n}\int_a^t\frac{(t-s)^{n-\alpha-1}}{\Gamma(\alpha)}(s)\mathrm{d}s, \quad (4)$$

where $n=[\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

Definition 3 (see [13]). For a function h given on the interval $[a,b]$, the Caputo fractional-order derivative of h is defined by

$$\left({}^{C}D^{\alpha}_{a^+}h\right)(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t\frac{(t-s)^{n-\alpha-1}}{\Gamma(\alpha)}h^{(n)}(s)\mathrm{d}s, \quad (5)$$

where $n=[\alpha]+1$ and $[\alpha]$ denotes the integer part of α.

3. Main Results

In this section, we will prove the existence of a mild solution for (2).

To obtain the existence of a mild solution, we will need the following assumptions:

(H_1): the function $u\longrightarrow(u/f_1(t,u,v))$ is increasing in $\mathbb{R}$ for every $t\in[0,t_1[$

(H_2): the function $v\longrightarrow(v/f_2(t,u,v))$ is increasing in $\mathbb{R}$ for every $t\in[0,t_1[$

(H_3): the functions f_i are continuous and bounded; that is, there exist positive numbers $L_i>0$ such that $|f_i(t,u,v)|\le L_i$ for all $(t,u,v)\in[0,1]\times\mathbb{R}\times\mathbb{R}\,(i=1,2)$

(H_4): for all $u,\overline{u},v,\overline{v}\in\mathbb{R}$ for all $t\in[0,1]$ there exist positive numbers $M_{g_i}>0$, such that

$$\left|g_i(t,u,v)-g_i(t,\overline{u},\overline{v})\right|\le M_{g_i}[|u-\overline{u}|+|v-\overline{v}|], \quad (i=1,2). \quad (6)$$

(H_5): there exist constants $A,B>0$ such that for all $u,\overline{u},v,\overline{v}\in\mathbb{R}$

$$\begin{aligned}&\left|I_i(u)-\left|I_i(\overline{u})\right|\right|\le A|u-\overline{u}|, && i=1,2,\dots,n,\\ &\left|I_j(v)-\left|I_j(\overline{v})\right|\right|\le B|v-\overline{v}|, && j=1,2,\dots,m.\end{aligned} \quad (7)$$

(H_6): for any $u,v\in C([0,1],R)$, there exist constants $K_\phi,K_\psi>0$, such that

$$\begin{aligned}\|\phi(u)-\phi(v)\|&\le K_\phi\|u-v\|,\\ \|\psi(u)-\psi(v)\|&\le K_\psi\|u-v\|.\end{aligned} \quad (8)$$

(H_7): for any $u,v\in C([0,1],R)$, there exist constants $M_\phi,M_\psi>0$ and $N_u,N_v>0$, such that

$$\begin{aligned}&\|\phi(u)\|\le M_\phi\|u-v\|,\\ &\|\varphi(u)\|\le M_\psi\|u-v\|,\\ &\left\|I_i(u)\right\|\le N_u\|u\|, \quad i=1,2,\dots,n,\\ &\left\|I_j(v)\right\|\le N_v\|v\|, \quad j=1,2,\dots,m.\end{aligned} \quad (9)$$

(H_8): there exist constants $C,D>0$, such that $|I_i(u_i)|\le C$, $i=1,2,\dots,n$ and $|I_j(v_j)|\le D$, $j=1,2,\dots,m$

(H_9): there exist constants $\rho,\mu>0$, such that $|\phi(u)|\le\rho$, $\forall u\in X$, $|\psi(v)|\le\mu$, and $\forall v\in Y$

(H_{10}): there exist constants $\rho_0,\delta_0>0$ and $\rho_i,\delta_i>0$ $(i=1,2)$, such that

$$\begin{aligned}\left|g_1(t,u,v)\right|&\le\rho_0+\rho_1\|u\|+\rho_2\|v\|,\\ \left|g_2(t,u,v)\right|&\le\delta_0+\delta_1\|u\|+\delta_2\|v\|,\end{aligned} \quad (10)$$

for all $(u,v)\in X\times Y$.

For brevity, let us set

$$\begin{aligned}\Delta_1&=L_1\left[K_\phi+nA+\frac{M_{g_1}}{\Gamma(\alpha+1)}\right],\\ \Delta_2&=L_2\left[K_\psi+mB+\frac{M_{g_2}}{\Gamma(\beta+1)}\right].\end{aligned} \quad (11)$$

Lemma 1. *Let $\alpha\in(0,1)$ and $h:[0,T_0]\longrightarrow\mathbb{R}$ be continuous. A function $u\in C([0,T_0],\mathbb{R})$ is a solution to the fractional integral equation:*

$$u(t)=u_0-\int_0^a\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s, \quad (12)$$

if and only if u is a solution to the following fractional Cauchy problems:

$$\begin{cases}D^\alpha u(t)=h(t)\ \text{a.e.} & t\in[0,T_0],\\ u(a)=u_0, & a>0.\end{cases} \quad (13)$$

Lemma 2. *Let us assume that hypotheses (H_1) and (H_3) hold. Let $\alpha\in(0,1)$ and $h:J\longrightarrow\mathbb{R}$ be continuous. A function u is a solution to the fractional integral equation:*

$$u(t)=f_1(t,u(t),v(t))\left[\phi(u)+\theta(t)\sum_{i=1}^{n}\frac{I_i(u(t_i^-))}{f_1(t,u(t_i),v(t_i))}+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s\right], \quad t\in[t_i,t_{i+1}], \quad (14)$$

where

$$\theta(t)=\begin{cases}0, & t\in[t_0,t_1],\\ 1, & t\overline{\in}[t_0,t_1[,\end{cases} \tag{15}$$

if and only if u is a solution of the following impulsive problem:

$$\begin{cases}D^{\alpha}\left(\dfrac{u(t)}{f_1(t,u(t),v(t))}\right)=h(t), & t\in[0,1],\ t\neq t_i,\ i=1,2,\dots,n,\quad 0<\alpha<1,\\ u(t_i^+)=u(t_i^-)+I_i(u(t_i^-)), & t_i\in(0,1),\ i=1,2,\dots,n,\\ \dfrac{u(0)}{f_1(0,u(0),v(0))}=\phi(u). & \end{cases} \tag{16}$$

Proof 1. Let us assume that u satisfies (16). If $t\in[t_0,t_1[$, then

$$D^{\alpha}\left(\frac{u(t)}{f_1(t,u(t),v(t))}\right)=h(t),\quad t\in[t_0,t_1[, \tag{17}$$

$$\frac{u(0)}{f_1(0,u(0),v(0))}=\phi(u). \tag{18}$$

Applying I^{α} on both sides of (17), we can obtain

$$\begin{aligned}\frac{u(t)}{f_1(t,u(t),v(t))}&=\frac{u(0)}{f_1(0,u(0),v(0))}+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s,\\ &=\phi(u)+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s,\end{aligned} \tag{19}$$

then

$$u(t)=f_1(t,u(t),v(t))\left[\phi(u)+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s\right]. \tag{20}$$

If $t\in[t_1,t_2[$, then

$$D^{\alpha}\left(\frac{u(t)}{f_1(t,u(t),v(t))}\right)=h(t),\quad t\in[t_1,t_2[, \tag{21}$$

$$u(t_1^+)=u(t_1^-)+I_1(u(t_1^-)). \tag{22}$$

According to Lemma 1 and the continuity of $f_1(t,u(t),v(t))$, we have

$$\begin{aligned}\frac{u(t)}{f_1(t,u(t),v(t))}&=\frac{u(t_1^+)}{f_1(t_1,u(t_1),v(t_1))}-\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s\\ &\quad+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s,\\ &=\frac{(u(t_1^-)+I_1(u(t_1^-)))}{f_1(t_1,u(t_1),v(t_1))}-\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s\\ &\quad+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s.\end{aligned} \tag{23}$$

Since

$$u(t_1^-)=f_1(t_1,u(t_1),v(t_1))\left[\phi(u)+\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s\right], \tag{24}$$

there exists

$$\begin{aligned}\frac{u(t)}{f_1(t,u(t),v(t))}&=\left(\phi(u)+\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s\right)\\ &\quad+\frac{I_1(u(t_1^-))}{f_1(t_1,u(t_1),v(t_1))}-\int_0^{t_1}\frac{(t_1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s\\ &\quad+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s,\\ &=\phi(u)+\frac{I_1(u(t_1^-))}{f_1(t_1,u(t_1),v(t_1))}\\ &\quad+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s,\end{aligned} \tag{25}$$

so

$$\begin{aligned}u(t)=f_1(t,u(t),v(t))\Bigg[\phi(u)&+\frac{I_1(u(t_1^-))}{f_1(t_1,u(t_1),v(t_1))}\\ &+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s\Bigg].\end{aligned} \tag{26}$$

If $t\in[t_2,t_3[$, then we have

$$\begin{aligned}\frac{u(t)}{f_1(t,u(t),v(t))}&=\frac{u(t_2^+)}{f_1(t_2,u(t_2),v(t_2))}-\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s\\ &\quad+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s,\\ &=\frac{(u(t_2^-)+I_2(u(t_2^-)))}{f_1(t_2,u(t_2),v(t_2))}-\int_0^{t_2}\frac{(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s\\ &\quad+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)\mathrm{d}s.\end{aligned} \tag{27}$$

For

$$u(t_2^-) = f_1(t_2, u(t_2), v(t_2))\left[\phi(u) + \frac{[u(t_1^-) + I_1(u(t_1^-))]}{f_1(t_1, u(t_1), v(t_1))} + \int_0^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\mathrm{d}s\right], \quad (28)$$

we have

$$\begin{aligned}\frac{u(t)}{f_1(t, u(t), v(t))} &= \phi(u) + \frac{(u(t_1^-) + I_1(u(t_1^-)))}{f_1(t_1, u(t_1), v(t_1))} + \int_0^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\mathrm{d}s + \frac{I_2(u(t_2^-))}{f_1(t_2, u(t_2), v(t_2))} - \int_0^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\mathrm{d}s + \int_0^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\mathrm{d}s, \\ &= \phi(u) + \frac{I_1(u(t_1^-))}{f_1(t_1, u(t_1), v(t_1))} + \frac{I_2(u(t_2^-))}{f_1(t_2, u(t_2), v(t_2))} + \int_0^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\mathrm{d}s,\end{aligned} \quad (29)$$

so

$$u(t) = f_1(t, u(t), v(t))\left(\phi(u) + \sum_{i=1}^{2} \frac{I_i(u(t_i^-))}{f_i(t_i, u(t_i), v(t_i))} + \int_0^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\mathrm{d}s\right). \quad (30)$$

If $t \in [t_i, t_{i+1}[$ $(i = 3, 4, \ldots, n)$, using the same method, we get

$$u(t) = f_1(t, u(t), v(t))\left(\phi(u) + \sum_{i=1}^{k} \frac{I_i(u(t_i^-))}{f_i(t_i, u(t_i), v(t_i))} + \int_0^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\mathrm{d}s\right). \quad (31)$$

Conversely, assume that u satisfies (14). If $t \in [t_0, t_1[$, then we have

$$u(t) = f_1(t, u(t), v(t))\left[\phi(u) + \int_0^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\mathrm{d}s\right]. \quad (32)$$

Then, divided by $f_1(t, u(t), v(t))$ and applying D^α on both sides of (32), (17) is satisfied.

Again, substituting $t = 0$ in (32), we have $(u(0)/f_1(0, u(0)v(0))) = \phi(u)$. Since $u \longrightarrow (u/f(t, u, v))$ is increasing in $\mathbb{R}$ for $t \in [t_0, t_1[$, the map $u \longrightarrow (u/f(t, u, v))$ is injective in $\mathbb{R}$. Then, we get (18).

If $t \in [t_1, t_2]$, then we have

$$u(t) = f_1(t, u(t), v(t))\left[\phi(u) + \frac{I_1(u(t_1^-))}{f_1(t_1, u(t_1), v(t_1))} + \int_0^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\mathrm{d}s\right]. \quad (33)$$

Then, divided by $f_1(t, u(t), v(t))$ and applying D^α on both sides of (33), (19) is satisfied. Again by (H_3), substituting $t = t_1$ in (32) and taking the limit of (33), (33) minus (32) gives (22).

If $t \in [t_i, t_{i+1}[$ $(i = 2, 3, \ldots, n)$, similarly we get

$$\begin{cases} D^\alpha\left(\dfrac{u(t)}{f_1(t, u(t), v(t))}\right) = h(t), & t \in [t_k, t_{k+1}[, \\ u(t_i^+) = u(t_i^-) + I_i(u(t_i^-)). \end{cases} \quad (34)$$

This completes the proof. □

Lemma 3. *Let g_1, g_2 are continuous, then $(u, v) \in X \times Y$ is a solution of (2) if and only if (u, v) is the solution of the integral equations:*

$$\begin{aligned} u(t) &= f_1(t, u(t), v(t))\left[\phi(u) + \theta(t)\sum_{i=1}^{n} \frac{I_i(u(t_i^-))}{f_1(t, u(t_i), v(t_i))} + \int_0^{t} \frac{(t - s)^{\alpha-1}}{\Gamma(\alpha)} g_1(t, u(t), v(t))\mathrm{d}s\right], \quad t \in [t_i, t_{i+1}], \\ v(t) &= f_2(t, u(t), v(t))\left[\psi(v) + \omega(t)\sum_{j=1}^{m} \frac{I_j(v(t_j^-))}{f_2(t, u(t_j), v(t_j))} + \int_0^{t} \frac{(t - s)^{\beta-1}}{\Gamma(\beta)} g_2(t, u(t), v(t))\mathrm{d}s\right], \quad t \in [t_j, t_{j+1}], \end{aligned} \quad (35)$$

where

$$\theta(t) = \begin{cases} 0, & t \in [t_0, t_1], \\ 1, & t \,\bar{\in}\, [t_0, t_1[, \end{cases} \qquad \omega(t) = \begin{cases} 0, & t \in [t_0, t_1], \\ 1, & t \,\bar{\in}\, [t_0, t_1[. \end{cases} \quad (36)$$

We define an operator $\Phi : X \times Y \longrightarrow X \times Y$ by

$$\Phi(u, v)(t) = (\Phi_1(u, v)(t), \Phi_2(u, v)(t)), \quad (37)$$

where

$$\Phi_1(u,v)(t) = f_1(t,u(t),v(t))\left[\phi(u) + \theta(t)\sum_{i=1}^{n}\frac{I_i(u(t_i^-))}{f_1(t,u(t_i),v(t_i))} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} g_1(s,u(s),v(s))ds\right], \tag{38}$$

$$\Phi_2(u,v)(t) = f_2(t,u(t),v(t))\left[\psi(v) + \omega(t)\sum_{j=1}^{m}\frac{I_j(v(t_j^-))}{f_2(t,u(t_j),v(t_j))} + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} g_2(s,u(s),v(s))ds\right].$$

Now, we are in a position to present our first result that deals with the existence and uniqueness of solutions for the problem (2). This result is based on Banach's contraction mapping principle.

Theorem 1. *Suppose that the condition $(H_1)-(H_7)$ holds and that $g_1, g_2 : [0,1]\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ are continuous functions. In addition, there exist positive constants λ_i, ζ_i, $i = 1,2$, such that*

$$\begin{aligned}|g_1(t,u,v) - g_1(t,\overline{u},\overline{v})| &\le \lambda_1|u-\overline{u}| + \zeta_1|v-\overline{v}|,\\ |g_2(t,u,v) - g_2(t,\overline{u},\overline{v})| &\le \lambda_2|u-\overline{u}| + \zeta_2|v-\overline{v}|.\end{aligned} \tag{39}$$

If $\max(\Delta_1,\Delta_2) < 1$, Δ_1 and Δ_2 are given by (11), then the impulsive coupled system (2) has a unique mild solution.

Proof 2. Let us set $\sup_{t\in J} g_1(t,0,0) = \kappa_1 < \infty$ and $\sup_{t\in J} |g_2(t,0,0)| = \kappa_1 < \infty$ and define a closed ball: $\overline{B} = \{(u,v) \in X\times Y : \|(u,v)\| \le r\}$, where

$$r \ge \max\left\{\frac{L_1\kappa_1}{1 - L_1\left(M_\phi + nN_u + (1/\Gamma(\alpha+1))(\lambda_1+\lambda_2)\right)}, \frac{L_2\kappa_2}{1 - L_2\left(M_\psi + nN_v + (1/\Gamma(\beta+1))(\zeta_1+\zeta_2)\right)}\right\}. \tag{40}$$

Then, we show that $\Phi\overline{B} \subset \overline{B}$. For $(u,v)\in\overline{B}$, we obtain

$$\begin{aligned}|\Phi_1(u,v)(t)| &\le L_1\left|\phi(u) + \theta(t)\sum_{i=1}^{n}\frac{I_i(u(t_i^-))}{f_1(t,u(t_i),v(t_i))} + \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g_1(s,u(s),v(s))ds\right|,\\ &\le L_1\left[M_\phi\|u\| + nN_u\|u\| + \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\left(|g_1(s,u(s),v(s)) - g_1(s,0,0)| + |g_1(s,0,0)|\right)ds\right],\\ &\le L_1\left[M_\phi\|u\| + nN_u\|u\| + \frac{1}{\Gamma(\alpha+1)}((\lambda_1+\lambda_2)\|u\| + \kappa_1)\right],\\ &\le L_1\left[\left(M_\phi + nN_u\right)r + \frac{1}{\Gamma(\alpha+1)}((\lambda_1+\lambda_2)r + \kappa_1)\right].\end{aligned} \tag{41}$$

Hence,

$$\|\Phi_1(u,v)(t)\| \le L_1\left[\left(M_\phi + nN_u\right)r + \frac{1}{\Gamma(\alpha+1)}((\lambda_1+\lambda_2)r + \kappa_1)\right]. \tag{42}$$

Working in a similar manner, one can find that

$$\|\Phi_2(u,v)(t)\| \le L_2\left[\left(M_\psi + nN_v\right)r + \frac{1}{\Gamma(\beta+1)}((\zeta_1+\zeta_2)r + \kappa_2)\right]. \tag{43}$$

From (42) and (43), it follows that $\|\Phi(u,v)\| \le r$.

Next, for (u,v), $(\overline{u},\overline{v}) \in X\times Y$, and for any $t\in[0,1]$, we have

$$\begin{aligned}|\Phi_1(u,v)(t) - \Phi_1(\overline{u},\overline{v})(t)| &= \Bigg|f_1(t,u(t),v(t))\left[\phi(u) + \theta(t)\sum_{i=1}^{n}\frac{I_i(u(t_i^-))}{f_1(t,u(t_i),v(t_i))} + \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g_1 \cdot (s,u(s),v(s))ds\right] - f_1(t,\overline{u}(t),\overline{v}(t))\\ &\quad\cdot\left[\phi(\overline{u}) + \theta(t)\sum_{i=1}^{n}\frac{I_i(\overline{u}(t_i^-))}{f_1(t,\overline{u}(t_i),\overline{v}(t_i))} + \int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g_1(s,\overline{u}(s),\overline{v}(s))ds\right]\Bigg|,\\ &\le L_1\left[K_\phi|u-\overline{u}| + nA|u-\overline{u}| + \frac{M_{g_1}}{\Gamma(\alpha+1)}(|u-\overline{u}| + |v-\overline{v}|)\right],\\ &\le L_1\left[K_\phi\|u-\overline{u}\| + nA\|u-\overline{u}\| + \frac{M_{g_1}}{\Gamma(\alpha+1)}(\|u-\overline{u}\| + \|v-\overline{v}\|)\right],\end{aligned} \tag{44}$$

which implies that

$$\left\|\Phi_1(u,v)-\Phi_1(\overline{u},\overline{v})\right\| \le L_1\left(K_\phi + nA + \frac{M_{g_1}}{\Gamma(\alpha+1)}\right)(\|u-\overline{u}\| + \|v-\overline{v}\|), = \Delta_1(\|u-\overline{u}\| + \|v-\overline{v}\|). \quad (45)$$

Similarly, we can show that

$$\left\|\Phi_2(u,v)-\Phi_2(\overline{u},\overline{v})\right\| \le \Delta_2(\|u-\overline{u}\| + \|v-\overline{v}\|). \quad (46)$$

From (45) and (46), we deduce that

$$\|\Phi(u,v)-\Phi(\overline{u},\overline{v})\| \le \max(\Delta_1,\Delta_2)(\|u-\overline{u}\| + \|v-\overline{v}\|). \quad (47)$$

□

In view of this condition $\max(\Delta_1,\Delta_2)<1$, it follows that Φ is a contraction. So, Banach's fixed point theorem applies, and hence the operator Φ has a unique fixed point. This, in turn, implies that the problem (2) has a unique solution on J. This completes the proof.

In our second result, we discuss the existence of solutions for the problem (2) by means of Leray–Schauder alternative.

For brevity, let us set

$$\mu_1 = \frac{L_1}{\Gamma(\alpha+1)}, \quad \mu_2 = \frac{L_2}{\Gamma(\beta+1)}, \quad (48)$$

$$\mu_0 = \min\{1-(\mu_1\rho_1+\mu_2\delta_1), 1-(\mu_1\rho_2+\mu_2\delta_2)\}. \quad (49)$$

Lemma 4 (Leray–Schauder alternative, see [14]). *Let $\mathcal{F}: G \longrightarrow G$ be a completely continuous operator (i.e., a map that is restricted to any bounded set in G is compact). Let $P(\mathcal{F}) = \{u \in G : u = \lambda\mathcal{F}u \text{ for some } 0<\lambda<1\}$. Then either the set $P(\mathcal{F})$ is unbounded or $\mathcal{F}$ has at least one fixed point.*

Theorem 2. *Let us assume that conditions $(H_1)-(H_3)$ and $(H_8)-(H_{10})$ hold. Furthermore, it is assumed that $\mu_1\rho_1+\mu_2\delta_1<1$ and $\mu_1\rho_2+\mu_2\delta_2<1$, where μ_1 and μ_2 are given by (48). Then, the boundary value problem (2) has at least one solution.*

Proof 3. We will show that the operator $\Phi: X\times Y \longrightarrow X\times Y$ satisfies all the assumptions of Lemma 4.

In the first step, we will prove that the operator Φ is completely continuous.

Clearly, it follows by the continuity of functions f_1, f_2, g_1, and g_2 that the operator Φ is continuous.

Let $S\subset X\times Y$ bounded. Then, we can find positive constants H_1 and H_2 such that $|g_1(t,u,v)|\le H_1$ abd $|g_2(t,u,v)|\le H_2$, $\forall (u,v)\in S$.

Thus, for any $u,v\in S$, we can get

$$\left|\Phi_1(u,v)(t)\right| \le L_1\left[\rho + \sum_{i=1}^{n} C + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}H_1 \mathrm{d}s\right], \le L_1\left[\rho + nC + \frac{H_1}{\Gamma(\alpha+1)}\right], \quad (50)$$

which yields

$$\left\|\Phi_1(u,v)\right\| \le L_1\left[\rho + nC + \frac{H_1}{\Gamma(\alpha+1)}\right]. \quad (51)$$

In a similar manner, one can show that

$$\left\|\Phi_2(u,v)\right\| \le L_2\left[\sigma + mD + \frac{H_2}{\Gamma(\beta+1)}\right]. \quad (52)$$

From the inequalities (51) and (52), we deduce that the operator Φ is uniformly bounded.

Now, we show that the operator Φ is equicontinuous.

We take $\tau_1,\tau_2\in J$ with $\tau_1<\tau_2$ and obtain

$$\begin{aligned}
&\left|\Phi_1(u(\tau_2),v(\tau_2)) - \Phi_1(u(\tau_1),v(\tau_1))\right| \\
&\le L_1\left|\left(\phi(u)+\theta(\tau_2)\sum_{i=1}^{n}\frac{I_i(u(t_i^-))}{f_1(t,u(t_i),v(t_i))} + H_1\int_0^{\tau_2}\frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s\right)\right. \\
&\quad \left. -\left(\phi(u)+\theta(\tau_1)\sum_{i=1}^{n}\frac{I_i(u(t_i^-))}{f_1(t,u(t_i),v(t_i))} + H_1\int_0^{\tau_1}\frac{(\tau_1-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s\right)\right| \\
&\le L_1\left(\left|(\theta(\tau_2)-\theta(\tau_1))\sum_{i=1}^{n}\frac{I_i(u(t_i^-))}{f_1(t,u(t_i),v(t_i))}\right| + H_1\left|\int_0^{\tau_2}\frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s - \int_0^{\tau_1}\frac{(\tau_1-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s\right|\right) \\
&\le L_1\left(\left|(\theta(\tau_2)-\theta(\tau_1))\sum_{i=1}^{n}\frac{I_i(u(t_i^-))}{f_1(t,u(t_i),v(t_i))}\right| + H_1\left|\int_0^{\tau_1}\frac{(\tau_1-s)^{\alpha-1}-(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s - \int_{\tau_2}^{\tau_2}\frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s\right|\right), \\
&\left|\Phi_2(u(\tau_2),v(\tau_2)) - \Phi_2(u(\tau_1),v(\tau_1))\right| \\
&\le L_2\left(\left|(\omega(\tau_2)-\omega(\tau_1))\sum_{j=1}^{m}\frac{I_j(u(t_j^-))}{f_2(t,u(t_j),v(t_j))}\right| + H_2\left|\int_0^{\tau_1}\frac{(\tau_1-s)^{\alpha-1}-(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s - \int_{\tau_2}^{\tau_2}\frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}s\right|\right),
\end{aligned} \quad (53)$$

which tend to 0 independently of (u, v). This implies that the operator $\Phi(u,v)$ is equicontinuous. Thus, by the above-mentioned findings, the operator $\Phi(u,v)$ is completely continuous.

In the next step, it will be established that the set $P = \{(u,v) \in X \times Y / (u,v) = \lambda\Phi(u,v), 0 \le \lambda \le 1\}$ is bounded.

Let $(u,v) \in P$. Then, we have $(u,v) = \lambda\Phi(u,v)$. Thus, for any $t \in [0,1]$, we can write

$$u(t) = \lambda\Phi_1(u,v)(t),\quad v(t) = \lambda\Phi_2(u,v)(t), \tag{54}$$

then

$$\begin{aligned}\|u\| &\le L_1\left[\rho + nC + \frac{1}{\Gamma(\alpha+1)}\left(\rho_0 + \rho_1\|u\| + \rho_2\|v\|\right)\right],\\ &\le L_1(\rho + nC) + \mu_1\left(\rho_0 + \rho_1\|u\| + \rho_2\|v\|\right),\end{aligned} \tag{55}$$

which implies that

$$\begin{aligned}\|v\| &\le L_2\left[\sigma + mD + \frac{1}{\Gamma(\beta+1)}\left(\delta_0 + \delta_1\|u\| + \delta_2\|v\|\right)\right],\\ &\le L_2(\sigma + mD) + \mu_2\left(\delta_0 + \delta_1\|u\| + \delta_2\|v\|\right).\end{aligned} \tag{56}$$

In consequence, we have

$$\begin{aligned}\|u\| + \|(v)\| \le{}& \left(L_1(\rho + nC) + L_2(\sigma + mD) + \mu_1\rho_0 + \mu_2\delta_0\right)\\ &+ \left(\mu_1\rho_1 + \mu_2\delta_1\right)\|u\| + \left(\mu_1\rho_2 + \mu_2\delta_2\right)\|v\|,\end{aligned} \tag{57}$$

which in view of (49), can be expressed as

$$\|(u,v)\| \le \frac{L_1(\rho + nC) + L_2(\sigma + mD) + \mu_1\rho_0 + \mu_2\delta_0}{\mu_0}. \tag{58}$$

This shows that the set is bounded. Hence, all the conditions of Lemma 4 are satisfied and consequently the operator Φ has at least one fixed point, which corresponds to a solution of the system (2). This completes the proof. □

4. Example

Consider the following coupled system of hybrid fractional differential equations:

$$\begin{cases} {}^{C}D^{1/2}\left(\dfrac{u(t)}{\left(t + \sqrt{u(t)+v(t)}\right)/(40+t^2)}\right) = \dfrac{e^{-t} + |\sin u(t)| + |\cos v(t)|}{20}, & t \in [0,1]\backslash\{t_1\},\ 0 < \dfrac{1}{2} < 1,\\ u(t_1^+) = u(t_1^-) + (-2u(t_1^-)), & t_1 \ne 0, 1,\\ {}^{C}D^{1/2}\left(\dfrac{v(t)}{\left(t^2 + \sqrt{u(t)-v(t)}\right)/(32+t)}\right) = \dfrac{e^{-2t} + |\sin(2u(t))| + \left|\cos^2(v(t))\right|}{20}, & t \in [0,1]\backslash\{t_1\},\ 0 < \dfrac{1}{2} < 1,\\ v(t_1^+) = v(t_1^-) + (-2v(t_1^-)), & t_1 \ne 0, 1,\\ \dfrac{u(0)}{f_1(0,u(0),v(0))} = \displaystyle\sum_{i=1}^{n} c_i u(t_i),\ \dfrac{v(0)}{f_2(0,u(0),v(0))} = \displaystyle\sum_{j=1}^{m} d_j u(t_j), \end{cases} \tag{59}$$

where $f_1(t,u,v) = ((t + \sqrt{u(t)+v(t)})/(40+t^2))$, $f_2(t,u,v) = ((t^2 + \sqrt{u(t)-v(t)})/(32+t))$, $g_1(t,u,v) = ((e^{-t} + |\sin u(t)| + |\cos v(t)|)/40)$, and $g_2(t,u,v) = ((e^{-2t} + |\sin(2u(t))| + |\cos^2(v(t))|)/20)$.

Note that

$$\begin{aligned} &\left|g_1(t,u_1,v_1) - g_1(t,u_2,v_2)\right| \le \frac{1}{40}\left|u_2 - u_1\right| + \frac{1}{40}\left|v_2 - v_1\right|,\\ &\left|g_2(t,u_1,v_1) - g_2(t,u_2,v_2)\right| \le \frac{1}{20}\left|u_2 - u_1\right| + \frac{1}{20}\left|v_2 - v_1\right|,\\ &\forall t \in [0,1], u_1, u_2, v_1, v_2 \in \mathbb{R},\\ &\Delta_1 = L_1\left[K_\phi + nA + \frac{M_{g_1}}{\Gamma(\alpha+1)}\right] \simeq 0.3354687 < 1,\\ &\Delta_2 = L_2\left[K_\psi + mB + \frac{M_{g_2}}{\Gamma(\beta+1)}\right] \simeq 0.2548789 < 1. \end{aligned} \tag{60}$$

Thus, all the assumptions in Theorem 2 are satisfied, and our results can be applied to the problem (59).

Data Availability

We make sure that this article is open to the public and that it is unrestricted to use, distribute, and reproduce in any medium, provided that the original work is stated correctly.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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