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*Research Article*

# Application of Optimal Homotopy Asymptotic Method to Some Well-Known Linear and Nonlinear Two-Point Boundary Value Problems

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The objective of this paper is to obtain an approximate solution for some well-known linear and nonlinear two-point boundary value problems. For this purpose, a semianalytical method known as optimal homotopy asymptotic method (OHAM) is used. Moreover, optimal homotopy asymptotic method does not involve any discretization, linearization, or small perturbations and that is why it reduces the computations a lot. OHAM results show the effectiveness and reliability of OHAM for application to two-point boundary value problems. The obtained results are compared to the exact solutions and homotopy perturbation method (HPM).

## 1. Introduction

Two-point boundary value problems (TPBVP) have many applications in the field of science and engineering [1, 2]. These problems arise in many physical situations like modeling of chemical reactions, heat transfer, viscous fluids, diffusions, deflection of beams, the solution of optimal control problems, etc. Due to the wide applications and importance of boundary value problems (BVP) in science and engineering we need solutions to these problems.

There are many techniques available for the solution of-of BVP like Adomian Decomposition Method (ADM) [3–7], Extended Adomian Decomposition Method (EADM)[8], Differential Transformation Method (DTM) [9], Variational Iteration Method (VIM) [10], Perturbation methods(PMs) [1, 11–13], and so on. Perturbation methods are easy to solve but they require small parameters which are sometimes not an easy task. Recently V. Marinca et al. presented optimal homotopy asymptotic method (OHAM) [14] for the solution of BVP, which did not require small parameters. The method can also be applied to solve the stationary solution of some partial differential equations, e.g., gKdv equation, nonlinear parabolic problems, and so on [15–20]. In OHAM, the concept of homotopy is used together with the perturbation techniques. Here, OHAM is applied to TPBVP to check the applicability of OHAM for TPBVP.

## 2. Basics of OHAM

Let us take the BVP whose general form is the following:

$$\mathscr{L}(w(\xi)) + \mathbb{N}(w(\xi)) + F(\xi) = 0, \quad B\left(w, \frac{dw}{d\xi}\right) \tag{1}$$

where $\mathscr{L}$ is a linear operator, $\xi$ is independent variable, $\mathbb{N}$ is the nonlinear operator, $F(\xi)$ is a known function, and $B$ is a boundary operator.

Homotopy on OHAM can be constructed as

$$\begin{gathered}(1-þ)(\mathscr{L}(\varphi(\xi,þ)+F(\xi)) \\ = H(þ)(\mathscr{L}(\varphi(\xi,þ)+F(\xi)+\mathbb{N}(\varphi(\xi,þ))), \\ B\left(\varphi(\xi,þ), \frac{\partial\varphi(\xi,þ)}{\partial\xi}\right)\end{gathered} \tag{2}$$

where $\text{þ} \in [0,1]$ is an embedding parameter, $\varphi(\xi,\text{þ})$ is an unknown function, $H(\text{þ})$ is a nonzero auxiliary function for $\text{þ} \neq 0$, and $H(\text{þ})$ is of the form

$$H(\text{þ}) = \text{þ}\mathbb{C}_1 + \text{þ}^2\mathbb{C}_2 + \text{þ}^3\mathbb{C}_3 + \cdots \quad (3)$$

Clearly when $\text{þ} = 0$ then $H(0) = 0$. And obviously, when $\text{þ} = 0$ then $\varphi(\xi,0) = w_0(\xi)$. When $\text{þ} = 1$ then $\varphi(\xi,1) = w(\xi)$. So as þ increases from 0 to 1, the solution $\varphi(\xi,\text{þ})$ varies from $w_0(\xi)$ to the exact solution $w(\xi)$, where $w_0(\xi)$ is obtained from (2) for $\text{þ} = 0$

$$\mathscr{L}(w_0(\xi)) + \digamma(\xi) = 0 \quad (4)$$

The proposed solution of (1) will be of the form

$$\varphi(\xi,\text{þ},\mathbb{C}_i) = w_0(\xi) + \sum_{k\geq 1} w_k(\xi,\mathbb{C}_i)\,\text{þ}^k, \quad i = 1,2,3,\ldots \quad (5)$$

Substituting this value of $\varphi(\xi,\text{þ},\mathbb{C}_i)$ into (1), after some calculations, we can obtain the governing equations of $w_0(\xi)$ by using (4) and $w_k(\xi)$, that is,

$$\mathscr{L}(w_1(\xi)) = \mathbb{C}_1\mathbb{N}_0(w_0(\xi)), \qquad B\left(w_1, \frac{dw_1}{d\xi}\right) = 0 \quad (6)$$

$$\begin{aligned}&\mathscr{L}(w_k(\xi) - w_{k-1}(\xi)) \\ &\quad = \mathbb{C}_k\mathbb{N}_0(w_0(\xi)) + \sum_{i=1}^{k-1}\mathbb{C}_i\mathscr{L}(w_{k-1}(\xi) \\ &\qquad + \mathbb{N}_{k-1}(w_0(\xi), w_1(\xi), w_2(\xi), \ldots, w_{k-1}(\xi)), \\ &\qquad\qquad k = 2,3,4,\ldots,\end{aligned} \quad (7)$$

$$B\left(w_k, \frac{dw_k}{d\xi}\right) = 0,$$

where $\mathbb{N}_m(w_0(\xi), w_1(\xi), w_2(\xi), \ldots, w_m(\xi))$ is the coefficient of $\text{þ}^m$ in the series expansion of $\mathbb{N}(\varphi(\xi,\text{þ},\mathbb{C}_i))$ with respect to the embedding parameter þ. And

$$\begin{aligned}\mathbb{N}(\varphi(\xi,\text{þ},\mathbb{C}_i)) &= \mathbb{N}_0(w_0(\xi)) \\ &\quad + \sum_{m\geq 1}\mathbb{N}_m(w_0, w_1, w_2, \ldots, w_m)\,\text{þ}^m, \\ &\qquad i = 1,2,3,\ldots,m\end{aligned} \quad (8)$$

where $\varphi(\xi,\text{þ},\mathbb{C}_i)$ is given by (5). The convergence of series (5) depends on the convergence of the constants $\mathbb{C}_i's$, if these constants are convergent at $\text{þ} = 1$, then the solution becomes

$$w(\xi,\mathbb{C}_i) = w_0(\xi) + \sum_{k\geq 1}(w_k(\xi,\mathbb{C}_i)). \quad (9)$$

Generally, the $mth$ order solution of the problem can be obtained in the form

$$w^{(m)}(\xi,\mathbb{C}_i) = w_0(\xi) + \sum_{k=1}^{m}(w_k(\xi,\mathbb{C}_i)), \qquad i = 1,2,3,\ldots,m \quad (10)$$

Putting this solution in (1) we get the following residual:

$$\begin{aligned}\mathfrak{R}(\xi,\mathbb{C}_i) &= \mathscr{L}\left(w^{(m)}(\xi,\mathbb{C}_i) + \digamma(\xi)\right) \\ &\quad + \mathbb{N}\left(w^{(m)}(\xi,\mathbb{C}_i)\right), \quad i = 1,2,3,\ldots,m\end{aligned} \quad (11)$$

If $\mathfrak{R}(\xi,\mathbb{C}_i) = 0$, then the solution is going to be exact, but generally, such a situation does not arise in nonlinear problems but the functional defined below can be minimized

$$J(\xi,\mathbb{C}_i) = \int_{x_0}^{x_1}\mathfrak{R}^2(\xi,\mathbb{C}_i)\,d\xi, \quad (12)$$

where $x_0$ and $x_1$ are two constants depending on the given problem. The values of $\mathbb{C}_i's$ can be optimally found by the condition

$$\frac{\partial J}{\partial \mathbb{C}_1} = \frac{\partial J}{\partial \mathbb{C}_2} = \cdots = \frac{\partial J}{\partial \mathbb{C}_m} = 0 \quad (13)$$

After knowing these constants, the solution (10) is well determined.

## 3. Examples

To check the applicability of OHAM for TPBVP, in this section four examples of TPBVP are presented in which one example is linear and the remaining are nonlinear.

*3.1. Example 1.* Let us consider the linear problem [1] of second order

$$\begin{aligned}&w''(\xi) = w'(\xi) - e^{(\xi-1)} - 1, \quad 0 < \xi < 1, \\ &w(0) = 0, \\ &w(1) = 1\end{aligned} \quad (14)$$

The exact solution of problem (14) is $\xi(1 - e^{\xi-1})$. Now according to OHAM $\mathscr{L}(w_0(\xi)) = w''(\xi) - w'(\xi)$, the nonlinear part $\mathbb{N}(w(\xi)) = 0$ and $\digamma(\xi) = e^{(\xi-1)} + 1$.

The zeroth-order problem is

$$\begin{aligned}w_0''(\xi) - w_0'(\xi) &= 1 + e^{\xi-1}, \\ w_0(0) &= 0, \\ w_0(1) &= 1\end{aligned} \quad (15)$$

The solution of (15) is

$$w_0(\xi) = \frac{\left(e - e^{\xi}\right)\xi}{\xi} \quad (16)$$

The first-order problem is

$$\begin{aligned}w_1''(\xi) - w_1'(\xi) &= -1 - e^{\xi-1} - \mathbb{C}_1 - \mathbb{C}_1 e^{\xi-1} + w_0'(\xi) \\ &\quad + \mathbb{C}_1 w_0'(\xi) - w_0''(\xi) \\ &\quad - \mathbb{C}_1 w_0''(\xi), \\ w_1(0) &= 0, \\ w_1(1) &= 0.\end{aligned} \quad (17)$$

Table 1: Comparison of the third-order OHAM solution with the exact solution and HPM.

| $\xi$ | **OHAM Solution** ($w^{(2)}$) | **Exact** | **HPM [1]** | $\lvert w^{(2)} - Exact \rvert$ |
|---|---|---|---|---|
| 0.1 | 0.059343 | 0.059343 | 0.05934820 | $1.38778 \times 10^{-17}$ |
| 0.3 | 0.151024 | 0.151024 | 0.15103441 | $2.77556 \times 10^{-17}$ |
| 0.5 | 0.196735 | 0.196735 | 0.19673826 | $2.77556 \times 10^{-17}$ |
| 0.7 | 0.181427 | 0.181427 | 0.18142196 | $5.55112 \times 10^{-17}$ |
| 0.9 | 0.0856463 | 0.0856463 | 0.08564186 | $5.55112 \times 10^{-17}$ |

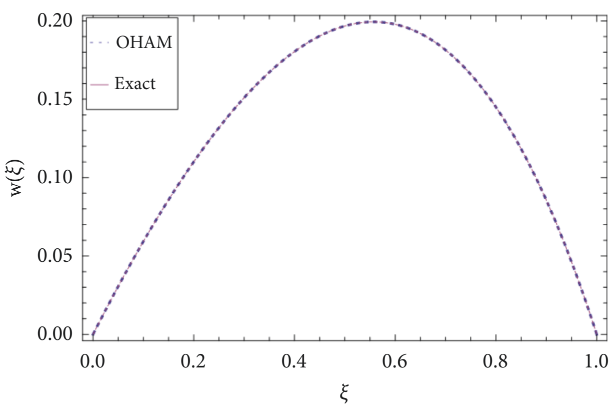


Figure 1: Comparison between exact solution (dashed line) and approximate solution (dotted line) for example 1.

The solution of (17) is

$$w_1(\xi) = 0 \tag{18}$$

The second-order problem is

$$\begin{aligned} w_2''(\xi) &= -\mathbb{C}_2 - \exp(\xi - 1)\,\mathbb{C}_2 + \mathbb{C}_2 w_0'(\xi) + w_1'(\xi) \\ &\quad + \mathbb{C}_1 w_1'(\xi) - w_2'(\xi) - \mathbb{C}_2 w_0''(\xi) - w_1''(\xi) \\ &\quad - \mathbb{C}_1 w_1''(\xi), \\ w_2(0) &= 0, \\ w_2(1) &= 0 \end{aligned} \tag{19}$$

The solution of (19) is

$$w_2(\xi) = 0 \tag{20}$$

And the third-order approximate solution of the bvp (14) is as follows:

$$w^{(2)}(\xi) = w_0(\xi) + w_1(\xi) + w_2(\xi) \tag{21}$$

$$w^{(2)}(\xi) = \frac{\left(e - e^{\xi}\right)\xi}{\xi} \tag{22}$$

Table 1 shows the comparison between the exact solution and the approximate solution obtained by OHAM. Figure 1 of the solution also shows well agreement with the exact solution.

*3.2. Example 2.* Consider the nonlinear two-point boundary value problem [1] of the type

$$\begin{aligned} w''(\xi) &= w^3(\xi) - w(\xi)\,w'(\xi), \quad \xi \in [1,2], \\ w(1) &= \frac{1}{2}, \\ w(2) &= \frac{1}{3} \end{aligned} \tag{23}$$

According to OHAM $\mathscr{L}(w(x)) = w''(\xi)$ and $\mathbb{N}(w(\xi)) = \mathrm{u}(\xi)\mathrm{u}'(\xi) - w^3(\xi)$, while $\mathcal{F}(\xi) = 0$. The exact solution of (23) is $1/(\xi + 1)$. Now proceeding with the same lines as above we have the following zeroth-order problem:

$$\begin{aligned} w_0''(\xi) &= 0, \\ w_0(1) &= \frac{1}{2}, \\ w_0(2) &= \frac{1}{3} \end{aligned} \tag{24}$$

The solution of (24) is

$$w_0(\xi) = \frac{4 - \xi}{6} \tag{25}$$

Now the first-order problem is

$$\begin{aligned} w_1''(\xi) &= \mathbb{C}_1 w_0^3(\xi) - \mathbb{C}_1 w_0(\xi)\,w_0'(\xi) - w_0''(\xi) \\ &\quad - \mathbb{C}_1 w_0''(\xi) \end{aligned} \tag{26}$$

$$\begin{aligned} w_1(1) &= 0, \\ w_1(2) &= 0 \end{aligned} \tag{27}$$

The solution of (26) is

$$\begin{aligned} &w_1(\xi) \\ &= \frac{-930\mathbb{C}_1 + 1649\xi\mathbb{C}_1 - 880\xi_1^2\mathbb{C} + 180\xi_1^3\mathbb{C} - 20\xi_1^4\mathbb{C} + \xi_{5\mathbb{C}1}}{4320} \end{aligned} \tag{28}$$

The second-order problem is

$$\begin{aligned} w_2''(\xi) &= \mathbb{C}_2 w_0^3(\xi) + 3\mathbb{C}_1 w_0^2(\xi)\,w_1(\xi) \\ &\quad - \mathbb{C}_2 w_0(\xi)\,w_0'(\xi) - \mathbb{C}_1 w_1(\xi)\,w_0'(\xi) \\ &\quad - \mathbb{C}_1 w_0(\xi)\,w_1'(\xi) - \mathbb{C}_2 w_0''(\xi) - w_1''(\xi) \\ &\quad - \mathbb{C}_1 w_1''(\xi), \\ w_2(1) &= 0, \\ w_2(2) &= 0. \end{aligned} \tag{29}$$

Table 2: Comparison of second-order OHAM solution with the exact solution for example 2.

| $\xi$ | **OHAM Solution** ($w^{(3)}$) | **Exact** | $\lvert w^{(3)} - Exact\rvert$ |
|---|---|---|---|
| 1.1 | 0.47619 | 0.47619 | $2.0597 \times 10^{-7}$ |
| 1.3 | 0.434783 | 0.434783 | $5.38284 \times 10^{-7}$ |
| 1.5 | 0.400001 | 0.4 | $1.39261 \times 10^{-6}$ |
| 1.7 | 0.370371 | 0.37037 | $6.7426 \times 10^{-7}$ |
| 1.9 | 0.344828 | 0.344828 | $8.10196 \times 10^{-8}$ |

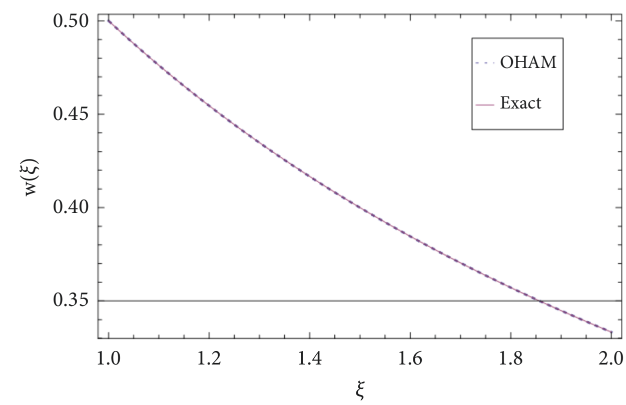

Figure 2: Comparison between exact solution (dashed line) and approximate solution (dotted line) for example 2.

The solution of (29) is

$$\begin{aligned} w_2(\xi) = {} & \frac{1}{130636800}(\xi-2)(\xi-1)\{30240\left(-465+\xi(127+(\xi-17)\xi)\mathbb{C}_1\right. \\ & -(8985375+\xi(4253423+\xi(-3664113+\xi(1100519+5\xi(-36795+\xi(3709+7(-33+\xi)\xi))))))\mathbb{C}_1^2 \\ & \left.+30240(-465+\xi(127+(-17+\xi)\xi))\mathbb{C}_2\right) \end{aligned} \tag{30}$$

The third-order problem is

$$\begin{aligned} & \mathbb{C}_3 w_0^3(\xi) + 3\mathbb{C}_2 w_0^2(\xi) w_1(\xi) + 3\mathbb{C}_1 w_0(\xi) w_1^2(\xi) \\ & \quad + 3\mathbb{C}_1 w_0^2(\xi) w_2(\xi) - 3\mathbb{C}_3 w_0(\xi) w_0'(\xi) \\ & \quad - \mathbb{C}_2 w_1(\xi) w_0'(\xi) - \mathbb{C}_1 w_2(\xi) w_0'(\xi) \\ & \quad - \mathbb{C}_2 w_0(\xi) w_1'(\xi) - \mathbb{C}_1 w_1(\xi) w_1'(\xi) \\ & \quad - \mathbb{C}_1 w_0(\xi) w_2'(\xi) - \mathbb{C}_3 w_0''(\xi) - \mathbb{C}_2 w_1''(\xi) \\ & \quad - w_2''(\xi) - \mathbb{C}_1 w_2''(\xi) + w_3''(\xi) = 0 \end{aligned} \tag{31}$$

The solution of the third-order problem results a large output, therefore not included here.

Now the third-order approximate solution is

$$w^{(3)}(\xi) = w_0(\xi) + w_1(\xi) + w_2(\xi) + w_3(\xi) \tag{32}$$

$\mathbb{C}_i's$ has the following values and then substituting in the above solution we will get the approximate solution. $w^{(3)}(\xi)$ is given in Appendix (A.1).

$$\begin{aligned} \mathbb{C}_1 &= -0.9637924142971654, \\ \mathbb{C}_2 &= -0.0002296939939480446, \\ \mathbb{C}_3 &= -0.000014314891134337846, \end{aligned} \tag{33}$$

The solution at the points given in Table 2 and the graph of the solution is shown in Figure 2. Here it is third-order OHAM solution while the HPM [1] gives the accuracy up to 9 decimal places in 7th order.

### 3.3. *Example 3.*

Now we consider higher order TPBVP of order four. The problem is

$$\frac{d^4 w(\xi)}{d\xi^4} = w^2(\xi) + F(\xi), \quad 0 \le \xi \le 1 \tag{34}$$

with the boundary conditions $w(0) = 0$, $w'(0) = 0$, $w(1) = 1$, and $w'(1) = 1$.

Table 3: Comparison of second-order OHAM solution with the exact solution for example 3.

| $\xi$ | **OHAM Solution** $(w^{(2)})$ | **Exact** | $\lvert w^{(2)} - Exact\rvert$ |
|---|---|---|---|
| 0.2 | 0.077119 | 0.0771200 | $6.152306 \times 10^{-11}$ |
| 0.4 | 0.279039 | 0.2790400 | $1.314346 \times 10^{-10}$ |
| 0.6 | 0.538559 | 0.5385600 | $1.001054 \times 10^{-10}$ |
| 0.8 | 0.788479 | 0.7884800 | $2.415356 \times 10^{-11}$ |

Where $\mathscr{L}(w(\xi)) = d^4w(\xi)/dx^4$, $\mathbb{N}(w(\xi)) = w^2(\xi)$, and $F(\xi) = -\xi^{10} + 4\xi^9 - 4\xi^8 - 4\xi^7 + 8\xi^6 - 4\xi^4 + 120\xi - 48$, the exact solution of problem (34) is $w_{exact} = \xi^5 - 2\xi^4 + 2\xi^2$. After solving this by the method described in Section 2, we have the following zeroth-order problem:

$$48 - 120\xi + 4\xi^4 - 8\xi^6 + 4\xi^7 + 4\xi^8 - 4\xi^9 + \xi^{10} + \frac{d^4 w_0(\xi)}{d\xi^4} = 0 \quad (35)$$

$$w_0(0) = 0, \quad w_0'(0) = 0, \quad w_0(1) = 1, \quad w_0'(1) = 1 \quad (36)$$

The solution to (35) is

$$w_0(\xi) = \frac{1}{1081080}\left(2155683\xi^2 + 8038\xi^3 - 2162160\xi^4 + 1081080\xi^5 - 2574\xi^8 + 1716\xi^{10} - 546\xi^{11} - 364\xi^{12} + 252\xi^{13} - 45\xi^{14}\right) \quad (37)$$

The first-order problem is

$$\begin{aligned} &-48 + 120\xi - 4\xi^4 + 8\xi^6 - 4\xi^7 - 4\xi^8 + 4\xi^9 - \xi^{10} \\ &- 48\mathbb{C}_1 + 120\xi\mathbb{C}_1 - 4\xi_1^4\mathbb{C} + 8\xi^6\mathbb{C}_1 - 4\xi^7\mathbb{C}_1 \\ &- 4\xi^8\mathbb{C}_1 + 4\xi^9\mathbb{C}_1 - \xi^{10}\mathbb{C}_1 + \mathbb{C}_1 w_0^2(\xi) \\ &- (1 + \mathbb{C}_1)\frac{d^4 w_0(\xi)}{d\xi^4} + \frac{d^4 w_1(\xi)}{d\xi^4} = 0 \end{aligned} \quad (38)$$

$$w_1(0) = 0, \quad w_1'(0) = 0, \quad w_1(1) = 0, \quad w_1'(1) = 0. \quad (39)$$

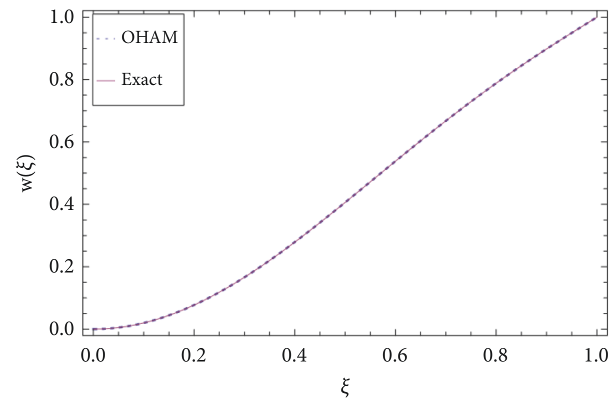


Figure 3: Comparison between exact solution (dashed line) and approximate solution (dotted line) for example 3.

The second-order problem is

$$\begin{aligned} &-48\mathbb{C}_2 + 120\xi\mathbb{C}_2 - 4\xi^4\mathbb{C}_2 + 8\xi^6\mathbb{C}_2 - 4\xi^7\mathbb{C}_2 - 4\xi^8\mathbb{C}_2 \\ &+ 4\xi^9\mathbb{C}_2 - \xi^{10}\mathbb{C}_2 + \mathbb{C}_2 w_0^2(\xi) + 2\mathbb{C}_1 w_0(\xi) w_1(\xi) \\ &- \mathbb{C}_2 w_0^{(4)}(\xi) - w_1^{(4)}(\xi) - \mathbb{C}_1 w_1^{(4)}(\xi) + w_2^{(4)}(\xi) = 0 \end{aligned} \quad (40)$$

$$w_2(0) = 0, \quad w_2'(0) = 0, \quad w_2(1) = 0, \quad w_2'(1) = 0. \quad (41)$$

The solutions of problem (38) and (40) are very large; therefore we did not write it here. The constants $\mathbb{C}_1$ and $\mathbb{C}_2$ have the values $-1.0011320722175725$ and $-1.079468959963785\times 10^{-6}$, respectively. Table 3 and Figure 3 show a good agreement with the exact values. The approximate solution $w_2(\xi)$ is given in Appendix (A.2).

*3.4. Example 4.* At last, consider the second-order nonlinear TPBVP[1]

$$w''(\xi) = w^2(\xi) + 2\pi^2\cos(2\pi\xi) - \sin^2(2\pi\xi), \quad 0 \le \xi \le 1 \quad (42)$$

$$w(0) = 0, \quad w(1) = 0. \quad (43)$$

Table 4: Comparison of third-order OHAM solution with the exact solution.

| $\xi$ | **OHAM Solution** $(w^{(3)})$ | **Exact** | $\lvert w^{(3)} - Exact\rvert$ |
|---|---|---|---|
| 0.1 | 0.0954915 | 0.0954915 | $5.59262 \times 10^{-9}$ |
| 0.3 | 0.654508 | 0.654508 | $1.23779 \times 10^{-8}$ |
| 0.5 | 0.999999 | 1. | $1.51565 \times 10^{-8}$ |
| 0.7 | 0.654508 | 0.654508 | $1.23779 \times 10^{-8}$ |
| 0.9 | 0.0954915 | 0.0954915 | $5.59262 \times 10^{-9}$ |

The exact solution of (42) is $\sin^2(\pi\xi)$. Solving (42) by the method depicted in Section 2, we have the following zeroth order problem:

$$-2\pi^2\cos(2\pi\xi) + \sin^4(\pi\xi) + w_0''(\xi) = 0, \quad w_0(0) = 0, \quad w_0(1) = 0 \tag{44}$$

The solution of (44) is given by

$$w_0(\xi) = \frac{1}{128\pi^2}\left(15 - 64\pi^2 + 24\pi^2\xi - 24\pi^2\xi^2 - 16\cos(2\pi\xi) - 64\pi^2\cos(2\pi\xi) + \cos(4\pi\xi)\right). \tag{45}$$

The first-, second-, and third-order problems are given in (46), (47), and (48) respectively.

$$\begin{aligned}&2\pi^2\cos(2\pi\xi)(1+\mathbb{C}_1) - \sin^4(\pi\xi)(1+\mathbb{C}_1) + \mathbb{C}_1 w_0^2(\xi) - w_0''(\xi) - \mathbb{C}_1 w_0''(\xi) + w_1''(\xi) = 0,\\ &w_1(0) = 0,\\ &w_1(1) = 0\end{aligned} \tag{46}$$

$$\begin{aligned}&\left\{2\pi^2\cos(2\pi\xi) - \sin^4(\pi\xi)\right\}\mathbb{C}_2 + \mathbb{C}_2 w_0^2(\xi) + 2\mathbb{C}_1 w_0(\xi) w_1(\xi) - \mathbb{C}_2 w_0''(\xi) - w_1''(\xi) - \mathbb{C}_1 w_1''(\xi) + w_2''(\xi) = 0,\\ &w_2(0) = 0,\\ &w_2(1) = 0\end{aligned} \tag{47}$$

$$\begin{aligned}&\left\{2\pi^2\cos(2\pi\xi) - \sin^4(\pi\xi)\right\}\mathbb{C}_3 + \mathbb{C}_3 w_0^2(\xi) + 2\mathbb{C}_2 w_0(\xi) w_1(\xi) + \mathbb{C}_1 w_1^2(\xi) + 2\mathbb{C}_1 w_0(\xi) w_2(\xi) + \mathbb{C}_3 w_0''(\xi) - \mathbb{C}_2 w_1''(\xi) - w_2''(\xi) - \mathbb{C}_1 w_2''(\xi) + w_3''(\xi) = 0\end{aligned} \tag{48}$$

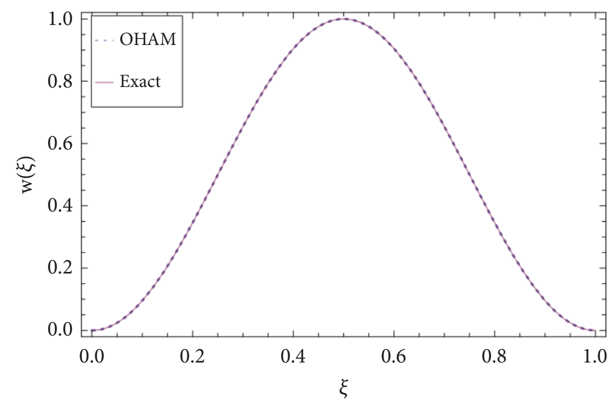


Figure 4: Comparison between exact solution (dashed line) and approximate solution (dotted line) for example 4.

$$w_3(0) = 0, \quad w_3(1) = 0 \tag{49}$$

The solutions of problem (46), (47), and (48) are very large and therefore cannot be written here but the table of values and the graph are shown in Table 4 and Figure 4, respectively. The approximate solution $w^{(3)}(\xi)$ is written in Appendix (A.3). The values of the constants $\mathbb{C}_i's$ can be found by (13) which are given as follows:

$$\mathbb{C}_1 = -0.9030981665320986, \quad \mathbb{C}_2 = -0.00569345107292796, \quad \mathbb{C}_3 = 0.00021560218552318884 \tag{50}$$

## 4. Conclusion

This paper reveals that OHAM is a very strong method for solving TPBVP and gives us a more accurate solution as compared to other methods. In these examples only second- and third-order solution gives us the accuracy up to 8 or 10 decimal places; therefore it is concluded that this method converges very fast to the exact solution and in some problems like example 1 it gives us the exact solution. The plots and tables show well agreement with the exact solution.

## Appendix

$$
\begin{aligned}
\overset{(3)}{w}(\xi) = {} & \frac{4-\xi}{6} + \frac{1}{4320}\left(-930\mathbb{C}_1 + 1649\xi\mathbb{C}_1 - 880\xi^2\mathbb{C}_1 + 180\xi^3\mathbb{C}_1 - 20\xi^4\mathbb{C}_1 + \xi^5\mathbb{C}_1\right) + \frac{1}{130636800}\left(-28123200\mathbb{C}_1\right. \\
& + 49865760\xi\mathbb{C}_1 - 26611200\xi^2\mathbb{C}_1 + 5443200\xi^3\mathbb{C}_1 - 604800\xi^4\mathbb{C}_1 + 30240\xi^5\mathbb{C}_1 - 17970750\mathbb{C}_1^2 + 18449279\xi\mathbb{C}_1^2 \\
& + 11103120\xi^2\mathbb{C}_1^2 - 17446800\xi^3\mathbb{C}_1^2 + 7333620\xi^4\mathbb{C}_1^2 - 1689534\xi^5\mathbb{C}_1^2 + 241920\xi^6\mathbb{C}_1^2 - 22080\xi^7\mathbb{C}_1^2 + 1260\xi^8\mathbb{C}_1^2 \\
& \left. - 35\xi^8\mathbb{C}_1^2 - 28123200\mathbb{C}_2 + 49865760\xi\mathbb{C}_2 - 26611200\xi^2\mathbb{C}_2 + 5443200\xi^3\mathbb{C}_2 - 604800\xi^4\mathbb{C}_2 + 30240\xi^5\mathbb{C}_2\right) \\
& + \frac{1}{13450364928000}\left(-2895564672000\mathbb{C}_1 + 5134178649600\xi\mathbb{C}_1 - 2739889152000\xi^2\mathbb{C}_1 + 560431872000\xi^3\mathbb{C}_1\right. \\
& - 62270208000\xi^4\mathbb{C}_1 + 3113510400\xi^5\mathbb{C}_1 - 3700536840000\mathbb{C}_1^2 + 3799075531680\xi\mathbb{C}_1^2 + 2286354470400\xi^2\mathbb{C}_1^2 \\
& - 3592645056000\xi^3\mathbb{C}_1^2 + 1510139030400\xi^4\mathbb{C}_1^2 - 347908841280\xi^5\mathbb{C}_1^2 + 49816166400\xi^6\mathbb{C}_1^2 - 4546713600\xi^7\mathbb{C}_1^2 \\
& + 259459200\xi^8\mathbb{C}_1^2 - 7207200\xi^9\mathbb{C}_1^2 - 1161287826270\mathbb{C}_1^3 + 270053629823\xi\mathbb{C}_1^3 + 1988069792280\xi^2\mathbb{C}_1^3 \\
& - 963479454080\xi^3\mathbb{C}_1^3 - 507564676190\xi^4\mathbb{C}_1^3 + 549609764847\xi^5\mathbb{C}_1^3 - 221549328000\xi^6\mathbb{C}_1^3 + 54219285120\xi^7\mathbb{C}_1^3 \\
& - 9069733530\xi^8\mathbb{C}_1^3 + 1086825025\xi^9\mathbb{C}_1^3 - 93716480\xi^{10}\mathbb{C}_1^3 + 5653440\xi^{11}\mathbb{C}_1^3 - 220220\xi^{12}\mathbb{C}_1^3 + 4235\xi^{13}\mathbb{C}_1^3 \\
& - 2895564672000\mathbb{C}_2 + 5134178649600\xi\mathbb{C}_2 - 2739889152000\xi^2\mathbb{C}_2 + 560431872000\xi^3\mathbb{C}_2 - 62270208000\xi^4\mathbb{C}_2 \\
& + 3113510400\xi^5\mathbb{C}_2 - 3700536840000\mathbb{C}_1\mathbb{C}_2 + 3799075531680\xi\mathbb{C}_1\mathbb{C}_2 + 2286354470400\xi^2\mathbb{C}_1\mathbb{C}_2 \\
& - 3592645056000\xi^3\mathbb{C}_1\mathbb{C}_2 + 1510139030400\xi^4\mathbb{C}_1\mathbb{C}_2 - 347908841280\xi^5\mathbb{C}_1\mathbb{C}_2 + 49816166400\xi^6\mathbb{C}_1\mathbb{C}_2 \\
& - 4546713600\xi^7\mathbb{C}_1\mathbb{C}_2 + 259459200\xi^8\mathbb{C}_1\mathbb{C}_2 - 7207200\xi^9\mathbb{C}_1\mathbb{C}_2 - 2895564672000\mathbb{C}_3 + 5134178649600\xi\mathbb{C}_3 \\
& \left. - 2739889152000\xi^2\mathbb{C}_3 + 560431872000\xi^3\mathbb{C}_3 - 62270208000\xi^4\mathbb{C}_3 + 3113510400\xi^5\mathbb{C}_3\right).
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
w^{(2)}(\xi) = {} & \left(2155683\xi^2 + 8038\xi^3 - 2162160\xi^4 + 1081080\xi^5 - 2574\xi^8 + 1716\xi^{10} - 546\xi^{11} - 364\xi^{12} + 252\xi^{13} - 45\xi^{14}\right)/ \\
& 1081080 + 1/236041030958866189056000\left(-14113828503813453911359\xi^2\mathbb{C}_1\right. \\
& + 17512915766704666962322\xi^3\mathbb{C}_1 - 5586404113887189501420\xi^8\mathbb{C}_1 - 231447740009522268000\xi^9\mathbb{C}_1 \\
& + 3735433519034813810160\xi^{10}\mathbb{C}_1 - 1179691619002973294400\xi^{11}\mathbb{C}_1 - 797705495192034374400\xi^{12}\mathbb{C}_1 \\
& + 550212193377310464000\xi^{13}\mathbb{C}_1 - 97319245468388853600\xi^{14}\mathbb{C}_1 + 2551023644304960\xi^{15}\mathbb{C}_1 \\
& - 856729299998872080\xi^{16}\mathbb{C}_1 + 279036341094724200\xi^{17}\mathbb{C}_1 + 247466505045614400\xi^{18}\mathbb{C}_1 \\
& - 155275084044250800\xi^{19}\mathbb{C}_1 - 36505192535330400\xi^{2}0\mathbb{C}_1 + 26398020790188000\xi^{2}1\mathbb{C}_1 \\
& - 8405056667479680\xi^{2}2\mathbb{C}_1 + 897927547083840\xi^{2}3\mathbb{C}_1 - 38159719228800\xi^{2}4\mathbb{C}_1 + 21095475553152\xi^{2}5\mathbb{C}_1 \\
& + 4049797421376\xi^{2}6\mathbb{C}_1 - 6052906241984\xi^{2}7\mathbb{C}_1 + 1221181635008\xi^{2}8\mathbb{C}_1 + 475889853600\xi^{2}9\mathbb{C}_1 \\
& \left. - 295593372480\xi^{3}0\mathbb{C}_1 + 60656299200\xi^{3}1\mathbb{C}_1 - 4738773375\xi^{3}2\mathbb{C}_1\right) \\
& + \left(\left(-326696825351660381778005627174034517747872000\xi^2\mathbb{C}_1\right.\right. \\
& + 405376470465654982768725806942188930704576000\xi^3\mathbb{C}_1 \\
& - 129310093901543591689242666969205624671360000\xi^8\mathbb{C}_1
\end{aligned}
$$

$$+ 4053764704656549827687258069421889307045760 0\xi^{3}\mathbb{C}_1$$

$$- 1293100939015435916892426669692056246713600 0\xi^{8}\mathbb{C}_1$$

$$- 535738703892407696306889771567874214400 00\xi^{9}\mathbb{C}_1$$

$$+ 864651481099634912241842030772133921612800 0\xi^{10}\mathbb{C}_1$$

$$- 273066593318814352267252241234258775552000 0\xi^{11}\mathbb{C}_1$$

$$- 184647172646597956123620297067672061952000 0\xi^{12}\mathbb{C}_1$$

$$+ 127359190171237623738628802659783557120000 0\xi^{13}\mathbb{C}_1$$

$$- 2252676411049710692001622000328823628800 00\xi^{14}\mathbb{C}_1 + 5904927396321076366168727485651968000\xi^{15}\mathbb{C}_1$$

$$- 1983095815708383589080913455027281664000\xi^{16}\mathbb{C}_1 + 6458933999995720474035815391174393600 00\xi^{17}\mathbb{C}_1$$

$$+ 57281779750567840321957112018161152000 0\xi^{18}\mathbb{C}_1$$

$$+ 862046688684628002096517676757223157085151 2\xi^{10}\mathbb{C}_1^2$$

$$- 270194245424160386743728359312394097826856 0\xi^{11}\mathbb{C}_1^2$$

$$- 185329624617078856768012091355453524798304 0\xi^{12}\mathbb{C}_1^2$$

$$+ 127359190171237623738628802659783557120000 0\xi^{13}\mathbb{C}_1^2$$

$$- 2231275510985037478060993038882542442718 80\xi^{14}\mathbb{C}_1^2 + 2418875311127281625362015955881564977 6\xi^{15}\mathbb{C}_1^2$$

$$- 3954294999119289026707499780027151742404\xi^{16}\mathbb{C}_1^2 + 128047673176899293655460050585704321603 4\xi^{17}\mathbb{C}_1^2$$

$$+ 1146299848262837493979094855639474837488\xi^{18}\mathbb{C}_1^2 - 7165835702539891212379500893151545530 20\xi^{19}\mathbb{C}_1^2$$

$$- 181534108291020667670791563651095064 36\xi^{2}0\mathbb{C}_1^2 + 122325062475997255309755518897728337880\xi^{2}1\mathbb{C}_1^2$$

$$- 385834789126385596106742950356068057 60\xi^{2}2\mathbb{C}_1^2 + 40647085107228344914023460357811779 20\xi^{2}3\mathbb{C}_1^2$$

$$- 30494448276899044715500318592159520 0\xi^{2}4\mathbb{C}_1^2 + 1721318243161025187183685420828224 00\xi^{2}5\mathbb{C}_1^2$$

$$+ 3122598976922878410874222728210650 4\xi^{2}6\mathbb{C}_1^2 - 486960649472624354043524692778784 96\xi^{2}7\mathbb{C}_1^2$$

$$+ 99559541104326748154962946138924 16\xi^{2}8\mathbb{C}_1^2 + 3768051722420250956380935649060800\xi^{2}9\mathbb{C}_1^2$$

$$- 234122003038524567782675624738688 0\xi^{3}0\mathbb{C}_1^2 + 474526294591146486759733985232768\xi^{3}1\mathbb{C}_1^2$$

$$- 40917013297075779490615187391192\xi^{3}2\mathbb{C}_1^2 + 386757107789543379419259821601 6\xi^{3}3\mathbb{C}_1^2$$

$$- 52161594559300577150208932796 0\xi^{3}4\mathbb{C}_1^2 - 5786518906322877500710520463 72\xi^{3}5\mathbb{C}_1^2$$

$$+ 26501954355192498287392175616 0\xi^{3}6\mathbb{C}_1^2 - 7980251045802826679782141248\xi^{3}7\mathbb{C}_1^2$$

$$- 2526892338582328717097671345 2\xi^{3}8\mathbb{C}_1^2 + 870098470139068992328601556 0\xi^{3}9\mathbb{C}_1^2$$

$$- 1340309891030782820607242784\xi^{4}0\mathbb{C}_1^2 + 135045166608803637462708432\xi^{4}1\mathbb{C}_1^2$$

$$- 18683464135506877900207872\xi^{4}2\mathbb{C}_1^2 - 327944394281858850752256 0\xi^{4}3\mathbb{C}_1^2$$

$$+ 382082367561732591275340 0\xi^{4}4\mathbb{C}_1^2 - 80519442024341119069670 4\xi^{4}5\mathbb{C}_1^2 - 1435825488225501086968 80\xi^{4}6\mathbb{C}_1^2$$

$$+ 1154363209750912502905 20\xi^{4}7\mathbb{C}_1^2 - 2779568815851054577350 0\xi^{4}8\mathbb{C}_1^2 + 3304269870772182097500\xi^{4}9\mathbb{C}_1^2$$

$$- 165213493538609104875\xi^{5}0\mathbb{C}_1^2 - 32669682535166038177800562717403451774787200\xi^{2}\mathbb{C}_2$$
$$+ 40537647046565498276872580694218893070457600\xi^{3}\mathbb{C}_2$$
$$- 12931009390154359168924266696920562467136000\xi^{8}\mathbb{C}_2$$
$$- 535738703892407696306889771567874214400000\xi^{9}\mathbb{C}_2$$
$$+ 864651481099634912241842030772133921612800 0\xi^{10}\mathbb{C}_2$$
$$- 2730665933188143522672522412342587755520000\xi^{11}\mathbb{C}_2$$
$$- 1846471726465979561236202970676720619520000\xi^{12}\mathbb{C}_2$$
$$+ 1273591901712376237386288026597835571200000\xi^{13}\mathbb{C}_2$$
$$- 225267641104971069200162200032882362880000\xi^{14}\mathbb{C}_2 + 5904927396321076366168727485651968000\xi^{15}\mathbb{C}_2$$
$$- 19830958157083835890809134550272816640 00\xi^{16}\mathbb{C}_2 + 64589339999957204740358153911743936000 0\xi^{17}\mathbb{C}_2$$
$$+ 572817797505678403219571120181611520000\xi^{18}\mathbb{C}_2 - 35941967836553127605666213363624064000 0\xi^{19}\mathbb{C}_2$$
$$- 84499613318393504219357505000184320 00\xi^{2}0\mathbb{C}_2 + 6110425378479954086446547070455040000 0\xi^{2}1\mathbb{C}_2$$
$$- 19455425077784486158148965087647744000\xi^{2}2\mathbb{C}_2 + 2078458576628113491233720925699072000\xi^{2}3\mathbb{C}_2$$
$$- 883293935801421228703444656230400 00\xi^{2}4\mathbb{C}_2 + 4883030065609908125858017125212160 0\xi^{2}5\mathbb{C}_2$$
$$+ 93741819274862205612958603819008 00\xi^{2}6\mathbb{C}_2 - 1401083520940562000888294907678720 0\xi^{2}7\mathbb{C}_2$$
$$+ 2826704059972590662056252066406400\xi^{2}8\mathbb{C}_2 + 1101555855990308389571183162880000\xi^{2}9\mathbb{C}_2$$
$$- 684218434127313503766165801984000\xi^{3}0\mathbb{C}_2 + 1404028707084421393473891993600 00\xi^{3}1\mathbb{C}_2$$
$$- \ 10968974274097042136514781200000\xi^{3}2\mathbb{C}_2)\Big)/5463709258346094058387175634104714600448000000$$
$$+ \Big(\Big(-638124094436097170281835052952590015088100643695530983999965982720000 0\xi^{2}\mathbb{C}_1\Big)\Big)/$$
$$10672048983002159682875334586237697538033908058707878788574740480000000 00$$

$$\text{(A.2)}$$

$$w^{(3)}(\xi) = \Big(15 + 64\pi^2 + 24\pi^2\xi - 24\pi^2\xi^2 - 16\cos[2\pi\xi] - 64\pi^2\cos[2\pi\xi] + \cos[4\pi\xi]\Big)/\Big(128\pi^2\Big) + 1/\Big(94371840\pi^6\Big)$$
$$\cdot (2312275\mathbb{C}_1 + 12037120\pi^2\mathbb{C}_1 + 11059200\pi^4\mathbb{C}_1 + 1018080\pi^2\xi\mathbb{C}_1 + 8824320\pi^4\xi\mathbb{C}_1 + 19224576\pi^6\xi\mathbb{C}_1$$
$$- 1018080\pi^2\xi^2\mathbb{C}_1 - 8478720\pi^4\xi^2\mathbb{C}_1 - 17694720\pi^6\xi^2\mathbb{C}_1 - 691200\pi^4\xi^3\mathbb{C}_1 - 2949120\pi^6\xi^3\mathbb{C}_1 + 345600\pi^4\xi^4\mathbb{C}_1$$
$$+ 1198080\pi^6\xi^4\mathbb{C}_1 + 331776\pi^6\xi^5\mathbb{C}_1 - 110592\pi^6\xi^6\mathbb{C}_1 - 2373120\cos[2\pi\xi]\,\mathbb{C}_1 - 12441600\pi^2\cos[2\pi\xi]\,\mathbb{C}_1$$
$$- 11796480\pi^4\cos[2\pi\xi]\,\mathbb{C}_1 - 1105920\pi^2\xi\cos[2\pi\xi]\,\mathbb{C}_1 - 4423680\pi^4\xi\cos[2\pi\xi]\,\mathbb{C}_1 + 1105920\pi^2\xi^2\cos[2\pi\xi]\,\mathbb{C}_1$$
$$+ 4423680\pi^4\xi^2\cos[2\pi\xi]\,\mathbb{C}_1 + 63360\cos[4\pi\xi]\,\mathbb{C}_1 + 414720\pi^2\cos[4\pi\xi]\,\mathbb{C}_1 + 737280\pi^4\cos[4\pi\xi]\,\mathbb{C}_1$$
$$+ 17280\pi^2\xi\cos[4\pi\xi]\,\mathbb{C}_1 - 17280\pi^2\xi^2\cos[4\pi\xi]\,\mathbb{C}_1 - 2560\cos[6\pi\xi]\,\mathbb{C}_1 - 10240\pi^2\cos[6\pi\xi]\,\mathbb{C}_1 + 45\cos[8\pi\xi]\,\mathbb{C}_1$$
$$+ 1105920\pi\sin[2\pi\xi]\,\mathbb{C}_1 + 4423680\pi^3\sin[2\pi\xi]\,\mathbb{C}_1 - 2211840\pi\xi\sin[2\pi\xi]\,\mathbb{C}_1 - 8847360\pi^3\xi\sin[2\pi\xi]\,\mathbb{C}_1$$
$$- 8640\pi\sin[4\pi\xi]\,\mathbb{C}_1 + 17280\pi\xi\sin[4\pi\xi]\,\mathbb{C}_1) + 1/\Big(487049291366400\pi^{10}\Big)\Big(11933558784000\pi^4\mathbb{C}_1\Big) + 1/$$
$$\Big(35659800916682342400\pi^{12}\Big)\,\xi\,\Big(384696643800268800\pi^8\mathbb{C}_1 + 3334400329855795200\pi^{10}\mathbb{C}_1$$
$$+ 7264291475800719360\pi^{12}\mathbb{C}_1 + 317056740153753600\pi^4\mathbb{C}_1^2 + 3055348438990848000\pi^6\mathbb{C}_1^2$$

$$
\begin{aligned}
&+ 9761324410664386560\pi^{8}\mathbb{C}_1^2 + 14367361192521891840\pi^{10}\mathbb{C}_1^2 + 15851242925787709440\pi^{12}\mathbb{C}_1^2 \\
&+ 222124734074698455\mathbb{C}_1^3 + 2171404189900081136\pi^{2}\mathbb{C}_1^3 + 7100075693557310464\pi^{4}\mathbb{C}_1^3 \\
&+ 10842199196917637120\pi^{6}\mathbb{C}_1^3 + 14250836120871567360\pi^{8}\mathbb{C}_1^3 + 11649501141852487680\pi^{10}\mathbb{C}_1^3 \\
&+ 8684066594856370176\pi^{12}\mathbb{C}_1^3 + 3846966438002688000\pi^{8}\mathbb{C}_2 + 3334400329855795200\pi^{10}\mathbb{C}_2 \\
&+ 7264291475800719360\pi^{12}\mathbb{C}_2 + 3170567401537536000\pi^{4}\mathbb{C}_1\mathbb{C}_2 + 3055348438990848000\pi^{6}\mathbb{C}_1\mathbb{C}_2 \\
&+ 9761324410664386560\pi^{8}\mathbb{C}_1\mathbb{C}_2 + 14367361192521891840\pi^{10}\mathbb{C}_1\mathbb{C}_2 + 15851242925787709440\pi^{12}\mathbb{C}_1\mathbb{C}_2 \\
&+ 3846966438002688000\pi^{8}\mathbb{C}_3 + 3334400329855795200\pi^{10}\mathbb{C}_3 + 7264291475800719360\pi^{12}\mathbb{C}_3\Big) + 1/ \\
&\left(3770462866155503616000000\pi^{14}\right)\Big(92382929312808960000000\pi^{8}\mathbb{C}_1 + 480922211281010688000000\pi^{10}\mathbb{C}_1 \\
&+ 441851117127598080000000\pi^{12}\mathbb{C}_1 + 24115667817114574848000\pi^{4}\mathbb{C}_1^2 + 127340270304721108992000\pi^{6}\mathbb{C}_1^2 \\
&+ 165958095733574860800000\pi^{8}\mathbb{C}_1^2 + 192368884512404275200000\pi^{10}\mathbb{C}_1^2 \\
&+ 883702234255196160000000\pi^{12}\mathbb{C}_1^2 + 594085555282615418420429\mathbb{C}_1^3 + 286243644005278442943161\!6\pi^{2}\mathbb{C}_1^3 \\
&+ 2597952006766286551449600\pi^{4}\mathbb{C}_1^3 + 3066666484153535692800000\pi^{6}\mathbb{C}_1^3 + 2139343103453036544000000\pi^{8}\mathbb{C}_1^3 \\
&+ 1442766633843032064000000\pi^{10}\mathbb{C}_1^3 + 441851117127598080000000\pi^{12}\mathbb{C}_1^3 + 92382929312808960000000\pi^{8}\mathbb{C}_2 \\
&+ 480922211281010688000000\pi^{10}\mathbb{C}_2 + 441851117127598080000000\pi^{12}\mathbb{C}_2 \\
&+ 24115667817114574848000\pi^{4}\mathbb{C}_1\mathbb{C}_2 + 127340270304721108992000\pi^{6}\mathbb{C}_1\mathbb{C}_2 \\
&+ 165958095733574860800000\pi^{8}\mathbb{C}_1\mathbb{C}_2 + 192368884512404275200000\pi^{10}\mathbb{C}_1\mathbb{C}_2 \\
&+ 883702234255196160000000\pi^{12}\mathbb{C}_1\mathbb{C}_2 + 92382929312808960000000\pi^{8}\mathbb{C}_3 + 480922211281010688000000\pi^{10}\mathbb{C}_3 \\
&+ 441851117127598080000000\pi^{12}\mathbb{C}_3\Big) + 1/\left(124684618589798400\pi^{12}\right)\left(-1345093160140800\pi^{8}\xi^{2}\mathbb{C}_1\right)
\end{aligned}
\tag{A.3}
$$

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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