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Research Article

Application of Residual Power Series Method to Fractional Coupled Physical Equations Arising in Fluids Flow

Anas Arafa[1] and Ghada Elmahdy[2]

[1]*Department of Mathematics and Computer Science, Faculty of Science, Port Said University, Port Said, Egypt*
[2]*Department of Basic science, Canal High Institute of Engineering and Technology, Suez, Egypt*

Correspondence should be addressed to Ghada Elmahdy; ghada.elmahdy91@gmail.com

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The approximate analytical solution of the fractional Cahn-Hilliard and Gardner equations has been acquired successfully via residual power series method (RPSM). The approximate solutions obtained by RPSM are compared with the exact solutions as well as the solutions obtained by homotopy perturbation method (HPM) and q-homotopy analysis method (q-HAM). Numerical results are known through different graphs and tables. The fractional derivatives are described in the Caputo sense. The results light the power, efficiency, simplicity, and reliability of the proposed method.

1. Introduction

Fractional differential equations (FDEs) have found applications in many problems in physics and engineering [1, 2]. Since most of the nonlinear FDEs cannot be solved exactly, approximate and numerical methods must be used. Some of the recent analytical methods for solving nonlinear problems include the Adomian decomposition method [3, 4], variational iteration method [5], homotopy perturbation method [6, 7], homotopy analysis method [8, 9], spectral collocation method [10], the tanh-coth method [11], exp-function method [12], Mittag-Leffler function method [13], differential quadrature method [14], and reproducing kernel Hilbert space method [15, 16].

The Gardner equation [17] (combined KdV-mKdV equation) is a useful model for the description of internal solitary waves in shallow water,

$$u_t + 6uu_x \pm 6u^2u_x + u_{xxx} = 0. \quad (1)$$

Those two models will be classified as positive Gardner equation and negative Gardner equation depending on the sign of the cubic nonlinear term [18, 19]. Gardner equation is widely used in various branches of physics, such as plasma physics, fluid physics, and quantum field theory [20, 21]. It also describes a variety of wave phenomena in plasma and solid state [22, 23].

The Cahn-Hilliard equation [24] is one type of partial differential equations (PDEs) and was first introduced in 1958 as a model for process of phase separation of a binary alloy under the critical temperature [25],

$$u_t = \gamma u_x + 6uu_x^2 + \left(3u^2 - 1\right) u_{xx} - u_{xxxx}, \quad \gamma \geq 0. \quad (2)$$

This equation is related to a number of interesting physical phenomena like the spinodal decomposition, phase separation, and phase ordering dynamics. On the other hand it becomes important in material sciences [26, 27].

The aim of this paper is to study the time-fractional Gardner equation [28–30] and time-fractional Cahn-Hilliard equation [31–37] of this form,

$$D_t^{\alpha} u(x,t) + 6\left(u - \varepsilon^2 u^2\right) u_x + u_{xxx} = 0, \quad (3)$$

$$D_t^{\alpha} u(x,t) - u_x - 6uu_x^2 - \left(3u^2 - 1\right) u_{xx} + u_{xxxx} = 0, \quad (4)$$

where $0 < \alpha \leq 1$, $-\infty < x < \infty$, and $0 \leq t < R$. Numerous methods have been used to solve this equations, for example, q-Homotopy analysis method [28], the new version of F-expansion method [29], reduced differential transform method [30], the generalized tanh-coth method [38], the generalized Kudryashov method [39], extended fractional Riccati expansion method [31], subequation method [32], homotopy analysis method [33], the Adomian decomposition method [34], improved $(\grave{G}/G)$–expansion method [35], homotopy perturbation method [36], and variational iteration method [37]. We solve Cahn-Hilliard equation and Gardner equation by RPSM.

The RPSM was first devised in 2013 by the Jordanian mathematician Omar Abu Arqub as an efficient method for determining values of coefficients of the power series solution for first and the second-order fuzzy differential equations [40]. The RPSM is an effective and easy to construct power series solution for strongly linear and nonlinear equations without linearization, perturbation, or discretization. In the last few years, the RPSM has been applied to solve a growing number of nonlinear ordinary and PDEs of different types, classifications, and orders. It has been successfully applied in the numerical solution of the generalized Lane-Emden equation [41], which is a highly nonlinear singular differential equation, in the numerical solution of higher-order regular differential equations [42], in approximate solution of the nonlinear fractional KdV-Burgers equation [43], in construct and predict the solitary pattern solutions for nonlinear time-fractional dispersive PDEs [44], and in predicting and representing the multiplicity of solutions to boundary value problems of fractional order [45]. The RPSM distinguishes itself from various other analytical and numerical methods in several important aspects [46]. Firstly, the RPSM does not need to compare the coefficients of the corresponding terms and a recursion relation is not required. Secondly, the RPSM provides a simple way to ensure the convergence of the series solution by minimizing the related residual error. Thirdly, the RPSM is not affected by computational rounding errors and does not require large computer memory and time. Fourthly, the RPSM does not require any converting while switching from the low-order to the higher-order and from simple linearity to complex nonlinearity; as a result, the method can be applied directly to the given problem by choosing an appropriate initial guess approximation.

2. Fundamental Concepts

Definition 1 (see [43]). The Caputo time-fractional derivatives of order $\alpha > 0$ of $u(x,t)$ is defined as

$$D_t^{\alpha} u(x,t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n u(x,\tau)}{\partial \tau^n} d\tau, & n-1 < \alpha < n, \\ \dfrac{\partial^n u(x,t)}{\partial t^n}, & \alpha = n \in N. \end{cases} \quad (5)$$

Definition 2 (see [47, 48]). A power series representation of the form

$$\sum_{n=0}^{\infty} C_n (t-t_0)^{n\alpha} = C_0 + C_1 (t-t_0)^{\alpha} + C_2 (t-t_0)^{2\alpha} + \ldots \quad (6)$$

where $0 \le n-1 < \alpha \le n, n \in N$ *and* $t \ge t_0$ is called fractional power series about t_0.

Theorem 3 (see [47, 48]). *Suppose that f has a fractional power series representation at t_0 of the form*

$$f(t) = \sum_{n=0}^{\infty} C_n (t-t_0)^{n\alpha}, \quad \text{where } 0 \le n-1 < \alpha \le n \text{ and } t_0 \le t < t_0 + R. \quad (7)$$

If $D^{n\alpha} f(t)$ are continuous on (t_0, t_0+R), $n = 0, 1, 2, 3, \ldots$, then coefficients C_n will take the form

$$C_n = \frac{D^{n\alpha} f(t_0)}{\Gamma(n\alpha+1)}. \quad (8)$$

Definition 4 (see [43]). A power series representation of the form

$$\sum_{n=0}^{\infty} f_n(x)(t-t_0)^{n\alpha} = f_0(x) + f_1(x)(t-t_0)^{\alpha} + f_2(x)(t-t_0)^{2\alpha} + \ldots \quad (9)$$

is called a multiple fractional power series about $t = t_0$.

Theorem 5 (see [43, 44]). *Suppose that $u(x,t)$ has a multiple fractional Power series representation at t_0 of the form*

$$u(x,t) = \sum_{n=0}^{\infty} f_n(x)(t-t_0)^{n\alpha}, \quad \text{where } x \in I,\ 0 \le n-1 < \alpha \le n \text{ and } t_0 \le t < t_0 + R. \quad (10)$$

If $D_t^{n\alpha} u(x,t)$ are continuous on $I \times (t_0, t_0+R)$, $n = 0, 1, 2, 3, \ldots$, then coefficients $f_n(x)$ will take the form

$$f_n(x) = \frac{D_t^{n\alpha} u(x,t_0)}{\Gamma(n\alpha+1)}. \quad (11)$$

Corollary 6 (see [44]). *Suppose that $u(x,y,t)$ has a multiple fractional Power series representation at t_0 of the form*

$$u(x,y,t) = \sum_{n=0}^{\infty} f_n(x,y)(t-t_0)^{n\alpha}, \quad (x,y) \in I_1 \times I_2,\ t_0 \le t < t_0 + R. \quad (12)$$

If $D_t^{n\alpha} u(x,y,t)$ are continuous on $I_1 \times I_2 \times (t_0, t_0 + R)$, $n = 0, 1, 2, 3, \ldots$, then $f_n(x,y)$ will take the form

$$f_n(x,y) = \frac{D_t^{n\alpha} u(x,y,t_0)}{\Gamma(n\alpha+1)}. \quad (13)$$

3. Basic Idea of RPSM

To give the approximate solution of nonlinear fractional order differential equations by means of the RPSM, we consider a general nonlinear fractional differential equation:

$$D^{\alpha} u(x,t) = N(u) + R(u) \quad (14)$$

where $N(u)$ is nonlinear term and $R(u)$ is a linear term. Subject to the initial condition

$$u(x,0) = f(x). \quad (15)$$

The RPSM proposes the solution for (14) as a fractional power series about the initial point $t = 0$,

$$u(x,t) = \sum_{n=0}^{\infty} f_n(x) \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \quad 0 < \alpha \le 1,\ -\infty < x < \infty,\ 0 \le t < R. \quad (16)$$

Next we let $u_k(x,t)$ denote the kth truncated series of $u(x,t)$,

$$u_k(x,t) = \sum_{n=0}^{k} f_n(x) \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}. \quad (17)$$

The 0th RPS approximate solution of $u(x,t)$ is

$$u_0(x,t) = u(x,0) = f(x). \quad (18)$$

Equation (17) can be written as

$$u_k(x,t) = f(x) + \sum_{n=1}^{k} f_n(x) \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}, \quad k = 1,2,3,\ldots. \quad (19)$$

We define the residual function for (14)

$$Res_u(x,t) = D_t^{\alpha} u(x,t) - N(u) - R(u). \quad (20)$$

Therefore, the kth residual function $Res_{u,k}$ is

$$Res_{u,k}(x,t) = D_t^{\alpha} u_k(x,t) - N(u_k) - R(u_k). \quad (21)$$

As in [40, 41], $Resu(x,t) = 0$ *and* $\lim_{k\to\infty} Res_k(x,t) = Res(x,t)$.*Therefore*, $D_t^{n\alpha} Res(x,t) = 0$ since the fractional derivative of a constant in the Caputo sense is zero and the fractional derivatives $D_t^{n\alpha}$ of $Res(x,t)$ and $Res_k(x,t)$ are matching at $t = 0$ for each $n = 0,1,2,\ldots,k$.; that is, $D_t^{n\alpha} Res(x,0) = D_t^{n\alpha} Res_k(x,0) = 0$, $n = 0,1,2,\ldots,k$.

To determine $f_1(x), f_2(x), f_3(x), \ldots$ we consider $k = 1,2,3,..$ in (19) and substitute it into (21), applying the fractional derivative $D_t^{(k-1)\alpha}$ in both sides, $k = 1,2,3,\ldots$, and finally we solve

$$D_t^{(k-1)\alpha} Res_{u,k}(x,0) = 0, \quad k = 1,2,3,\ldots. \quad (22)$$

4. Applications

To illustrate the basic idea of RPSM, we consider the following two time-fractional Gardner and Cahn-Hilliard equations.

4.1. Time-Fractional Gardner Equation. Consider the time-fractional homogeneous Gardner equation

$$D_t^{\alpha} u(x,t) + 6\left(u - \varepsilon^2 u^2\right) u_x + u_{xxx} = 0. \quad (23)$$

Subject to the initial Condition

$$u(x,0) = \frac{1}{2} + \frac{1}{2} \tanh\left[\frac{x}{2}\right]. \quad (24)$$

The exact solution when $\varepsilon = 1, \alpha = 1$ is

$$u(x,t) = \frac{1}{2} + \frac{1}{2} \tanh\left[\frac{x-t}{2}\right]. \quad (25)$$

We define the residual function for (23) as

$$Res_u(x,t) = D_t^{\alpha} u(x,t) + 6\left(u - \varepsilon^2 u^2\right) u_x + u_{xxx}, \quad (26)$$

therefore, for the kth residual function $Res_{u,k}(x,t)$,

$$Res_{u,k}(x,t) = D_t^{\alpha} u_k + 6\left(u_k - \varepsilon^2 u_k^2\right) u_{kx} + u_{kxxx}. \quad (27)$$

To determine $f_1(x)$, we consider $(k = 1)$ in (27)

$$Res_{u,1}(x,t) = D_t^{\alpha} u_1 + 6u_1 u_{1x} - 6\varepsilon^2 u_1^2 u_{1x} + u_{1xxx}. \quad (28)$$

But from (19) at $k = 1$,

$$u_1(x,t) = f(x) + f_1(x) \frac{t^{\alpha}}{\Gamma(1+\alpha)}, \quad (29)$$

$$\begin{aligned} Res_{u,1}(x,t) &= f_1 + 6ff_x - 6\varepsilon^2 f_x f^2 + f_{xxx} + \left[6ff_{1x} + 6f_1 f_x - 12\varepsilon^2 f_x f f_1 - 6\varepsilon^2 f_{1x} f^2 + f_{1xxx}\right] \\ &\cdot \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \left[6f_1 f_{1x} - 6\varepsilon^2 f_x f_1^2 - 12\varepsilon^2 f_{1x} f f_1\right] \\ &\cdot \frac{t^{2\alpha}}{\Gamma(1+\alpha)^2} - 6\varepsilon^2 f_{1x} f_1^2 \frac{t^{3\alpha}}{\Gamma(1+\alpha)^3}. \end{aligned} \quad (30)$$

Now depending on the result of (22) In the case of k=1, we have $Res_{u_1}(x,0) = 0$,

$$f_1 = -6ff_x + 6\varepsilon^2 f_x f^2 - f_{xxx}, \quad (31)$$

$$f_1(x) = \frac{1}{8} \operatorname{sech}\left[\frac{x}{2}\right]^4 \left(-1 + \left(-4 + 3\varepsilon^2\right)\cosh[x] + 3\left(-1 + \varepsilon^2\right)\sinh[x]\right). \quad (32)$$

To determine $f_2(x)$, we consider $(k = 2)$ in (27)

$$Res_{u,2}(x,t) = D_t^{\alpha} u_2 + 6u_2 u_{2x} - 6\varepsilon^2 u_2^2 u_{2x} + u_{2xxx}. \quad (33)$$

But from (19) at $k = 2$,

$$u_2(x,t) = f(x) + f_1(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + f_2(x) \cdot \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}, \quad (34)$$

$$\begin{aligned} Res_{u,2}(x,t) &= f_1 + 6ff_x - 6\varepsilon^2 f_x f^2 + f_{xxx} + \left[f_2 + 6ff_{1x} + 6f_1 f_x - 12\varepsilon^2 f_x ff_1 - 6\varepsilon^2 f_{1x} f^2 + f_{1xxx}\right] \\ &\cdot \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \left[6ff_{2x} + 6f_2 f_x - 12\varepsilon^2 f_x ff_2 - 6\varepsilon^2 f_{2x} f^2 + f_{2xxx}\right]\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \left[6f_1 f_{1x} - 6\varepsilon^2 f_x f_1^2 - 12\varepsilon^2 f_{1x} ff_1\right]\frac{t^{2\alpha}}{\Gamma(1+\alpha)^2} + \left[6f_1 f_{2x} \right. \\ &\left. + 6f_2 f_{1x} - 12\varepsilon^2 f_x f_1 f_2 - 12\varepsilon^2 f_{1x} ff_2 - 12\varepsilon^2 f_{2x} ff_1\right]\frac{t^{3\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} - 6\varepsilon^2 f_{1x} f_1^2 \\ &\cdot \frac{t^{3\alpha}}{\Gamma(1+\alpha)^3} + \left[6f_2 f_{2x} - 6\varepsilon^2 f_x f_2^2 - 12\varepsilon^2 f_{2x} ff_2\right] \cdot \frac{t^{4\alpha}}{\Gamma(1+2\alpha)^2} + \left[-12\varepsilon^2 f_{1x} f_1 f_2 - 6\varepsilon^2 f_{2x} f_1^2\right] \\ &\cdot \frac{t^{4\alpha}}{\Gamma(1+\alpha)^2\Gamma(1+2\alpha)} + \left[-6\varepsilon^2 f_{1x} f_2^2 - 12\varepsilon^2 f_{2x} f_1 f_2\right]\frac{t^{5\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)^2} - 6\varepsilon^2 f_{2x} f_2^2 \\ &\cdot \frac{t^{6\alpha}}{\Gamma(1+2\alpha)^3}. \end{aligned} \quad (35)$$

Applying D_t^{α} on both sides and solving the equation $D_t^{\alpha} Res_{u,2}(x,0) = 0$, then we get

$$f_2 = -6ff_{1x} - 6f_1 f_x + 12\varepsilon^2 f_x ff_1 + 6\varepsilon^2 f_{1x} f^2 - f_{1xxx}, \quad (36)$$

$$\begin{aligned} f_2(x) &= \frac{-1}{64}\operatorname{sech}\left[\frac{x}{2}\right]^7 \left(-24\left(-1+\varepsilon^2\right)\cosh\left[\frac{x}{2}\right] - 6\left(22 - 37\varepsilon^2 + 15\varepsilon^4\right)\cosh\left[\frac{3x}{2}\right] + 24\cosh\left[\frac{5x}{2}\right] \right. \\ &- 42\varepsilon^2\cosh\left[\frac{5x}{2}\right] + 18\varepsilon^4\cosh\left[\frac{5x}{2}\right] + 206\sinh\left[\frac{x}{2}\right] - 204\varepsilon^2\sinh\left[\frac{x}{2}\right] - 129\sinh\left[\frac{3x}{2}\right] + 222\varepsilon^2\sinh\left[\frac{3x}{2}\right] \\ &- 90\varepsilon^4\sinh\left[\frac{3x}{2}\right] + 25\sinh\left[\frac{5x}{2}\right] \left. - 42\varepsilon^2\sinh\left[\frac{5x}{2}\right] + 18\varepsilon^4\sinh\left[\frac{5x}{2}\right]\right). \end{aligned} \quad (37)$$

The solution in series form is given by

$$u(x,t) = f(x) + f_1(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + f_2(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \ldots \quad (38)$$

4.2. Time-Fractional Cahn-Hilliard Equation. Consider the time-fractional Cahn-Hilliard equation

$$D_t^{\alpha} u(x,t) - u_x - 6uu_x^2 - \left(3u^2 - 1\right)u_{xx} + u_{xxxx} = 0. \quad (39)$$

Subject to the initial condition

$$u(x,0) = \tanh\left[\frac{\sqrt{2}}{2}x\right]. \quad (40)$$

The exact solution when $\alpha = 1$ is

$$u(x,t) = \tanh\left[\frac{\sqrt{2}}{2}(x+t)\right]. \quad (41)$$

We define the residual function for (39) as

$$Res_u(x,t) = D_t^{\alpha} u(x,t) - u_x - 6uu_x^2 - \left(3u^2 - 1\right)u_{xx} + u_{xxxx}, \quad (42)$$

therefore, for the kth residual function $Res_{u,k}(x,t)$,

$$Res_{u,k}(x,t) = D_t^{\alpha} u_k - u_{kx} - 6u_k u_{kx}^2 - \left(3u_k^2 - 1\right)u_{kxx} + u_{kxxxx}. \quad (43)$$

To determine $f_1(x)$, we consider $(k = 1)$ in (43)

$$Res_{u,1}(x,t) = D_t^{\alpha} u_1 - u_{1x} - 6u_1 u_{1x}^2 - \left(3u_1^2 - 1\right)u_{1xx} + u_{1xxxx}. \quad (44)$$

From (19) at $k = 1$,

$$u_1(x,t) = f(x) + f_1(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)}, \quad (45)$$

$$\begin{aligned} Res_{u,1}(x,t) &= f_1 - f_x - 6ff_x^2 - 3f^2 f_{xx} + f_{xx} + f_{xxxx} + \left[-f_{1x} - 6f_1 f_x^2 - 12ff_x f_{1x} - 6ff_1 f_{xx} - 3f^2 f_{1xx} + f_{1xx} + f_{1xxxx}\right]\frac{t^{\alpha}}{\Gamma(1+\alpha)} + \left[-6ff_{1x}^2 \right. \\ &\left. - 12f_1 f_x f_{1x} - 3f_1^2 f_{xx} - 6ff_1 f_{1xx}\right]\frac{t^{2\alpha}}{\Gamma(1+\alpha)^2} + \left[-6f_1 f_{1x}^2 - 3f_1^2 f_{1xx}\right]\frac{t^{3\alpha}}{\Gamma(1+\alpha)^3}. \end{aligned} \quad (46)$$

If we put $Res_{u,1}(x,0)=0$, then

$$f_1(x)=f_x+6f(x)f_x^2+3f^2f_{xx}-f_{xx}-f_{xxxx}, \tag{47}$$

$$f_1(x)=\frac{\operatorname{sech}\left[x/\sqrt{2}\right]^2}{\sqrt{2}}. \tag{48}$$

Similarity, to determine $f_2(x)$, we substitute

$$\begin{aligned}u_2(x,t)=&f(x)+f_1(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)}\\&+f_2(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)},\end{aligned} \tag{49}$$

into (43) where $k=2$,

$$\begin{aligned}Res_{u,2}(x,t)=&\left[f_1-f_x-6ff_x^2-3f^2f_{xx}+f_{xx}\right.\\&\left.+f_{xxxx}\right]+\left[f_2-f_{1x}-6f_1f_x^2-12ff_xf_{1x}\right.\\&\left.-6ff_1f_{xx}-3f^2f_{1xx}+f_{1xx}+f_{1xxxx}\right]\frac{t^{\alpha}}{\Gamma(1+\alpha)}\\&+\left[-f_{2x}-12ff_xf_{2x}-6f_2f_x^2-6ff_2f_{xx}\right.\\&\left.-3f_{2xx}f^2+f_{2xx}+f_{2xxxx}\right]\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+\left[-6ff_{1x}^2\right.\\&\left.-12f_1f_xf_{1x}-3f_1^2f_{xx}-6ff_1f_{1xx}\right]\frac{t^{2\alpha}}{\Gamma(1+\alpha)^2}\\&+\left[-12ff_{1x}f_{2x}-12f_2f_xf_{1x}-6f_1f_2f_{xx}\right.\\&\left.-6ff_2f_{1xx}-12f_1f_xf_{2x}-6f_{2xx}ff_1\right]\\&\cdot\frac{t^{3\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}+\left[-6f_1f_{1x}^2-3f_1^2f_{1xx}\right]\\&\cdot\frac{t^{3\alpha}}{\Gamma(1+\alpha)^3}+\left[-6ff_{2x}^2-12f_2f_xf_{2x}-3f_2^2f_{xx}\right.\\&\left.-6f_{2xx}ff_2\right]\frac{t^{4\alpha}}{\Gamma(1+2\alpha)^2}+\left[-12f_1f_{1x}f_{2x}-6f_2f_{1x}^2\right.\\&\left.-6f_1f_2f_{1xx}-3f_{2xx}f_1^2\right]\frac{t^{4\alpha}}{\Gamma(1+\alpha)^2\Gamma(1+2\alpha)}\\&+\left[-6f_1f_{2x}^2-12f_2f_{1x}f_{2x}-3f_2^2f_{1xx}-6f_{2xx}f_1f_2\right]\\&\cdot\frac{t^{5\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)^2}+\left[-6f_2f_{2x}^2-3f_{2xx}f_2^2\right]\\&\cdot\frac{t^{6\alpha}}{\Gamma(1+2\alpha)^3}.\end{aligned} \tag{50}$$

Solving the equation $D_t^{\alpha}Res_{u,2}(x,0)=0$, we find that

$$\begin{aligned}f_2(x)=&f_{1x}+6f_1f_x^2+12ff_xf_{1x}+6ff_1f_{xx}\\&+3f^2f_{1xx}-f_{1xx}-f_{1xxxx},\end{aligned} \tag{51}$$

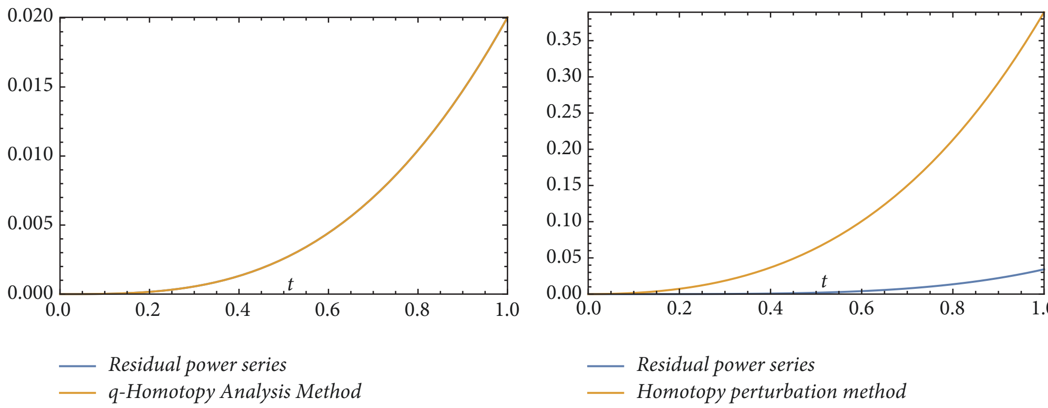
$$f_2(x)=-\operatorname{sech}\left[\frac{x}{\sqrt{2}}\right]^2\tanh\left[\frac{x}{\sqrt{2}}\right]. \tag{52}$$

To determine $f_3(x)$, we substitute

$$\begin{aligned}u_3(x,t)=&f(x)+f_1(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)}\\&+f_2(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+f_3(x)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)},\end{aligned} \tag{53}$$

into (43) where k=3,

$$\begin{aligned}Res_{u,3}(x,t)=&f_1-f_x-6ff_x^2-3f^2f_{xx}+f_{xx}\\&+f_{xxxx}+\left[f_2-f_{1x}-12ff_xf_{1x}-6f_1f_x^2\right.\\&\left.-6ff_1f_{xx}-3f^2f_{1xx}+f_{1xx}+f_{1xxxx}\right]\frac{t^{\alpha}}{\Gamma(1+\alpha)}\\&+\left[f_3-f_{2x}-12ff_xf_{2x}-6f_2f_x^2-6ff_2f_{xx}\right.\\&\left.-3f_{2xx}f^2+f_{2xx}+f_{2xxxx}\right]\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+\left[-6ff_{1x}^2\right.\\&\left.-12f_1f_xf_{1x}-3f_1^2f_{xx}-6ff_1f_{1xx}\right]\frac{t^{2\alpha}}{\Gamma(1+\alpha)^2}\\&+\left[-12ff_xf_{3x}-f_{3x}-6f_3f_x^2-6ff_3f_{xx}\right.\\&\left.-3f^2f_{3xx}+f_{3xx}+f_{3xxxx}\right]\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\\&+\left[-12ff_{1x}f_{2x}-12f_1f_xf_{2x}-12f_2f_xf_{1x}\right.\\&\left.-6f_1f_2f_{xx}-6ff_2f_{1xx}-6f_{2xx}ff_1\right]\\&\cdot\frac{t^{3\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}+\left[-6f_1f_{1x}^2-3f_1^2f_{1xx}\right]\\&\cdot\frac{t^{3\alpha}}{\Gamma(1+\alpha)^3}+\left[-12f_1f_{1x}f_{2x}-6f_2f_{1x}^2\right.\\&\left.-6f_1f_2f_{1xx}-3f_{2xx}f_1^2\right]\frac{t^{4\alpha}}{\Gamma(1+\alpha)^2\Gamma(1+2\alpha)}\\&+\left[-6ff_{2x}^2-12f_2f_xf_{2x}-3f_2^2f_{xx}-6f_{2xx}ff_2\right]\\&\cdot\frac{t^{4\alpha}}{\Gamma(1+2\alpha)^2}+\left[-12ff_{1x}f_{3x}-12f_1f_xf_{3x}\right.\\&\left.-12f_3f_xf_{1x}-6f_1f_3f_{xx}-6ff_3f_{1xx}-6ff_1f_{3xx}\right]\\&\cdot\frac{t^{4\alpha}}{\Gamma(1+\alpha)\Gamma(1+3\alpha)}+\left[-6f_1f_{2x}^2-12f_2f_{1x}f_{2x}\right.\end{aligned}$$



(a) (b)

Figure 1: (a) Absolute errors of fractional Gardner equation, at $x = 2, \varepsilon = 1$. (b) Absolute errors of fractional Chan-Hilliard equation at $x = 2$.

$$\begin{aligned}
&- 3f_2^2 f_{1xx} - 6f_{2xx}f_1f_2\Big] \frac{t^{5\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)^2} \\
&+ \Big[-12ff_{2x}f_{3x} - 12f_2f_xf_{3x} - 12f_3f_xf_{2x} \\
&- 6f_2f_3f_{xx} - 6ff_3f_{2xx} - 6ff_2f_{3xx}\Big] \\
&\cdot \frac{t^{5\alpha}}{\Gamma(1+2\alpha)\Gamma(1+3\alpha)} + \Big[-12f_1f_{1x}f_{3x} - 6f_3f_{1x}^2 \\
&- 3f_1^2f_{3xx} - 6f_1f_3f_{1xx}\Big] \frac{t^{5\alpha}}{\Gamma(1+\alpha)^2\Gamma(1+3\alpha)} \\
&+ \Big[-12f_1f_{2x}f_{3x} - 12f_2f_{1x}f_{3x} - 12f_3f_{1x}f_{2x} \\
&- 6f_2f_3f_{1xx} - 6f_1f_3f_{2xx} - 6f_1f_2f_{3xx}\Big] \\
&\cdot \frac{t^{6\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)\Gamma(1+3\alpha)} + \Big[-6f_2f_{2x}^2 \\
&- 3f_{2xx}f_2^2\Big] \frac{t^{6\alpha}}{\Gamma(1+2\alpha)^3} + \Big[-6ff_{3x}^2 - 12f_3f_xf_{3x} \\
&- 3f_3^2f_{xx} - 6ff_3f_{3xx}\Big] \frac{t^{6\alpha}}{\Gamma(1+3\alpha)^2} + \Big[-6f_1f_{3x}^2 \\
&- 12f_3f_{1x}f_{3x} - 3f_3^2f_{1xx} - 6f_1f_3f_{3xx}\Big] \\
&\cdot \frac{t^{7\alpha}}{\Gamma(1+\alpha)\Gamma(1+3\alpha)^2} + \Big[-12f_2f_{2x}f_{3x} - 6f_3f_{2x}^2 \\
&- 6f_2f_3f_{2xx} - 3f_2^2f_{3xx}\Big] \frac{t^{7\alpha}}{\Gamma(1+3\alpha)\Gamma(1+2\alpha)^2} \\
&+ \Big[-6f_2f_{3x}^2 - 12f_3f_{2x}f_{3x} - 3f_3^2f_{2xx} - 6f_{3xx}f_3f_2\Big] \\
&\cdot \frac{t^{8\alpha}}{\Gamma(1+2\alpha)\Gamma(1+3\alpha)^2} + \Big[-6f_3f_{3x}^2 - 3f_3^2f_{3xx}\Big] \\
&\cdot \frac{t^{9\alpha}}{\Gamma(1+3\alpha)^3}.
\end{aligned} \tag{54}$$

Applying $D_t^{2\alpha}$ on both sides and then solving the equation $D_t^{2\alpha} Res_{u,3}(x,0) = 0$, we get

$$\begin{aligned}
f_3(x) &= \Big[f_{2x} + 12ff_xf_{2x} + 6f_2f_x^2 + 6ff_2f_{xx} \\
&+ 3f_{2xx}f^2 - f_{2xx} - f_{2xxxx}\Big] + \Big[6ff_{1x}^2 + 12f_1f_xf_{1x} \\
&+ 3f_1^2f_{xx} + 6ff_1f_{1xx}\Big] \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2},
\end{aligned} \tag{55}$$

$$\begin{aligned}
f_3(x) &= \frac{1}{8}\operatorname{sech}\left[\frac{x}{\sqrt{2}}\right]^6 \Big(-4\sqrt{2} \\
&+ \left(264 - 96\cosh\left[\sqrt{2}x\right] + \sqrt{2}\sinh\left[2\sqrt{2}x\right]\right) \\
&\cdot \tanh\left[\frac{x}{\sqrt{2}}\right]\Big) + \Big(\frac{-21}{2}\operatorname{sech}\left[\frac{x}{\sqrt{2}}\right]^6 \tanh\left[\frac{x}{\sqrt{2}}\right] \\
&+ 12\operatorname{sech}\left[\frac{x}{\sqrt{2}}\right]^4 \tanh\left[\frac{x}{\sqrt{2}}\right]^3\Big) \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}.
\end{aligned} \tag{56}$$

The solution in series form is given by

$$\begin{aligned}
u(x,t) &= f(x) + f_1(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)} + f_2(x)\frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \\
&+ f_3(x)\frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \dots
\end{aligned} \tag{57}$$

5. Numerical Results

This section deals with the approximate analytical solutions obtained by RPSM for Gardner and Cahn-Hilliard equations. In classical case($\alpha \longrightarrow 1$), Figure 1 and Tables 1 and 2 describe the comparison between RPSM with q-HAM [28] and HPM [36]. In fractional case, Figures 2, 3, and 4 describe the geometrical behavior of the solutions obtained by RPSM for different fractional value α of the two equations.

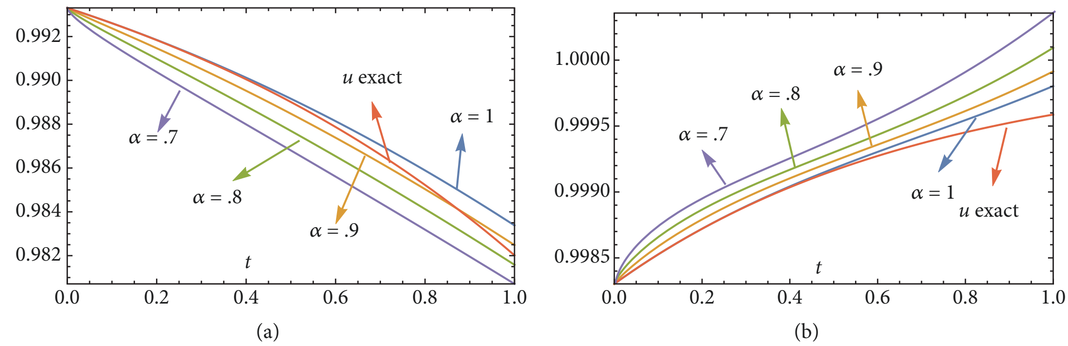


FIGURE 2: (a) Fractional Gardner equation at $x = 5, \varepsilon = 1$. (b) Fractional Chan-Hilliard equation at $x = 5$.

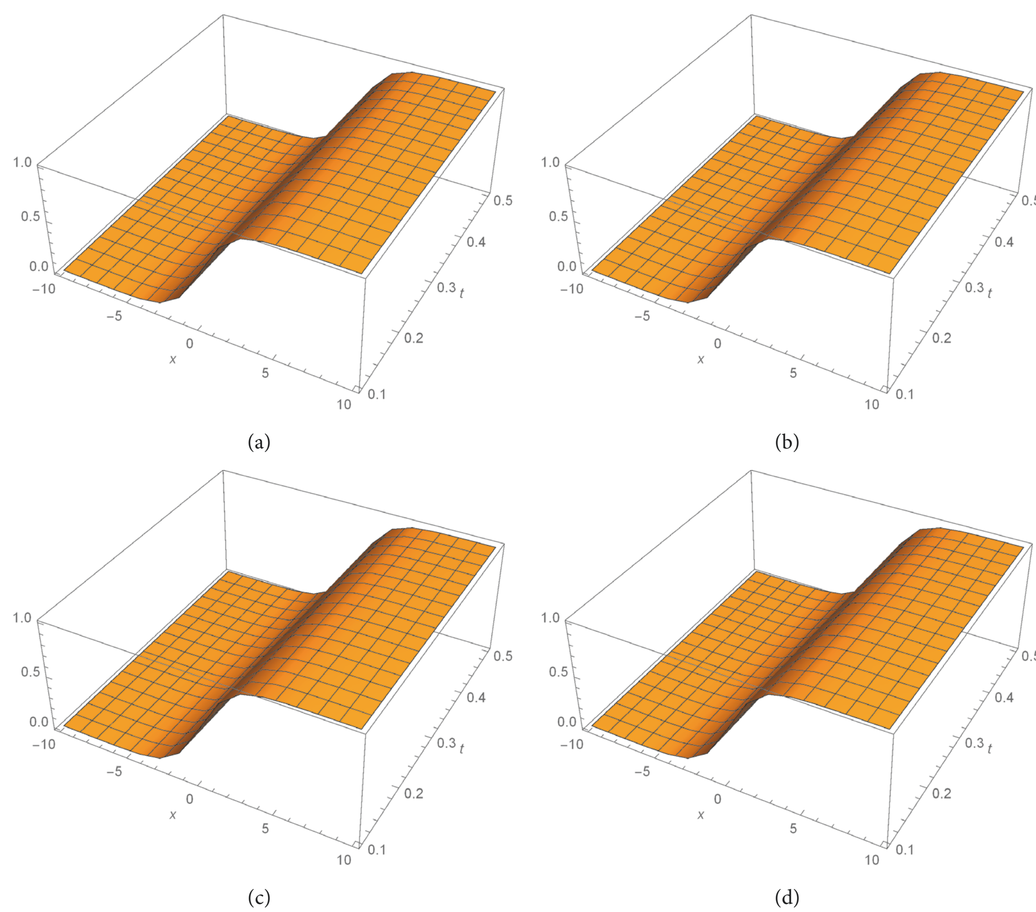


FIGURE 3: The approximate solution for fractional Gardner equation at $\varepsilon = 1$: (a) $\alpha = 1$, (b) Exact solution, (c) $\alpha = .99$, and (d) $\alpha = .95$.

6. Conclusions

This work has used the RPSM for finding the solution of the time-fractional Gardner and Cahn-Hilliard equations. A very good agreement between the results obtained by the RPSM and q-HAM [28] was observed in Figure 1(a) and Table 1. Figure 1(b) and Table 2 indicate that the mentioned method achieves a higher level of accuracy than HPM [36]. Consequently, the work emphasized that the method introduces a significant improvement in this field over existing techniques.

Data Availability

[1] The [approximate solution obtained by q-homotopy analysis method] data used to support the findings of this study have been deposited in the [article] repository ([doi.org/10.1016/j.asej.2014.03.014]) [28]. [2] The [approximate solution obtained by homotopy perturbation method] data used to support the findings of this study have been deposited in the [article] repository ([doi.org/10.1080/10288457.2013.867627]) [36].

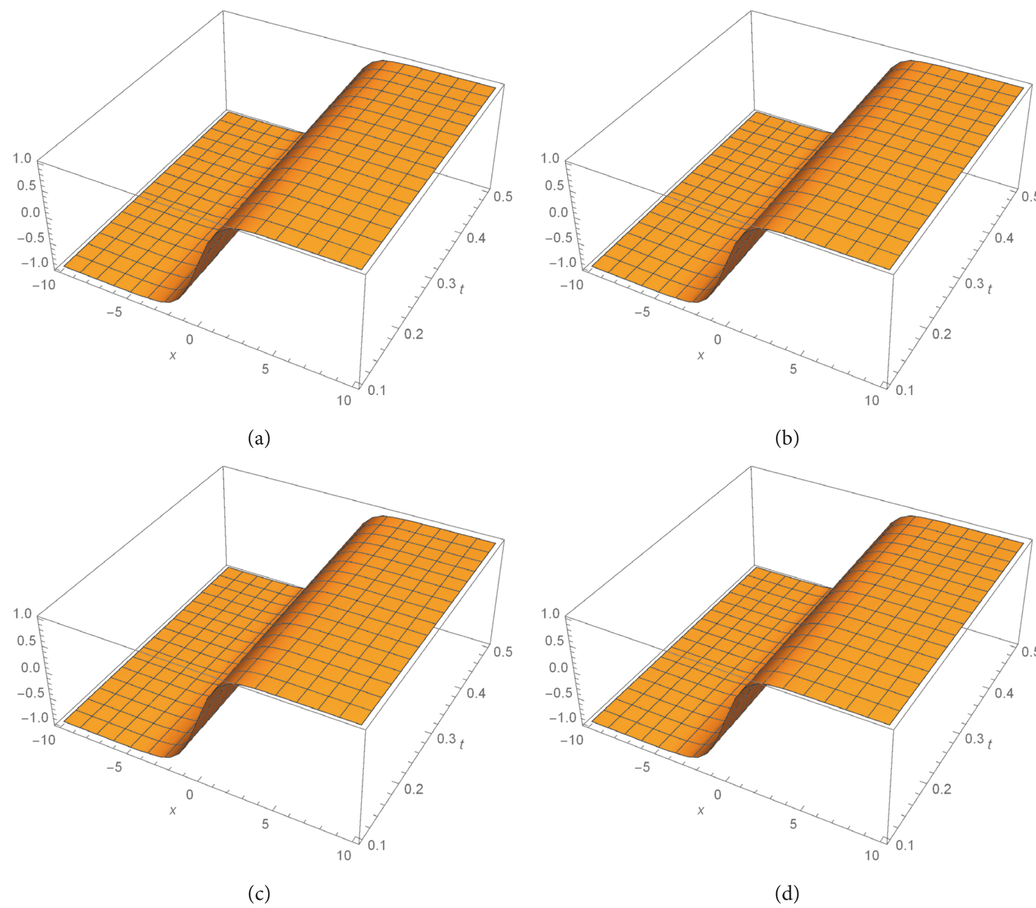


Figure 4: The approximate solution for fractional Chan-Hilliard equation: (a) $\alpha = 1$, (b) Exact solution, (c) $\alpha = .99$, and (d) $\alpha = .95$.

Table 1: The absolute errors $|u_{exact} - u_3|$ for Gardner equation when $t = .2, \varepsilon = 1, \alpha \longrightarrow 1$.

x	$\lvert u_{exact} - u_{RPS}\rvert$	$\lvert u_{exact} - u_{qHAM}\rvert$
.1	166.002×10^{-6}	166.002×10^{-6}
.2	162.707×10^{-6}	162.707×10^{-6}
.3	156.257×10^{-6}	156.257×10^{-6}
.4	146.917×10^{-6}	146.917×10^{-6}
.5	135.064×10^{-6}	135.064×10^{-6}

Table 2: The absolute errors $|u_{exact} - u_4|$ for Cahn-Hilliard when $t = .2, \alpha \longrightarrow 1$.

x	$\lvert u_{exact} - u_{RPS}\rvert$	$\lvert u_{exact} - u_{HPM}\rvert$
.1	25.5541×10^{-6}	4.68338×10^{-3}
.2	41.5291×10^{-6}	7.28902×10^{-3}
.3	54.2246×10^{-6}	9.6162×10^{-3}
.4	62.8898×10^{-6}	11.5931×10^{-3}
.5	67.2637×10^{-6}	13.174×10^{-3}

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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