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Research Article

Global and Local Structures of Bifurcation Curves of ODE with Nonlinear Diffusion

Tetsutaro Shibata

Laboratory of Mathematics, Graduate School of Engineering, Hiroshima University, Higashi-Hiroshima, 739-8527, Japan

Correspondence should be addressed to Tetsutaro Shibata; shibata@amath.hiroshima-u.ac.jp

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We consider the nonlinear eigenvalue problem $[D(u)u']' + \lambda f(u) = 0$, $u(t) > 0$, $t \in I := (0, 1)$, $u(0) = u(1) = 0$, where $D(u) = u^k$, $f(u) = u^{2n-k-1} + \sin u$, and $\lambda > 0$ is a bifurcation parameter. Here, $n \in \mathbb{N}$ and k $(0 \le k < 2n - 1)$ are constants. This equation is related to the mathematical model of animal dispersal and invasion, and λ is parameterized by the maximum norm $\alpha = \|u_\lambda\|_\infty$ of the solution u_λ associated with λ and is written as $\lambda = \lambda(\alpha)$. Since $f(u)$ contains both power nonlinear term u^{2n-k-1} and oscillatory term $\sin u$, it seems interesting to investigate how the shape of $\lambda(\alpha)$ is affected by $f(u)$. The purpose of this paper is to characterize the total shape of $\lambda(\alpha)$ by n and k. Precisely, we establish three types of shape of $\lambda(\alpha)$, which seem to be new.

1. Introduction

This paper is concerned with the following nonlinear eigenvalue problems:

$$\left[D(u(t))\,u(t)'\right]' + \lambda f(u(t)) = 0, \quad t \in I := (0,1), \tag{1}$$

$$u(t) > 0, \quad t \in I, \tag{2}$$

$$u(0) = u(1) = 0, \tag{3}$$

where $D(u) = u^k$, $f(u) = u^{2n-k-1} + \sin u$, and $\lambda > 0$ is a bifurcation parameter. Here, $n \in \mathbb{N}$ and k $(0 \le k < 2n - 1)$ are constants. Bifurcation problems have a long history and there are so many results concerning the asymptotic properties of bifurcation diagrams. We refer to [1–8] and the references therein. Moreover, bifurcation problems with nonlinear diffusion have been proposed in the field of population biology, and several model equations of logistic type have been considered. We refer to [9] and the references therein. In particular, the case $D(u) = u^k$ $(k > 0)$ has been derived from a model equation of animal dispersal and invasion in [10, 11]. In this situation, λ is a parameter which represents the habitat size and diffusion rate. On the other hand, there are several papers which treat the asymptotic behavior of oscillatory bifurcation curves. We refer to [7, 12–19] and the references therein. Our equation (1) contains both nonlinear diffusion term and oscillatory nonlinear terms. The purpose of this paper is to find the difference between the structures of bifurcation curves of the equations with only oscillatory term and those with both nonlinear diffusion term and the oscillatory term in (1). To clarify our intention, let $k = 2$ and $n = 2$. Then (1) is given as

$$\left(u^2 u'\right)' + \lambda(u + \sin u) = 0. \tag{4}$$

The corresponding equation without nonlinear diffusion is the case $k = 0$ and $n = 1$, namely,

$$u'' + \lambda(u + \sin u) = 0. \tag{5}$$

It should be mentioned that, by using a generalized time-map argument in [9], for any given $\alpha > 0$, there exists a unique classical solution pair (λ, u_α) of (1)–(3) satisfying $\alpha = \|u_\alpha\|_\infty$. Furthermore, λ is parameterized by α as $\lambda = \lambda(\alpha)$ and is continuous in $\alpha > 0$. For (5), the following asymptotic formula for $\lambda(\alpha)$ as $\alpha \longrightarrow \infty$ has been obtained.

Theorem 1 (see [12]). *Consider (5) with (2)–(3). Then as $\alpha \longrightarrow \infty$,*

$$\lambda(\alpha) = \pi^2 - 4\frac{\pi}{\alpha}\sqrt{\frac{\pi}{2\alpha}}\sin\left(\alpha - \frac{\pi}{4}\right) + o\left(\alpha^{-3/2}\right). \tag{6}$$

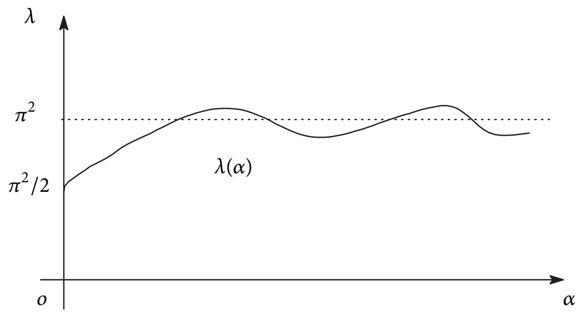

Figure 1: The graph of $\lambda(\alpha)$ for (5) ($k = 0, n = 1$).

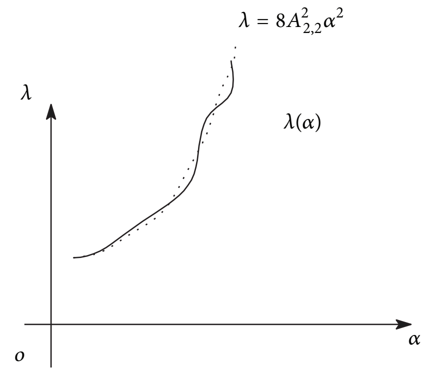

Figure 2: The graph of $\lambda(\alpha)$ for $k = n = 2$.

For (5) with (2)–(3), the asymptotic behavior of $\lambda(\alpha)$ as $\alpha \longrightarrow 0$ is as follows. For a solution pair $(\lambda(\alpha), u_\alpha)$ satisfying $\|u_\alpha\|_\infty = \alpha$, put $v_\alpha(t) := u_\alpha(t)/\alpha$ and let $\alpha \longrightarrow 0$. Then we easily obtain the function $v_0 \in C^2(I)$ which satisfies $-v_0''(t) = 2\lambda(0)v_0(t)$, $v_0(t) > 0$ for $t \in I$ with $v_0(0) = v_0(1) = 0$. This implies $\lambda(0) = \pi^2/2$. By this fact and Theorem 1, the bifurcation curve $\lambda(\alpha)$ starts from $\pi^2/2$ and tends to π^2 with oscillation and intersects the line $\lambda = \pi^2$ infinitely many times for $\alpha \gg 1$.

Since (4) includes both the nonlinear diffusion function and oscillatory term, it seems interesting how the nonlinear diffusion functions give effect to the structures of bifurcation curves.

Now we state our main results.

Theorem 2. *Consider (1) with (2)–(3). Then as $\alpha \longrightarrow \infty$,*

$$\lambda(\alpha) = 4n\alpha^{2k+2-2n}\left\{A_{k,n}^2 - 2A_{k,n}\sqrt{\frac{\pi}{2n}}\alpha^{k+(1/2)-2n}\sin\left(\alpha - \frac{\pi}{4}\right) + o\left(\alpha^{k+(1/2)-2n}\right)\right\}, \tag{7}$$

where

$$A_{k,n} = \int_0^1 \frac{s^k}{\sqrt{1-s^{2n}}}ds. \tag{8}$$

By Theorem 2, we obtain the global behavior of $\lambda(\alpha)$ as $\alpha \longrightarrow \infty$ for $n = k = 2$ and see that the asymptotic behavior of $\lambda(\alpha)$ is completely different from that for $k = 0, n = 1$ by comparing Figures 1 and 2.

Now we establish the asymptotic behavior of $\lambda(\alpha)$ as $\alpha \longrightarrow 0$ to obtain a complete understanding of the structure of $\lambda(\alpha)$. Let

$$B_0 := \int_0^1 \frac{s^k}{\sqrt{1-s^{k+2}}}ds, \tag{9}$$

$$B_1 := \frac{k+2}{12(k+4)}\int_0^1 \frac{s^k\left(1-s^{k+4}\right)}{\left(1-s^{k+2}\right)^{3/2}}ds, \tag{10}$$

$$B_2 = \frac{k+2}{2n}\int_0^1 \frac{s^k\left(1-s^{2n}\right)}{\left(1-s^{k+2}\right)^{3/2}}ds, \tag{11}$$

$$B_3 = \frac{n}{k+2}\int_0^1 \frac{s^k\left(1-s^{k+2}\right)}{\left(1-s^{2n}\right)^{3/2}}ds. \tag{12}$$

Theorem 3. *Consider (1) with (2)–(3). Then the following asymptotic formulas hold as $\alpha \longrightarrow 0$.*

(i) *Assume that $k + 4 < 2n$. Then*

$$\lambda(\alpha) = 2(k+2)\alpha^k\left\{B_0^2 + 2B_0B_1\alpha^2 + o\left(\alpha^2\right)\right\}. \tag{13}$$

(ii) *Assume that $2n = k + 4$. Then*

$$\lambda(\alpha) = 2(k+2)\alpha^k\left\{B_0^2 - 10B_0B_1\alpha^2 + o\left(\alpha^2\right)\right\}. \tag{14}$$

(iii) *Assume that $k + 2 < 2n < k + 4$. Then*

$$\lambda(\alpha) = 2(k+2)\alpha^k\left\{B_0^2 - B_0B_2\alpha^{2n-k-2} + o\left(\alpha^{2n-k-2}\right)\right\}. \tag{15}$$

(iv) *Assume that $2n = k + 2$. Then*

$$\lambda(\alpha) = (k+2)\alpha^k\left\{B_0^2 + B_0B_1\alpha^2 + o\left(\alpha^2\right)\right\}. \tag{16}$$

(v) *Assume that $k + 1 < 2n < k + 2$. Then*

$$\lambda(\alpha) = 4n\alpha^{2(k+1-n)}\left\{A_{k,n}^2 - 2A_{k,n}B_3\alpha^{k+2-2n} + o\left(\alpha^{k+2-2n}\right)\right\}. \tag{17}$$

The rough images of the graphs of $\lambda(\alpha)$ for $k = 1, n = 2$, $n = k = 2$, and $k = 1, n = 3$ are given in Figures 3, 4, and 5.

The proofs depend on the generalized time-map argument in [9] and stationary phase method (cf. Lemma 4). It should be mentioned that if we apply Lemma 4 to our situation, careful consideration about the regularity of the functions is necessary.

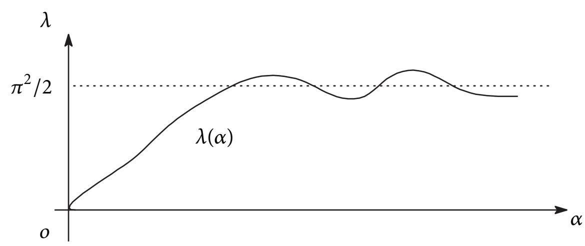

Figure 3: The graph of $\lambda(\alpha)$ for $k = 1, n = 2$.

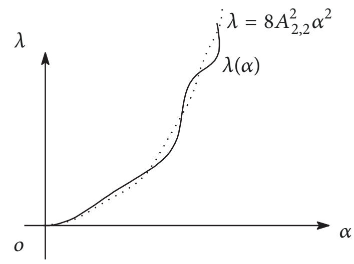

Figure 4: The graph of $\lambda(\alpha)$ for $k = n = 2$.

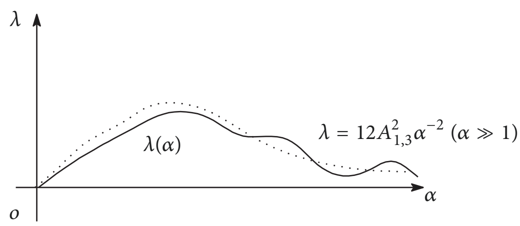

Figure 5: The graph of $\lambda(\alpha)$ for $k = 1, n = 3$.

2. Proof of Theorem 2

We put

$$\Lambda := \left\{\alpha > 0 \mid f(\alpha) > 0, \int_u^\alpha f(t) D(t)\, dt > 0 \text{ for all } u \in [0, \alpha)\right\}. \tag{18}$$

It was shown in [9, (2.7)] that if $\alpha \in \Lambda$, then $\lambda(\alpha)$ is well defined. In our situation, it is clear that, for $t > 0$, $D(t) > 0$, $f(t) > 0$, so $f(t)D(t) > 0$. Therefore, $\Lambda \equiv \mathbb{R}_+$. By this and the generalized time-map obtained in [9] (cf. (24)) and the time-map argument in [8, Theorem 2.1], we see that, for any given $\alpha > 0$, there exists a unique classical solution pair (λ, u_α) of (1)–(3) satisfying $\alpha = \|u_\alpha\|_\infty$. Furthermore, λ is parameterized by α as $\lambda = \lambda(\alpha)$ and is continuous in $\alpha > 0$. For $u \geq 0$, we put

$$\begin{aligned} G(u) &:= \int_0^u f(y) D(y)\, dy = \frac{1}{2n} u^{2n} + G_1(u) \\ &:= \frac{1}{2n} u^{2n} + \int_0^u y^k \sin y\, dy. \end{aligned} \tag{19}$$

It is known from [9] that if $(u_\alpha, \lambda(\alpha)) \in C^2(\overline{I}) \times \mathbb{R}_+$ satisfies (1)–(3), then

$$u_\alpha(t) = u_\alpha(1 - t), \quad 0 \leq t \leq 1, \tag{20}$$

$$u_\alpha\left(\frac{1}{2}\right) = \max_{0 \leq t \leq 1} u_\alpha(t) = \alpha, \tag{21}$$

$$u_\alpha'(t) > 0, \quad 0 < t < \frac{1}{2}. \tag{22}$$

In what follows, we denote by C various positive constants independent of $\alpha \gg 1$. For $0 \leq s \leq 1$ and $\alpha \gg 1$, we have

$$\begin{aligned} \left|\frac{G_1(\alpha) - G_1(\alpha s)}{\alpha^{2n}(1 - s^{2n})}\right| &= \left|\frac{\int_{\alpha s}^{\alpha} w^k \sin w\, dw}{\alpha^{2n}(1 - s^{2n})}\right| \\ &\leq C \frac{\alpha^{k+1}(1 - s^{k+1})}{\alpha^{2n}(1 - s^{2n})} \leq C\alpha^{k+1-2n} \\ &\ll 1. \end{aligned} \tag{23}$$

By this, (19), and Taylor expansion, we have from [9, (2.5)] that

$$\begin{aligned} \sqrt{\frac{\lambda(\alpha)}{2}} &= \int_0^\alpha \frac{D(u)}{\sqrt{G(\alpha) - G(u)}} du \\ &= \int_0^\alpha \frac{u^k}{\sqrt{(1/2n)(\alpha^{2n} - u^{2n}) + G_1(\alpha) - G_1(u)}} du \\ &= \sqrt{2n}\alpha^{k+1-n} \int_0^1 \frac{s^k}{\sqrt{1 - s^{2n} + (2n/\alpha^{2n})(G_1(\alpha) - G_1(\alpha s))}} ds \\ &= \sqrt{2n}\alpha^{k+1-n} \int_0^1 \frac{s^k}{\sqrt{1 - s^{2n}}} \left\{1 - \frac{n}{\alpha^{2n}} \frac{G_1(\alpha) - G_1(\alpha s)}{(1 - s^{2n})} (1 + o(1))\right\} ds \\ &= \sqrt{2n}\alpha^{k+1-n} \left\{\int_0^1 \frac{s^k}{\sqrt{1 - s^{2n}}} ds - \frac{n}{\alpha^{2n}} L(\alpha)(1 + o(1))\right\}, \end{aligned} \tag{24}$$

where

$$L(\alpha) := \int_0^1 \frac{s^k}{(1 - s^{2n})^{3/2}} (G_1(\alpha) - G_1(\alpha s))\, ds. \tag{25}$$

We see from (24) and (25) that if we obtain the precise asymptotic formula for $L(\alpha)$ as $\alpha \longrightarrow \infty$, then we obtain Theorem 2. To do this, we apply the stationary phase method to our situation. By combining [13, Lemma 2] and [7, Lemmas 2.24], we have the following equality.

Lemma 4 (see [13, Lemma 2 and 10, Lemma 2.24]). *Assume that the function $f(r) \in C^2[0,1]$, $w(r) \in C^3[0,1]$, and*

$$\begin{aligned} &w'(r) < 0, \quad r \in (0, 1], \\ &w'(0) = 0, \\ &w''(0) < 0. \end{aligned} \tag{26}$$

Then as $\mu \longrightarrow \infty$

$$\int_0^1 f(r) e^{i\mu w(r)} dr = \frac{1}{2} e^{i(\mu w(0)-(\pi/4))} \sqrt{\frac{2\pi}{\mu |w''(0)|}} f(0) + O\left(\frac{1}{\mu}\right). \tag{27}$$

In particular, by taking the imaginary part of (27), as $\mu \longrightarrow \infty$,

$$\int_0^1 f(r) \sin(\mu w(r)) dr = \frac{1}{2} \sqrt{\frac{2\pi}{\mu |w''(0)|}} f(0) \sin\left(w(0)\mu - \frac{\pi}{4}\right) + O\left(\frac{1}{\mu}\right). \tag{28}$$

We note that, to obtain (27), we have to be careful about the regularity of f and w.

Lemma 5. *As $\alpha \longrightarrow \infty$,*

$$L(\alpha) = \sqrt{\frac{\pi}{2}} \frac{1}{n^{3/2}} \alpha^{k+(1/2)} \sin\left(\alpha - \frac{\pi}{4}\right) + O\left(\alpha^k\right). \tag{29}$$

Proof. We put $s = \sin\theta$ and

$$Y(\theta) := Y_1(\theta)\left(G_1(\alpha) - G_1(\alpha \sin\theta)\right) := \frac{\sin^k\theta}{\left(1 + \sin^2\theta + \cdots + \sin^{2n-2}\theta\right)^{3/2}} \left(G_1(\alpha) - G_1(\alpha\sin\theta)\right). \tag{30}$$

By integration by parts, (25) and (30), we have

$$\begin{aligned} L(\alpha) &= \int_0^1 \frac{s^k \left(G_1(\alpha) - G_1(\alpha s)\right)}{\left(1-s^2\right)^{3/2} \left(1 + s^2 + \cdots + s^{2n-2}\right)^{3/2}} ds \\ &= \int_0^{\pi/2} \frac{1}{\cos^2\theta} \frac{\sin^k\theta \left(G_1(\alpha) - G_1(\alpha\sin\theta)\right)}{\left(1 + \sin^2\theta + \cdots + \sin^{2n-2}\theta\right)^{3/2}} d\theta \\ &:= L_1(\alpha) - L_2(\alpha) \\ &= \left[\tan\theta Y(\theta)\right]_0^{\pi/2} \\ &\quad - \int_0^{\pi/2} \tan\theta \left\{Y_1(\theta)\left(G_1(\alpha) - G_1(\alpha\sin\theta)\right)\right\}' d\theta. \end{aligned} \tag{31}$$

By l'Hôpital's rule, we obtain

$$\lim_{\theta \longrightarrow \pi/2} \frac{G_1(\alpha) - G_1(\alpha\sin\theta)}{\cos\theta} = \lim_{\theta \longrightarrow \pi/2} \frac{\alpha\cos\theta (\alpha\sin\theta)^k \sin(\alpha\sin\theta)}{\sin\theta} = 0. \tag{32}$$

This implies that $L_1(\alpha) = 0$. Next,

$$\begin{aligned} L_2(\alpha) &= \int_0^{\pi/2} \tan\theta \left\{Y_1'(\theta)\left(G_1(\alpha) - G_1(\alpha\sin\theta)\right)\right. \\ &\quad \left. - Y_1(\theta)\alpha\cos\theta(\alpha\sin\theta)^k \sin(\alpha\sin\theta)\right\} d\theta. \\ &:= L_{21}(\alpha) - L_{22}(\alpha). \end{aligned} \tag{33}$$

We first calculate $L_{21}(\alpha)$. Assume that $k > 0$. Then

$$Y_1'(\theta) = \frac{\sin^{k-1}\theta\cos\theta}{\left(1 + \sin^2\theta + \cdots + \sin^{2n-2}\theta\right)^{3/2}} \times \left[k - \frac{3\left(\sin^2\theta + 2\sin^4\theta + \cdots + (n-1)\sin^{2n-2}\theta\right)}{1 + \sin^2\theta + \cdots + \sin^{2n-2}\theta}\right]. \tag{34}$$

This implies that, for $\alpha \gg 1$,

$$\left|\tan\theta Y_1'(\theta)\right| \le C\left|\sin^k\theta\right| \le C. \tag{35}$$

By direct calculation, we also obtain (35) for the case where $k = 0$. By integration by parts, we obtain

$$\begin{aligned} \left|G_1(\alpha) - G_1(\alpha\sin\theta)\right| &= \left|\int_{\alpha\sin\theta}^{\alpha} w^k \sin w dw\right| \\ &\le \left|\left[-w^k\cos w\right]_{\alpha\sin\theta}^{\alpha}\right| + \left|\int_{\alpha\sin\theta}^{\alpha} kw^{k-1}\cos w dw\right| \\ &\le C\alpha^k. \end{aligned} \tag{36}$$

By (35) and (36), for $\alpha \gg 1$, we obtain

$$\left|L_{21}(\alpha)\right| = \left|\tan\theta Y_1'(\theta)\left(G_1(\alpha) - G_1(\alpha\sin\theta)\right)\right| \le C\alpha^k. \tag{37}$$

Since

$$L_{22}(\alpha) = \alpha^{k+1} \int_0^{\pi/2} Y_1(\alpha) \sin^{k+1}\theta \sin(\alpha\sin\theta) d\theta, \tag{38}$$

by putting $\theta = (\pi/2)(1-r)$, we obtain

$$L_{22}(\alpha) = \frac{\pi}{2}\alpha^{k+1} \int_0^1 \frac{\cos^{2k+1}(\pi/2)r}{\left(1 + \cos^2(\pi/2)r + \cdots + \cos^{2n-2}(\pi/2)r\right)^{3/2}} \sin\left(\alpha\cos\frac{\pi}{2}r\right) dr. \tag{39}$$

Let

$$f(r) := \frac{\cos^{2k+1}(\pi/2)r}{\left(1+\cos^2(\pi/2)r+\cdots+\cos^{2n-2}(\pi/2)r\right)^{3/2}},$$
$$w(r) := \cos\frac{\pi}{2}r, \tag{40}$$
$$\mu := \alpha.$$

Case 1. Assume that $k > 1/2$ or $k = 0$. Then clearly $f(r) \in C^2[0,1]$, and we are able to apply Lemma 4 to (39). Then we obtain

$$L_{22}(\alpha) = \sqrt{\frac{\pi}{2}}\frac{1}{n^{3/2}}\alpha^{k+(1/2)}\sin\left(\alpha-\frac{\pi}{4}\right)+O\left(\alpha^k\right). \tag{41}$$

By this, (33), and (37), we obtain (29).

Case 2. Assume that $0 < k < 1/2$. Then $f(r) \in C^{1+2k}[0,1]$ with $0 < 2k < 1$. Therefore, $f(r)$ does not satisfy the condition in Lemma 4. However, we found in [14] that we can still apply Lemma 4 to (39) in this situation and obtain (41). For completeness, the reason will be explained in the Appendix. By this, (33), and (41), we obtain (29). Thus the proof is complete. □

By (24) and Lemma 5, we obtain Theorem 2 immediately. Thus the proof is complete.

3. Proof of Theorem 3

In this section, let $0 < \alpha \ll 1$. The proofs of Theorem 3 (i)-(v) are similar. Therefore, we only prove (i) and (iv).

Proof of Theorem 3 (i). We assume that $2n > k+4$. Then by Taylor expansion, for $0 \le s \le 1$, we have

$$\begin{aligned} G(\alpha) - G(\alpha s) &= \frac{1}{2n}\alpha^{2n}\left(1-s^{2n}\right)+\frac{1}{k+2}\alpha^{k+2}\left(1-s^{k+2}\right) \\ &\quad -\frac{1}{6(k+4)}\alpha^{k+4}\left(1-s^{k+4}\right)(1+o(1)). \end{aligned} \tag{42}$$

By this, (24), Taylor expansion, and putting $u = \alpha s$, we obtain

$$\begin{aligned} \sqrt{\frac{\lambda(\alpha)}{2}} &= \int_0^\alpha \frac{u^k}{\sqrt{(1/2n)\left(\alpha^{2n}-u^{2n}\right)+(1/(k+2))\left(\alpha^{k+2}-u^{k+2}\right)-(1/6(k+4))\left(\alpha^{k+4}-u^{k+4}\right)(1+o(1))}}du \\ &= \sqrt{k+2}\,\alpha^{k/2}\int_0^1 \frac{s^k}{\sqrt{1-s^{k+2}}\sqrt{1-((k+2)/6(k+4))\left(\left(1-s^{k+4}\right)/\left(1-s^{k+2}\right)\right)\alpha^2+o\left(\alpha^2\right)}}ds \\ &= \sqrt{k+2}\,\alpha^{k/2}\int_0^1 \frac{s^k}{\sqrt{1-s^{k+2}}}\left(1+\frac{k+2}{12(k+4)}\frac{1-s^{k+4}}{1-s^{k+2}}\alpha^2+o\left(\alpha^2\right)\right)ds = \sqrt{k+2}\,\alpha^{k/2}\left\{B_0+B_1\alpha^2+o\left(\alpha^2\right)\right\}. \end{aligned} \tag{43}$$

This implies (13). Thus the proof is complete. □

Proof of Theorem 3 (iv). We assume that $2n = k+2$. Then by (42), for $0 \le s \le 1$, we have

$$\begin{aligned} G(\alpha) - G(w) &= \frac{2}{k+2}\left(\alpha^{k+2}-w^{k+2}\right) \\ &\quad -\frac{1}{6(k+4)}\left(\alpha^{k+4}-w^{k+4}\right)(1+o(1)). \end{aligned} \tag{44}$$

By this, (24), and putting $w = \alpha s$, we obtain

$$\begin{aligned} \sqrt{\frac{\lambda(\alpha)}{2}} &= \int_0^\alpha \frac{w^k}{\sqrt{(2/(k+2))\left(\alpha^{k+2}-w^{k+2}\right)-(1/6(k+4))\left(\alpha^{k+4}-w^{k+4}\right)(1+o(1))}}dw \\ &= \sqrt{\frac{k+2}{2}}\alpha^{k/2}\int_0^1 \frac{s^k}{\sqrt{1-s^{k+2}}}\left\{1-\frac{k+2}{12(k+4)}\frac{1-s^{k+4}}{1-s^{k+2}}\alpha^2+o\left(\alpha^2\right)\right\}^{-1/2}ds \\ &= \sqrt{\frac{k+2}{2}}\alpha^{k/2}\int_0^1 \frac{s^k}{\sqrt{1-s^{k+2}}}\left\{1+\frac{k+2}{24(k+4)}\frac{1-s^{k+4}}{1-s^{k+2}}\alpha^2+o\left(\alpha^2\right)\right\}ds. \end{aligned} \tag{45}$$

This implies

$$\sqrt{\lambda} = \sqrt{k+2}\alpha^{k/2}\left\{B_0 + \frac{1}{2}B_1\alpha^2 + o\left(\alpha^2\right)\right\}. \quad (46)$$

This implies (16). Thus the proof is complete. □

Appendix

In this section, by following the argument in [14], we show that Case 2 in Lemma 5 holds for completeness. We put

$$f(x) = f_1(x) f_2(x) := \cos^{2k+1}\frac{\pi}{2}x \cdot \frac{1}{\left(1+\cos^2(\pi/2)x + \cdots + \cos^{2n-2}(\pi/2)x\right)^{3/2}}. \quad (A.1)$$

Note that $0 < 2k < 1$. We see that $f_2(x) \in C^2[0,1]$. The essential point of the proof of (27) in this case is to show Lemma 2.24 in [7] (see also [7, Lemma 2.25]). Namely, as $\mu \longrightarrow \infty$,

$$\Phi(\mu) := \int_0^1 f(x) e^{-i\mu x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\mu}}e^{-i(\pi/4)} f(0) + O\left(\frac{1}{\mu}\right). \quad (A.2)$$

We put $h(x) = (f(x) - f(0))/x$. Then we have $f(x) = f(0) + xh(x)$. We know from [7, Lemma 2.24] that, for $\mu \gg 1$,

$$\int_0^1 e^{-i\mu x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\mu}}e^{-i\pi/4} + O\left(\frac{1}{\mu}\right). \quad (A.3)$$

By (A.2) and (A.3), we obtain

$$\begin{aligned}\Phi(\mu) &= f(0)\int_0^1 e^{-i\mu x^2} dx + \int_0^1 x e^{-i\mu x^2} h(x)\,dx \\ &= \frac{1}{2} f(0)\sqrt{\frac{\pi}{\mu}}e^{-i\pi/4} + O\left(\frac{1}{\mu}\right) \\ &\quad + \int_0^1 x e^{-i\mu x^2} h(x)\,dx.\end{aligned} \quad (A.4)$$

We put

$$\Phi_1(\mu) := \int_0^1 x e^{-i\mu x^2} h(x)\,dx. \quad (A.5)$$

Now we prove that $h(x) \in C^1[0,1]$, because if it is proved, then by integration by parts, we easily show that $\Phi_1(\mu) = O(1/\mu)$ and our conclusion (A.2) follows immediately from (A.4) and (A.5). For $0 \le x \le 1$, we have

$$\begin{aligned}h(x) &= \frac{f(x) - f(0)}{x} \\ &= f_2(x)\frac{f_1(x) - f_1(0)}{x} + f_1(0)\frac{f_2(x) - f_2(0)}{x} \\ &:= f_2(x) h_1(x) + f_1(0) h_2(x).\end{aligned} \quad (A.6)$$

Then we have $h_2(x) \in C^1[0,1]$. Furthermore, by direct calculation, we can show that $h_1(x) \in C^1[0,1]$. It is reasonable, because by Taylor expansion, for $0 < x \ll 1$, we have

$$h_1(x) = -\frac{(2k+1)\pi^2}{8}x + O\left(x^3\right). \quad (A.7)$$

Thus the proof is complete.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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