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Research Article

About a Problem for Loaded Parabolic-Hyperbolic Type Equation with Fractional Derivatives

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An existence and uniqueness of solution of local boundary value problem with discontinuous matching condition for the loaded parabolic-hyperbolic equation involving the Caputo fractional derivative and Riemann-Liouville integrals have been investigated. The uniqueness of solution is proved by the method of integral energy and the existence is proved by the method of integral equations. Let us note that, from this problem, the same problem follows with continuous gluing conditions (at $\lambda = 1$); thus an existence theorem and uniqueness theorem will be correct and on this case.

1. Introduction and Formulation of a Problem

Development of the theory of the equations with fractional derivatives is stimulated with development of the theory of the integer order differential equations. About applications to physics, biology, mathematical modeling, and so forth, one can find works [1–3]. Notice works [4–7], devoted to the studying of BVPs for parabolic-hyperbolic equations, involving fractional derivatives. BVPs for the mixed type equations involving the Caputo and the Riemann-Liouville fractional differential operators were investigated in works [8, 9].

Note that with intensive research on problem of optimal control of the agroeconomic system, regulating the label of ground waters and soil moisture, it has become necessary to investigate a new class of equations called "loaded equations." It was given the most general definition of "loaded equations" and various loaded equations are classified in detail by Nakhushev [10]. After this work very interesting results on the theory of boundary value problems for the loaded equations parabolic, parabolic-hyperbolic, and elliptic-hyperbolic types were published; for example, see [11–13].

In this direction, some local and nonlocal problems for the loaded elliptic-hyperbolic type equations of the second and third order in double-connected domains were investigated (see [14–17]).

BVPs for the loaded mixed type equations with fractional derivative have not been investigated yet.

In the given work, we consider the following equation:

$$0 = \begin{cases} u_{xx} - {}_C D_{oy}^{\alpha} u + p(x) \int_x^1 (t-x)^{\beta-1} u(t,0)\, dt, & \text{at } y > 0 \\ u_{xx} - u_{yy} + q(x+y) \int_{x+y}^1 (t-x-y)^{\gamma-1} u(t,0)\, dt, & \text{at } y < 0, \end{cases} \tag{1}$$

with the following operation [18]:

$$ {}_C D_{oy}^{\alpha} f = \frac{1}{\Gamma(1-\alpha)} \int_0^y (y-t)^{-\alpha} f'(t)\, dt, \quad (2) $$

where $0 < \alpha, \beta, \gamma < 1$.

Let Ω be domain, bounded with segments $\overline{A_1A_2} = \{(x,y): x = 1,\ 0 \le y \le h\}$, $\overline{B_1B_2} = \{(x,y): x = 0,\ 0 \le y \le h\}$, and $\overline{B_2A_2} = \{(x,y): y = h, 0 \le x \le 1\}$ at $y > 0$ and characteristics A_1C: $x - y = 1$; B_1C: $x + y = 0$ of (1) at $y < 0$, where $A_1(1;0)$, $A_2(1;h)$, $B_1(0;0)$, $B_2(0;h)$, $C(1/2;-1/2)$.

Let us enter designations

$$ D_{xa}^{-\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_x^a (t-x)^{\beta-1} f(t)\, dt, \quad 0 < \beta < 1, \quad (3) $$

$$ \begin{aligned} \Omega^+ &= \Omega \cap (y > 0), \\ \Omega^- &= \Omega \cap (y < 0), \\ I_1 &= \left\{x: \frac{1}{2} < x < 1\right\}, \\ I_2 &= \{y: 0 < y < h\}. \end{aligned} \quad (4) $$

In the domain of Ω the following problem is investigated.

Problem 1. To find a solution $u(x,y)$ of (1) from the class of functions,

$$ \begin{aligned} W = \Big\{ & u(x,y): u(x,y) \in C(\overline{\Omega}) \cap C^2(\Omega^-),\ u_{xx} \\ & \in C(\Omega^+),\ {}_C D_{oy}^{\alpha} u \in C(\Omega^+) \Big\}, \end{aligned} \quad (5) $$

satisfying boundary conditions

$$ u(x,y)|_{A_1A_2} = \varphi(y), \quad y \in \overline{I}_2, \quad (6) $$

$$ u(x,y)|_{B_1B_2} = \psi(y), \quad y \in \overline{I}_2, \quad (7) $$

$$ u(x,y)|_{A_1C} = \omega(x), \quad x \in \overline{I_1}, \quad (8) $$

and gluing condition

$$ \lim_{y \to +0} y^{1-\alpha} u_y(x,y) = \lambda u_y(x,-0), \quad (x,0) \in A_1B_1, \quad (9) $$

where $\varphi(y)$, $\psi(y)$, and $\omega(x)$ are given functions, $\lambda = \text{const}$ ($\lambda \ne 0$), and besides $\omega(1) = \varphi(0)$.

2. The Uniqueness of Solution of Problem 1

It is known that (1) at $y \le 0$ on the characteristics coordinates $\xi = x + y$ and $\eta = x - y$ totally looks like

$$ u_{\xi\eta} = -\frac{q(\xi)}{4} \int_\xi^1 (t-\xi)^{\gamma-1} u(t,0)\, dt. \quad (10) $$

Let us enter designations $u(x,0) = \tau(x)$, $0 \le x \le 1$; $u_y(x,-0) = \nu^-(x)$, $0 < x < 1$;

$$ \lim_{y \to +0} y^{1-\alpha} u_y(x,y) = \nu^+(x), \quad 0 < x < 1. \quad (11) $$

It is known that solution of the Cauchy problem for (1) in the domain of Ω^- can be represented as follows:

$$ \begin{aligned} u(x,y) &= \frac{\tau(x+y) + \tau(x-y)}{2} - \frac{1}{2} \int_{x+y}^{x-y} \nu^-(t)\, dt \\ &- \frac{1}{4} \int_{x+y}^{x-y} q(\xi)\, d\xi \int_\xi^{x-y} d\eta \int_\xi^1 (t-\xi)^{\gamma-1} \tau(t)\, dt. \end{aligned} \quad (12) $$

After using condition (8) and taking (3) into account from (12) we will get

$$ \begin{aligned} \nu^-(x) = {} & \frac{x-1}{2} \Gamma(\gamma)\, q(x)\, D_{x1}^{-\gamma} \tau(x) - \tau'(x) \\ & + \omega'\left(\frac{x+1}{2}\right). \end{aligned} \quad (13) $$

Considering designations and gluing condition (9) we have

$$ \nu^+(x) = \lambda \nu^-(x). \quad (14) $$

Further from (1) at $y \to +0$ taking into account (2), (14), and

$$ \lim_{y \to 0} D_{0y}^{\alpha-1} f(y) = \Gamma(\alpha) \lim_{y \to 0} y^{1-\alpha} f(y), \quad (15) $$

we get [8]

$$ \tau''(x) - \lambda \Gamma(\alpha)\, \nu^-(x) + \Gamma(\beta)\, p(x)\, D_{x1}^{-\beta} \tau(x) = 0. \quad (16) $$

Theorem 2. *Satisfying conditions*

$$ \begin{aligned} p(0) &\le 0, \\ p'(x) &\le 0; \\ \lambda q(0) &\le 0, \\ \lambda\left((x-1)\, q(x)\right)' &\ge 0, \end{aligned} \quad (17) $$

the solution $u(x,y)$ of Problem 1 is unique.

Proof. It is known that if homogeneous problem has only trivial solution, then we can state that original problem has unique solution. For this aim we assume that Problem 1 has two solutions; then denoting difference of these solutions as $u(x,y)$ we will get appropriate homogenous problem.

We multiply (16) to $\tau(x)$ and integrated it from 0 to 1:

$$ \begin{aligned} & \int_0^1 \tau''(x)\, \tau(x)\, dx - \lambda \Gamma(\alpha) \int_0^1 \tau(x)\, \nu^-(x)\, dx \\ & + \Gamma(\beta) \int_0^1 \tau(x)\, p(x)\, D_{x1}^{-\beta} \tau(x)\, dx = 0. \end{aligned} \quad (18) $$

We will investigate the integral

$$ \begin{aligned} I = {} & \lambda \Gamma(\alpha) \int_0^1 \tau(x)\, \nu^-(x)\, dx \\ & - \Gamma(\beta) \int_0^1 \tau(x)\, p(x)\, D_{x1}^{-\beta} \tau(x)\, dx. \end{aligned} \quad (19) $$

Taking (13) into account, $\omega(x) = 0$, we get

$$\begin{aligned} I &= \frac{\lambda\Gamma(\alpha)\Gamma(\gamma)}{2}\int_0^1 \tau(x)(x-1)q(x)D_{x1}^{-\gamma}\tau(x)\,dx \\ &\quad - \lambda\Gamma(\alpha)\int_0^1 \tau(x)\tau'(x)\,dx - \Gamma(\beta) \\ &\quad \cdot \int_0^1 \tau(x)p(x)D_{x1}^{-\beta}\tau(x)\,dx = \frac{\lambda\Gamma(\alpha)}{2} \\ &\quad \cdot \int_0^1 q(x)\tau(x)(1-x)\,dx\int_x^1 (t-x)^{\gamma-1}\tau(t)\,dt \\ &\quad - \frac{\lambda\Gamma(\alpha)}{2}\int_0^1 d\left(\tau^2(x)\right) - \int_0^1 \tau(x)p(x)\,dx \\ &\quad \cdot \int_x^1 (t-x)^{\beta-1}\tau(t)\,dt. \end{aligned} \tag{20}$$

Considering $\tau(1) = 0$, $\tau(0) = 0$ (deduced from conditions (6) and (7) in homogeneous case) and on a base of the formula (see [19, p. 188]),

$$\begin{aligned} &|x-t|^{-\gamma} \\ &\quad = \frac{1}{\Gamma(\gamma)\cos(\pi\gamma/2)}\int_0^\infty z^{\gamma-1}\cos[z(x-t)]\,dz, \\ &\qquad 0 < \gamma < 1. \end{aligned} \tag{21}$$

After some simplifications from (20) we will get

$$\begin{aligned} I &= \frac{\lambda\Gamma(\alpha)q(0)}{4\Gamma(1-\gamma)\sin(\pi\gamma/2)} \\ &\quad \cdot \int_0^\infty z^{-\gamma}\left[\left(\int_0^1 \tau(t)\cos zt\,dt\right)^2\right. \\ &\quad \left. + \left(\int_0^1 \tau(t)\sin zt\,dt\right)^2\right]dz \\ &\quad + \frac{\lambda\Gamma(\alpha)}{4\Gamma(1-\gamma)\sin(\pi\gamma/2)}\int_0^\infty z^{-\gamma}dz \\ &\quad \cdot \int_0^1 \frac{\partial}{\partial x}[(x-1)q(x)]\left[\left(\int_x^1 \tau(t)\cos zt\,dt\right)^2\right. \\ &\quad \left. + \left(\int_x^1 \tau(t)\sin zt\,dt\right)^2\right]dx \\ &\quad - \frac{p(0)}{2\Gamma(1-\beta)\sin(\pi\beta/2)} \\ &\quad \cdot \int_0^\infty z^{-\beta}\left[\left(\int_0^1 \tau(t)\cos zt\,dt\right)^2\right. \\ &\quad \left. + \left(\int_0^1 \tau(t)\sin zt\,dt\right)^2\right]dz \\ &\quad - \frac{1}{2\Gamma(1-\beta)\sin(\pi\beta/2)}\int_0^\infty z^{-\beta}dz\int_0^1 \frac{\partial}{\partial x}[p(x)] \\ &\quad \cdot \left[\left(\int_x^1 \tau(t)\cos zt\,dt\right)^2\right. \\ &\quad \left. + \left(\int_x^1 \tau(t)\sin zt\,dt\right)^2\right]dx. \end{aligned} \tag{22}$$

Thus, owing to (17) from (22) it is concluded that $\tau(x) \equiv 0$. Hence, based on the solution of the first boundary problem for (1) [9, 20] taking into account (6) and (7) we will get $u(x,y) \equiv 0$ in $\overline{\Omega^+}$. Further, from functional relations (13), taking into account $\tau(x) \equiv 0$, we get that $\nu^-(x) \equiv 0$. Consequently, based on the solution (12) we obtain $u(x,y) \equiv 0$ in closed domain $\overline{\Omega^-}$. □

3. The Existence of Solution of Problem 1

Theorem 3. *Satisfying conditions (17) and*

$$\begin{aligned} &\varphi(y), \psi(y) \in C\left(\overline{I_2}\right) \cap C^1(I_2), \\ &\omega(x) \in C^1\left(\overline{I_1}\right) \cap C^3(I_1), \\ &p(x) \in C\left(\overline{A_1B_1}\right) \cap C^2(A_1B_1), \\ &q(x+y) \in C\left(\overline{\Omega^-}\right) \cap C^2(\Omega^-), \end{aligned} \tag{23}$$

the solution of the investigating problem exists.

Taking (13) into account from (16) we will obtain

$$\begin{aligned} &\tau''(x) + \frac{\lambda}{2}\Gamma(\alpha)\Gamma(\gamma)(1-x)q(x)D_{x1}^{-\gamma}\tau(x) \\ &\quad + \lambda\Gamma(\alpha)\tau'(x) \\ &\quad = \lambda\Gamma(\alpha)\omega'\left(\frac{x+1}{2}\right) - \Gamma(\beta)p(x)D_{x1}^{-\beta}\tau(x). \end{aligned} \tag{24}$$

From here,

$$\tau''(x) + \lambda\Gamma(\alpha)\tau'(x) = f(x), \tag{25}$$

where

$$\begin{aligned} f(x) &= \frac{\lambda}{2}\Gamma(\alpha)\Gamma(\gamma)(x-1)q(x)D_{x1}^{-\gamma}\tau(x) \\ &\quad + \lambda\Gamma(\alpha)\omega'\left(\frac{x+1}{2}\right) - \Gamma(\beta)p(x)D_{x1}^{-\beta}\tau(x). \end{aligned} \tag{26}$$

Solution of (25) together with conditions

$$\begin{aligned} \tau(0) &= \psi(0), \\ \tau(1) &= \varphi(0) \end{aligned} \tag{27}$$

has a form

$$\tau(x) = (1-x)\psi(0) + x\varphi(0) + \int_0^1 G(x,t) f_1(t)\,dt, \quad (28)$$

where $f_1(x) = f(x) + \lambda\Gamma(\alpha)[\psi(0) - \varphi(0)]$. One has

$$G(x,t) = \begin{cases} \dfrac{\left(e^{\lambda\Gamma(\alpha)x} - 1\right)\left(e^{\lambda\Gamma(\alpha)t} - e^{\lambda\Gamma(\alpha)}\right)}{e^{\lambda\Gamma(\alpha)x}\left(e^{\lambda\Gamma(\alpha)} - 1\right)\lambda\Gamma(\alpha)}; & 0 \le t \le x \\ \dfrac{\left(e^{\lambda\Gamma(\alpha)t} - 1\right)\left(e^{\lambda\Gamma(\alpha)t} - e^{\lambda\Gamma(\alpha)}\right)}{e^{\lambda\Gamma(\alpha)x}\left(e^{\lambda\Gamma(\alpha)} - 1\right)\lambda\Gamma(\alpha)}; & t \le x \le 1. \end{cases} \quad (29)$$

$G(x,t)$ is Green's functions of the boundary value problem for (25) with conditions (27). Further, considering (26) and using (3) from (28) we will get

$$\begin{aligned} \tau(x) = {} & \frac{\lambda\Gamma(\alpha)}{2}\int_0^x \tau(t)\,dt \\ & \cdot \int_0^t (s-1)(t-s)^{\gamma-1} K_1(x,s) q(s)\,ds \\ & - \int_0^x \tau(t)\,dt \int_0^t (t-s)^{\beta-1} K_1(x,s) p(s)\,ds \\ & + \frac{\lambda\Gamma(\alpha)}{2}\int_x^1 \tau(t)\,dt \\ & \cdot \int_0^x (s-1)(t-s)^{\gamma-1} K_1(x,s) q(s)\,ds \\ & - \int_x^1 \tau(t)\,dt \int_0^x (t-s)^{\beta-1} K_1(x,s) p(s)\,ds \\ & + \frac{\lambda\Gamma(\alpha)}{2}\int_x^1 \tau(t)\,dt \\ & \cdot \int_t^x (s-1)(t-s)^{\gamma-1} K_2(x,s) q(s)\,ds \\ & - \int_x^1 \tau(t)\,dt \int_x^t (t-s)^{\beta-1} K_2(x,s) p(s)\,ds \\ & + F(x), \end{aligned} \quad (30)$$

where

$$\begin{aligned} F(x) = {} & \lambda\Gamma(\alpha)\int_0^1 G(x,t)\,\omega'\left(\frac{t+x}{2}\right)dt \\ & - \lambda\Gamma(\alpha)\left(\psi(0) - \varphi(0)\right)\int_0^1 G(x,t)\,dt, \\ K_1(x,t) = {} & \frac{\left(e^{\lambda\Gamma(\alpha)t} - 1\right)\left(e^{\lambda\Gamma(\alpha)x} - e^{\lambda\Gamma(\alpha)}\right)}{e^{\lambda\Gamma(\alpha)x}\left(e^{\lambda\Gamma(\alpha)} - 1\right)\lambda\Gamma(\alpha)}; \\ K_2(x,t) = {} & \frac{\left(e^{\lambda\Gamma(\alpha)x} - 1\right)\left(e^{\lambda\Gamma(\alpha)t} - e^{\lambda\Gamma(\alpha)}\right)}{e^{\lambda\Gamma(\alpha)x}\left(e^{\lambda\Gamma(\alpha)} - 1\right)\lambda\Gamma(\alpha)}. \end{aligned} \quad (31)$$

Folding separately the integrals with limits $\int_0^x$ and $\int_x^1$ we rewrite integral equation (30) as follows:

$$\tau(x) = \int_0^1 K(x,t)\,\tau(t)\,dt + F(x). \quad (32)$$

Here

$$K(x,t) = \begin{cases} \displaystyle\int_0^t K_1(x,s)\left[\frac{\lambda\Gamma(\alpha)}{2}(1-s)(t-s)^{\gamma-1} q(s) - (t-s)^{\beta-1} p(s)\right]ds; & 0 \le t \le x \\ \displaystyle\int_0^x K_1(x,s)\left[\frac{\lambda\Gamma(\alpha)}{2}(s-1)(t-s)^{\gamma-1} q(s) - (t-s)^{\beta-1} p(s)\right]ds + \int_x^t K_2(x,s)\left[\frac{\lambda\Gamma(\alpha)}{2}(s-1)(t-s)^{\gamma-1} q(s) - (t-s)^{\beta-1} p(s)\right]ds; & x \le t \le 1. \end{cases} \quad (33)$$

For better understanding, we present the statement of equivalence.

Remark 4. Satisfying all conditions of Theorem 3, the existence and the uniqueness of solutions of Problem 1 in the class of W are equivalent to the unique solvability of the Fredholm type integral equation (32) in the class of $C^1[0,1] \cap C^2(0,1)$.

Now we need to investigate the integral equation (32); for this aim we will estimate the kernel and the right side.

Owing to class (23) of the given functions and after some evaluations from (33) we will conclude that

$$\begin{aligned} \left|K_1(x,t)\right| = {} & \frac{\left(e^{\lambda\Gamma(\alpha)t} - 1\right)}{\left(e^{\lambda\Gamma(\alpha)} - 1\right)\lambda\Gamma(\alpha)}\left|1 - e^{\lambda\Gamma(\alpha)(1-x)}\right| \\ \le {} & \frac{\left|1 - e^{\lambda\Gamma(\alpha)}\right|}{\lambda\Gamma(\alpha)} = \text{const} < \infty \quad \text{at } 0 \le t \le x, \\ \left|K(x,t)\right| = {} & \left|\int_0^t K_1(x,s)\left[\frac{\lambda\Gamma(\alpha)}{2}(1-s)(t-s)^{\gamma-1} q(s)\right.\right. \end{aligned}$$

$$- (t-s)^{\beta-1} p(s) \Big] ds \Big| \le \text{const} \left| \int_0^t (t - s)^{\sigma-1} ds \right| \le \text{const} \cdot t^{\sigma} \le \text{const}$$
$$\text{at } 0 \le t \le x, \ \text{where } \sigma = \min\{\gamma, \beta\}. \tag{34}$$

Accordingly we can get $|K(x,t)| \le \text{const}$ at $x \le t \le 1$.

Taking into account $\omega(x) \in C^1(\overline{I_1}) \cap C^3(I_1)$ and $|G(x,t)| \le \text{const}$ and $|G_x(x,t)| \le \text{const}$ from $F(x) = \lambda\Gamma(\alpha)\int_0^1 G(x,t)\omega'((t+x)/2)dt - \lambda\Gamma(\alpha)(\psi(0) - \varphi(0))\int_0^1 G(x,t)dt$ we will get $|F(x)| \le \text{const}$ and $|F'(x)| \le \text{const}$. Hence on the base of theory of Fredholm integral equations of the second kind the integral equation (32) is solvability and we can write a solution of this equation via resolvent-kernel:

$$\tau(x) = F(x) - \int_0^1 \mathfrak{R}(x,t) F(t)\, dt, \tag{35}$$

where $\mathfrak{R}(x,t)$ is the resolvent-kernel of $K(x,t)$.

We will find unknown functions $\nu^-(x)$ and $\nu^+(x)$ accordingly from (13) and (14):

$$\begin{aligned} \nu^-(x) &= \frac{1-x}{2} q(x) \int_x^1 (t-x)^{1-\gamma} dt \int_0^1 \mathfrak{R}(t,s) F(s)\, ds \\ &- \frac{1-x}{2} q(x) \int_x^1 (t-x)^{1-\gamma} F(t)\, dt - F'(x) \\ &+ \int_0^1 \frac{\partial \mathfrak{R}(x,t)}{\partial x} F(t)\, dt + \omega'\left(\frac{x+1}{2}\right), \end{aligned} \tag{36}$$

and $\nu^+(x) = \lambda \nu^-(x)$.

We write solution of Problem 1 in the domain Ω^+ as follows [18, 21]:

$$\begin{aligned} u(x,y) = &\int_0^y G_\xi(x,y,0,\eta)\psi(\eta)\, d\eta \\ &- \int_0^y G_\xi(x,y,1,\eta)\varphi(\eta)\, d\eta \\ &+ \int_0^1 G_0(x-\xi,y)\tau(\xi)\, d\xi \\ &- \int_0^y \int_0^1 G(x,y,0,\eta) p(\xi)\, d\xi\, d\eta \\ &\cdot \int_\xi^1 (t-\xi)^{\beta-1} \tau(t)\, dt. \end{aligned} \tag{37}$$

Here $G_0(x-\xi,y) = (1/\Gamma(1-\alpha))\int_0^y \eta^{-\alpha} G(x,y,\xi,\eta)d\eta$,

$$\begin{aligned} G(x,y,\xi,\eta) = &\frac{(y-\eta)^{\alpha/2-1}}{2} \\ &\cdot \sum_{n=-\infty}^{\infty} \left[e_{1,\alpha/2}^{1,\alpha/2}\left(-\frac{|x-\xi+2n|}{(y-\eta)^{\alpha/2}} \right) \right. \\ &\left. - e_{1,\alpha/2}^{1,\alpha/2}\left(-\frac{|x+\xi+2n|}{(y-\eta)^{\alpha/2}} \right) \right] \end{aligned} \tag{38}$$

is Green's function of the first boundary problem for (1) with the Riemann-Liouville fractional differential operator (note that it is true for the Caputo operator too) (see [20]), and

$$e_{1,\delta}^{1,\delta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\delta - \delta n)} \tag{39}$$

is the Wright type function [18].

Solution of Problem 1 in the domain Ω^- will be found by formula (12). Hence, Theorem 3 is proved.

Competing Interests

The authors declare that they have no competing interests.

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