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Research Article

The Dynamical Analysis of a Prey-Predator Model with a Refuge-Stage Structure Prey Population

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We proposed and analyzed a mathematical model dealing with two species of prey-predator system. It is assumed that the prey is a stage structure population consisting of two compartments known as immature prey and mature prey. It has a refuge capability as a defensive property against the predation. The existence, uniqueness, and boundedness of the solution of the proposed model are discussed. All the feasible equilibrium points are determined. The local and global stability analysis of them are investigated. The occurrence of local bifurcation (such as saddle node, transcritical, and pitchfork) near each of the equilibrium points is studied. Finally, numerical simulations are given to support the analytic results.

1. Introduction

The development of the qualitative analysis of ordinary differential equations is deriving to study many problems in mathematical biology. The modeling for the population dynamics of a prey-predator system is one of the important and interesting goals in mathematical biology, which has received wide attention by several authors [1–6]. In the natural world many kinds of prey and predator species have a life history that is composed of at least two stages: immature and mature, and each stage has different behavioral properties. So, some works of stage structure prey-predator models have been provided in a good number of papers in the literatures [7–12]. Zhang et al. in [9] and Cui and Takeuchi in [11] proposed two mathematical models of prey stage structure; in these models the predator species consumes exclusively the immature prey. Indeed, there are many factors that impact the dynamics of prey-predator interactions such as disease, harvesting, prey refuge, delay, and many other factors. Several prey species have gone to extinction, and this extinction must be caused by external effects such as overutilization, overpredation, and environmental factors (pollution, famine). Prey may avoid becoming attacked by predators either by protecting themselves or by living in a refuge where it will be out of sight of predators. Some theoretical and empirical studies have shown and tested the effects of prey refuges and making an opinion that the refuges used by prey have a stabilizing influence on prey-predator interactions; also prey extinction can be prevented by the addition of refuges; for instance, we can refer to [13–23]. Therefore, it is important and worthwhile to study the effects of a refuge on the prey population with prey stage structure. Consequently, in this paper, we proposed and analyzed the prey-predator model involving a stage structure in prey population together with a preys refuge property as a defensive property against the predation.

2. Mathematical Model

In this section a prey-predator model with a refuge-stage structure prey population is proposed for study. Let $x(T)$ represent the population size of the immature prey at time T; $y(T)$ represents the population size of the mature prey at time T, while $z(T)$ denotes the population size of the predator species at time T. Therefore in order to describe the dynamics of this model mathematically the following hypotheses are adopted:

(1) The immature prey grows exponentially depending completely on its parents with growth rate $r > 0$. There is an intraspecific competition between their individuals with intraspecific competition rate $\delta_1 > 0$. The immature prey individual becomes mature with grownup rate $\beta > 0$ and faces natural death with a rate $d_1 > 0$.

(2) There is an intraspecific competition between the individuals of mature prey population with intraspecific competition rate $\delta_2 > 0$. Further the mature prey species faces natural death rate too with a rate $d_2 > 0$.

(3) The environment provides partial protection of prey species against the predation with a refuge rate $0 < m < 1$; therefore there is $1-m$ of prey species available for predation.

(4) There is an intraspecific competition between the individuals of predator population with intraspecific competition rate $\delta_3 > 0$. Further the predator species faces natural death rate too with a rate $d_3 > 0$.

The predator consumes the prey in both the compartments according to the mass action law represented by Lotka–Volterra type of functional response with conversion rates $0 < e_1 < 1$ and $0 < e_2 < 1$ for immature and mature prey, respectively. Consequently, the dynamics of this model can be represented mathematically with the following set of differential equations.

$$\begin{aligned}
\dot{x} &= ry - \delta_1 x^2 - d_1 x - \beta x - \gamma_1 (1-m) xz \\
\dot{y} &= \beta x - \delta_2 y^2 - d_2 y - \gamma_2 (1-m) yz \\
\dot{z} &= e_1\gamma_1 (1-m) xz + e_2\gamma_2 (1-m) yz - \delta_3 z^2 - d_3 z
\end{aligned} \tag{1}$$

with initial conditions, $x(0) > 0$, $y(0) > 0$, and $z(0) > 0$.

In order to simplify the analysis of system (1) the number of parameters in system (1) is reduced using the following dimensionless variables in system (1):

$$\begin{aligned}
y_1 &= \frac{e_1\gamma_1(1-m)}{d_2} x, \\
y_2 &= \frac{e_2\gamma_2(1-m)}{d_2} y, \\
y_3 &= \frac{\delta_3}{d_2} z, \\
t &= d_2 T.
\end{aligned} \tag{2}$$

Accordingly, the dimensionless form of system (1) can be written as

$$\begin{aligned}
\dot{y}_1 &= a_1 y_2 - a_2 y_1^2 - a_3 y_1 - a_4 y_1 y_3 \\
\dot{y}_2 &= b_1 y_1 - b_2 y_2^2 - y_2 - b_3 y_2 y_3 \\
\dot{y}_3 &= y_1 y_3 + y_2 y_3 - y_3^2 - b_4 y_3,
\end{aligned} \tag{3}$$

where the dimensionless parameters are given by

$$\begin{aligned}
a_1 &= \frac{e_1\gamma_1 r}{e_2\gamma_2 d_2}, \\
a_2 &= \frac{\delta_1}{e_1\gamma_1(1-m)}, \\
a_3 &= \frac{d_1+\beta}{d_2}, \\
a_4 &= \frac{\gamma_1(1-m)}{\delta_3}, \\
b_1 &= \frac{e_2\gamma_2\beta}{e_1\gamma_1 d_2}, \\
b_2 &= \frac{\delta_2}{e_2\gamma_2(1-m)}, \\
b_3 &= \frac{\gamma_2(1-m)}{\delta_3}, \\
b_4 &= \frac{d_3}{d_2}.
\end{aligned} \tag{4}$$

According to the equations given in system (3), all the interaction functions are continuous and have a continuous partial derivatives. Therefore they are Lipschitzain and hence the solution of system (3) exists and is unique. Moreover the solution of system (3) is bounded as shown in the following theorem.

Theorem 1. *All the solutions of system (3) that initiate in the positive octant are uniformly bounded.*

Proof. Let $W = y_1 + y_2 + y_3$ be solutions of system (3) with initial conditions, $y_1(0) > 0$, $y_2(0) > 0$, and $y_3(0) > 0$. Then by differentiation W with respect to t we get

$$\begin{aligned}
\frac{dW}{dt} &\le a_1 y_2\left(1 - \frac{y_2}{a_1/b_2}\right) + b_1 y_1\left(1 - \frac{y_1}{b_1/a_2}\right) - \sigma_1 W \\
&\le \sigma_2 - \sigma_1 W;
\end{aligned} \tag{5}$$

here $\sigma_1 = \min\{1, a_3, b_4\}$ and $\sigma_2 = a_1^2/4b_2 + b_1^2/4a_2$. Now by using comparison theorem, we get

$$0 < W \le \left(W_0 e^{-\sigma_1 t} + \frac{\sigma_2}{\sigma_1}\left(1 - e^{-\sigma_1 t}\right)\right). \tag{6}$$

Thus for $t \to \infty$ we obtain $0 < W \le \sigma_2/\sigma_1$. Hence, all solutions of system (3) in R_+^3 are uniformly bounded and therefore we have finished the proof. □

3. Local Stability Analysis

It is observed that system (3) has at most three biologically feasible equilibrium points; namely, E_i, $i = 0, 1, 2$. The existence conditions for each of these equilibrium points are discussed below:

(1) The trivial equilibrium points $E_0 = (0,0,0)$ exist always.

(2) The predator free equilibrium point is denoted by $E_1 = (\tilde{y}_1, \tilde{y}_2, 0)$, where

$$\tilde{y}_1 = \frac{1}{b_1}\left(b_2\tilde{y}_2^2 + \tilde{y}_2\right) \tag{7}$$

while $\tilde{y}_2$ is a positive root of the following third-order polynomial

$$A_1 y_2^3 + A_2 y_2^2 + A_3 y_2 + A_4 = 0; \tag{8}$$

here, $A_1 = a_2 b_2^2/b_1^2$, $A_2 = 2a_2 b_2/b_1^2$, $A_3 = (a_2 + a_3 b_1 b_2)/b_1^2$, and $A_4 = (a_3 - a_1 b_1)/b_1$ exist uniquely in the interior of $y_1 y_2$-plane if and only if the following condition holds:

$$a_3 < a_1 b_1. \tag{9}$$

(3) The interior (positive) equilibrium point is given by $E_2 = (y_1^*, y_2^*, y_3^*)$, where

$$y_1^* = \frac{\left[(b_2 + b_3) y_2^* + (1 - b_3 b_4)\right] y_2^*}{b_1 - b_3 y_2^*}, \qquad y_3^* = y_1^* + y_2^* - b_4 \tag{10}$$

while y_2^* is a positive root of the following third-order polynomial:

$$B_1 y_2^3 + B_2 y_2^2 + B_3 y_2 + B_4 = 0; \tag{11}$$

here,

$$\begin{aligned} B_1 &= a_2 (b_2 + b_3)^2 + a_4 b_2 (b_2 + b_3) > 0, \\ B_2 &= \left[2a_2 (b_2 + b_3) + 2a_4 b_2 + a_4 b_3\right](1 - b_3 b_4) \\ &\quad - \left[a_3 b_2 b_3 + b_3^2 (a_3 + a_1)\right], \\ B_3 &= 2a_1 b_1 b_3 + (a_2 + a_4)(1 - b_3 b_4)^2 \\ &\quad + b_1 (a_3 - a_4 b_4)(b_2 + b_3) \\ &\quad + \left[b_1 a_4 - b_3 (a_3 - a_4 b_4)\right](1 - b_3 b_4), \\ B_4 &= b_1 \left[a_3 - a_1 b_1 - a_3 b_3 b_4 - a_4 b_4 (1 - b_3 b_4)\right]. \end{aligned} \tag{12}$$

Clearly, (11) has a unique positive root represented by y_2^* if the following set of conditions hold:

$$B_4 < 0 \quad \text{with} \quad (B_2 > 0 \text{ or } B_3 < 0) \tag{13}$$

Therefore, E_2 exists uniquely in int. R_+^3 if in addition to condition (13) the following conditions are satisfied.

$$\begin{aligned} &y_1^* + y_2^* > b_4 \\ &\frac{b_3 b_4 - 1}{b_2 + b_3} < y_2^* < \frac{b_1}{b_3} \\ &\quad \text{or } \frac{b_1}{b_3} < y_2^* < \frac{b_3 b_4 - 1}{b_2 + b_3}. \end{aligned} \tag{14}$$

Now to study the local stability of these equilibrium points, the Jacobian matrix $J(y_1, y_2, y_3)$ for the system (3) at any point (y_1, y_2, y_3) is determined as

$$\begin{pmatrix} -2a_2 y_1 - a_3 - a_4 y_3 & a_1 & -a_4 y_1 \\ b_1 & -2b_2 y_2 - 1 - b_3 y_3 & -b_3 y_2 \\ y_3 & y_3 & y_1 + y_2 - 2y_3 - b_4 \end{pmatrix}. \tag{15}$$

Thus, system (3) has the following Jacobian matrix near $E_0 = (0,0,0)$.

$$J(E_0) = \begin{pmatrix} -a_3 & a_1 & 0 \\ b_1 & -1 & 0 \\ 0 & 0 & -b_4 \end{pmatrix}. \tag{16}$$

Then the characteristic equation of $J(E_0)$ is given by

$$(\lambda + b_4)\left[\lambda^2 + (a_3 + 1)\lambda + a_3 - a_1 b_1\right] = 0. \tag{17}$$

Clearly, all roots of (17) have negative real parts if and only if the following condition holds:

$$a_3 > a_1 b_1. \tag{18}$$

So, E_0 is locally asymptotically stable under condition (18) and saddle point otherwise. Therefore, E_0 is locally asymptotically stable whenever E_1 does not exist and unstable whenever E_1 exists.

The Jacobian matrix of system (3) around the equilibrium point $E_1 = (\tilde{y}_1, \tilde{y}_2, 0)$ reduced to

$$J(E_1) = \begin{pmatrix} -2a_2\tilde{y}_1 - a_3 & a_1 & -a_4\tilde{y}_1 \\ b_1 & -2b_2\tilde{y}_2 - 1 & -b_3\tilde{y}_2 \\ 0 & 0 & \tilde{y}_1 + \tilde{y}_2 - b_4 \end{pmatrix} = (a_{ij})_{3\times 3}. \tag{19}$$

Then the characteristic equation of $J(E_1)$ is written by

$$\left[\lambda - a_{33}\right]\left[\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21}\right] = 0. \tag{20}$$

Straightforward computation shows that all roots of (20) have negative real part provided that

$$\tilde{y}_1 + \tilde{y}_2 < b_4 \tag{21}$$

$$(2a_2\tilde{y}_1 + a_3)(2b_2\tilde{y}_2 + 1) > a_1 b_1. \tag{22}$$

So, E_1 is locally asymptotically stable if the above two conditions hold.

Finally the Jacobian matrix of system (3) around the interior equilibrium point $E_2 = (y_1^*, y_2^*, y_3^*)$ is written

$$J(E_2) = (b_{ij})_{3\times 3}; \tag{23}$$

here

$$\begin{aligned} b_{11} &= -\left(2a_2 y_1^* + a_4 y_3^* + a_3\right), \\ b_{12} &= a_1, \\ b_{13} &= -a_4 y_1^*, \\ b_{21} &= b_1, \\ b_{22} &= -\left(2b_2 y_2^* + b_3 y_3^* + 1\right), \\ b_{23} &= -b_3 y_2^*, \\ b_{31} &= y_3^*, \\ b_{32} &= y_3^*, \\ b_{33} &= -y_3^*. \end{aligned} \tag{24}$$

Hence, the characteristic equation of $J(E_2)$ becomes

$$\lambda^3 + D_1\lambda^2 + D_2\lambda + D_3 = 0 \tag{25}$$

with

$$\begin{aligned} D_1 &= -\left(b_{11} + b_{22} + b_{33}\right) > 0, \\ D_2 &= b_{11}b_{22} - b_{12}b_{21} + b_{11}b_{33} - b_{13}b_{31} + b_{22}b_{33} \\ &\quad - b_{23}b_{32}, \\ D_3 &= b_{33}\left(b_{12}b_{21} - b_{11}b_{22}\right) + b_{11}b_{23}b_{32} - b_{12}b_{23}b_{31} \\ &\quad + b_{22}b_{13}b_{31} - b_{21}b_{13}b_{32}. \end{aligned} \tag{26}$$

Consequently, $\Delta = D_1D_2 - D_3$ can be written as

$$\begin{aligned} \Delta &= \left(b_{11} + b_{22}\right)\left(b_{12}b_{21} - b_{11}b_{22}\right) - b_{11}b_{22}b_{33} \\ &\quad - \left(b_{11} + b_{22}\right)\left(b_{11}b_{33} - b_{13}b_{31}\right). \\ &\quad - b_{22}^2 b_{33} - b_{11}b_{33}^2 - b_{22}b_{33}^2 + b_{13}b_{31}\left(b_{33} + b_{21}\right) \\ &\quad + b_{23}b_{32}\left(b_{33} + b_{12}\right) \end{aligned} \tag{27}$$

Since $D_1 > 0$, then according to Routh-Hurwitz criterion E_2 is locally asymptotically stable if and only if $D_3 > 0$ and $\Delta = D_1D_2 - D_3 > 0$. According to the form of D_3 and the signs of Jacobian elements the last four terms are positive, while the first term will be positive under the sufficient condition (28) below. However Δ becomes positive if and only if in addition to condition (28) the second sufficient condition given by (29) holds.

$$\left(2a_2 y_1^* + a_4 y_3^* + a_3\right)\left(2b_2 y_2^* + b_3 y_3^* + 1\right) > a_1 b_1 \tag{28}$$

$$\begin{aligned} y_3^* &> b_1, \\ y_3^* &> a_1. \end{aligned} \tag{29}$$

Therefore under these two sufficient conditions E_2 is locally asymptotically stable.

4. Global Stability

In this section the global stability for the equilibrium points of system (3) is investigated by using the Lyapunov method as shown in the following theorems.

Theorem 2. *Assume that the vanishing equilibrium point E_0 is locally asymptotically stable; then it is globally asymptotically stable in R_+^3 if and only if the following condition holds:*

$$\frac{a_4 b_1}{a_3} < b_3 < \frac{a_4}{a_1}. \tag{30}$$

Proof. Consider the following positive definite real valued function:

$$V_0\left(y_1, y_2, y_3\right) = \frac{1}{a_4} y_1 + \frac{1}{b_3} y_2 + y_3. \tag{31}$$

Straightforward computation shows that the derivative of V_0 with respect to t is given by

$$\frac{dV_0}{dt} < \left(\frac{a_4 b_1 - a_3 b_3}{a_4 b_3}\right) y_1 + \left(\frac{a_1 b_3 - a_4}{a_4 b_3}\right) y_2 - b_4 y_3. \tag{32}$$

Therefore, by using condition (30), we obtain dV_0/dt which is negative definite in R_+^3, and then V_0 is a Lyapunov function with respect to E_0. Hence E_0 is globally asymptotically stable in R_+^3 and the proof is complete. □

Theorem 3. *Assume that the predator free equilibrium point $E_1 = (\tilde{y}_1, \tilde{y}_2, 0)$ is locally asymptotically stable; then it is globally asymptotically stable in R_+^3 if the following condition holds:*

$$\begin{aligned} &\left(\frac{a_1}{a_4 y_1} + \frac{b_1}{b_3 y_2}\right)^2 \\ &\quad < 4\left(\frac{a_1 \tilde{y}_2}{a_4 y_1 \tilde{y}_1} + \frac{a_2}{a_4}\right)\left(\frac{b_1 \tilde{y}_1}{b_3 y_2 \tilde{y}_2} + \frac{b_2}{b_3}\right). \end{aligned} \tag{33}$$

Proof. Consider the following positive definite real valued function:

$$\begin{aligned} V_1\left(y_1, y_2, y_3\right) &= \frac{1}{a_4}\left(y_1 - \tilde{y}_1 - \tilde{y}_1 \ln \frac{y_1}{\tilde{y}_1}\right) \\ &\quad + \frac{1}{b_3}\left(y_2 - \tilde{y}_2 - \tilde{y}_2 \ln \frac{y_2}{\tilde{y}_2}\right) + y_3. \end{aligned} \tag{34}$$

Straightforward computation shows that the derivative of V_1 with respect to t is given by

$$\begin{aligned} \frac{dV_1}{dt} &= -\left(\frac{a_1 \tilde{y}_2}{a_4 y_1 \tilde{y}_1} + \frac{a_2}{a_4}\right)\left(y_1 - \tilde{y}_1\right)^2 \\ &\quad - \left(\frac{b_1 \tilde{y}_1}{b_3 y_2 \tilde{y}_2} + \frac{b_2}{b_3}\right)\left(y_2 - \tilde{y}_2\right)^2 \\ &\quad + \left(\frac{a_1}{a_4 y_1} + \frac{b_1}{b_3 y_2}\right)\left(y_1 - \tilde{y}_1\right)\left(y_2 - \tilde{y}_2\right) - y_3^2 \\ &\quad - \left(b_4 - \tilde{y}_1 - \tilde{y}_2\right) y_3. \end{aligned} \tag{35}$$

Now using condition (33) gives us that

$$\frac{dV_1}{dt} < -\left[\sqrt{\frac{a_1\tilde{y}_2}{a_4 y_1\tilde{y}_1} + \frac{a_2}{a_4}}\,(y_1 - \tilde{y}_1) - \sqrt{\frac{b_1\tilde{y}_1}{b_3 y_2\tilde{y}_2} + \frac{b_2}{b_3}}\,(y_2 - \tilde{y}_2)\right]^2 - (b_4 - \tilde{y}_1 - \tilde{y}_2)\, y_3. \tag{36}$$

Clearly dV_1/dt is negative definite due to local stability condition (21). Hence V_1 is a Lyapunov function with respect to E_1, and then E_1 is globally asymptotically stable, which completes the proof. □

Theorem 4. *Assume that the interior equilibrium point $E_2 = (y_1^*, y_2^*, y_3^*)$ is locally asymptotically stable in R_+^3; then it is globally asymptotically stable if and only if the following condition holds:*

$$\left(\frac{y_2^* a_1 b_3}{y_1^* b_1} - a_4\right)^2 < 4\frac{y_2^* a_1 a_2 b_3 b_4}{y_1^* b_1}. \tag{37}$$

Proof. Consider the following positive definite real valued function around E_2:

$$\begin{aligned} V_1(y_1, y_2, y_3) = {} & \left(y_1 - y_1^* - y_1^* \ln\frac{y_1}{y_1^*}\right) \\ & + \frac{y_2^* a_1}{y_1^* b_1}\left(y_2 - y_2^* - y_2^* \ln\frac{y_2}{y_2^*}\right) \\ & + \frac{y_2^* a_1 b_3}{y_1^* b_1}\left(y_3 - y_3^* - y_3^* \ln\frac{y_3}{y_3^*}\right). \end{aligned} \tag{38}$$

Our computation for the derivative of V_2 with respect to t gives that

$$\begin{aligned} \frac{dV_2}{dt} = {} & -\frac{a_1}{y_1 y_2 y_1^*}(y_2 y_1^* - y_1 y_2^*)^2 - a_2(y_1 - y_1^*)^2 \\ & - \frac{y_2^* a_1 b_2}{y_1^* b_1}(y_2 - y_2^*)^2 \\ & + \left(\frac{y_2^* a_1 b_3}{y_1^* b_1} - a_4\right)(y_1 - y_1^*)(y_3 - y_3^*) \\ & - \frac{y_2^* a_1 b_3 b_4}{y_1^* b_1}(y_3 - y_3^*)^2. \end{aligned} \tag{39}$$

Now by using the condition (37) we obtain that

$$\begin{aligned} \frac{dV_2}{dt} & < -\frac{a_1}{y_1 y_2 y_1^*}(y_2 y_1^* - y_1 y_2^*)^2 - \frac{y_2^* a_1 b_2}{y_1^* b_1}(y_2 - y_2^*)^2 \\ & - \left[\sqrt{a_2}(y_1 - y_1^*) - \sqrt{\frac{y_2^* a_1 b_3 b_4}{y_1^* b_1}}(y_3 - y_3^*)\right]^2. \end{aligned} \tag{40}$$

According to the above inequality we have dV_2/dt which is negative definite; therefore, E_2 is globally asymptotically stable in R_+^3 and hence the proof is complete. □

5. Local Bifurcation

In this section the local bifurcation near the equilibrium points of system (3) is investigated using Sotomayor's theorem for local bifurcation [24]. It is well known that the existence of nonhyperbolic equilibrium point is a necessary but not sufficient condition for bifurcation to occur. Now rewrite system (3) in the form

$$\dot{Y} = f(Y), \tag{41}$$

where

$$Y = (y_1, y_2, y_3)^T, \quad f = (f_1, f_2, f_3)^T. \tag{42}$$

Then according to Jacobian matrix of system (3) given in (15), it is simple to verify that for any nonzero vector $U = (u_1, u_2, u_3)^T$ we have

$$\begin{aligned} & D^2 f(y_1, y_2, y_3)(U, U) \\ & = \frac{\partial^2 f}{\partial y_1^2}u_1^2 + \frac{\partial^2 f}{\partial y_1 \partial y_2}u_1 u_2 + \frac{\partial^2 f}{\partial y_2 \partial y_1}u_2 u_1 + \frac{\partial^2 f}{\partial y_2^2}u_2^2 \\ & + \frac{\partial^2 f}{\partial y_1 \partial y_3}u_1 u_3 + \frac{\partial^2 f}{\partial y_3 \partial y_1}u_3 u_1 + \frac{\partial^2 f}{\partial y_2 \partial y_3}u_2 u_3 \\ & + \frac{\partial^2 f}{\partial y_3 \partial y_2}u_3 u_2 + \frac{\partial^2 f}{\partial y_3^2}u_3^2. \end{aligned} \tag{43}$$

Consequently, we obtain that

$$D^2 f(y_1, y_2, y_3)(U, U) = \begin{pmatrix} -2a_2 u_1^2 - 2a_4 u_1 u_3 \\ -2b_2 u_2^2 - 2b_3 u_2 u_3 \\ 2u_1 u_3 + 2u_2 u_3 - 2u_3^2 \end{pmatrix} \tag{44}$$

and therefore

$$D^3 f(y_1, y_2, y_3)(U, U, U) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{45}$$

Thus system (3) has no pitchfork bifurcation due to (45). Moreover, the local bifurcation near the equilibrium points is investigated in the following theorems:

Theorem 5. *System (3) undergoes a transcritical bifurcation near the vanishing equilibrium point, but saddle node bifurcation cannot occur, when the parameter a_3 passes through the bifurcation value $a_3^* = a_1 b_1$.*

Proof. According to the Jacobian matrix $J(E_0)$ given by (16), system (3) at the equilibrium point E_0 with $a_3 = a_3^*$ has zero eigenvalue, say $\lambda_0^* = 0$, and the Jacobian matrix $J(E_0, a_3^*)$ becomes

$$J(E_0, a_3^*) = J_0^* = \begin{pmatrix} -a_3^* & a_1 & 0 \\ b_1 & -1 & 0 \\ 0 & 0 & -b_4 \end{pmatrix}. \tag{46}$$

Now let $U^{[0]} = (u_1^{[0]}, u_2^{[0]}, u_3^{[0]})^T$ be the eigenvector corresponding to the eigenvalue $\lambda_0^* = 0$. Thus $J_0^* U^{[0]} = \mathbf{0}$ gives $U^{[0]} = (u_1^{[0]}, b_1 u_1^{[0]}, 0)^T$, where $u_1^{[0]}$ represents any nonzero real number. Also, let $W^{[0]} = (w_1^{[0]}, w_2^{[0]}, w_3^{[0]})^T$ represents the eigenvector corresponding to the eigenvalue $\lambda_0^* = 0$ of J_0^{*T}. Hence $J_0^{*T} W^{[0]} = \mathbf{0}$ gives that $W^{[0]} = (w_1^{[0]}, a_1 w_1^{[0]}, 0)^T$, where $w_1^{[0]}$ denotes any nonzero real number. Now, since

$$\begin{aligned}\frac{df}{da_3} &= f_{a_3}(Y, a_3) = \left(\frac{df_1}{da_3}, \frac{df_2}{da_3}, \frac{df_3}{da_3}\right)^T \\ &= (-y_1, 0, 0)^T\end{aligned} \tag{47}$$

thus $f_{a_3}(E_0, a_3^*) = (0,0,0)^T$, which gives $(W^{[0]})^T f_{a_3}(E_0, a_3^*) = 0$. So, according to Sotomayor's theorem for local bifurcation, system (3) has no saddle node bifurcation at $a_3 = a_3^*$. Also, since

$$Df_{a_3}(E_0, a_3) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{48}$$

then,

$$\begin{aligned}&\left(W^{[0]}\right)^T \left(Df_{a_3}(E_0, a_3^*) U^{[0]}\right) \\ &\quad = \left(w_1^{[0]}, a_1 w_1^{[0]}, 0\right)\left(-u_1^{[0]}, 0, 0\right)^T = -u_1^{[0]} w_1^{[0]} \neq 0.\end{aligned} \tag{49}$$

Moreover, by substituting E_0, a_3^*, and $U^{[0]}$ in (44) we get that

$$\begin{aligned}&D^2 f(E_0, a_3^*)\left(U^{[0]}, U^{[0]}\right) \\ &\quad = \left(-2a_2\left(u_1^{[0]}\right)^2, -2b_2 b_1^2 \left(u_1^{[0]}\right)^2, 0\right)^T.\end{aligned} \tag{50}$$

Hence, it is obtain that

$$\begin{aligned}&\left(W^{[0]}\right)^T D^2 f(E_0, a_3^*)\left(U^{[0]}, U^{[0]}\right) \\ &\quad = -2\left(a_2 + a_1 b_2 b_1^2\right)\left(u_1^{[0]}\right)^2 w_1^{[0]} \neq 0.\end{aligned} \tag{51}$$

Thus, according to Sotomayor's theorem system (3) has a transcritical bifurcation at E_0 as the parameter a_3 passes through the value a_3^*; thus the proof is complete. □

Theorem 6. *Assume that condition (22) holds; then system (3) undergoes a transcritical bifurcation near the predator free equilibrium point E_1, but saddle node bifurcation cannot occur, when the parameter b_4 passes through the bifurcation value $b_4^* = \tilde{y}_1 + \tilde{y}_2$.*

Proof. According to the Jacobian matrix $J(E_1)$ given by (19), system (3) at the equilibrium point E_1 with $b_4 = b_4^*$ has zero eigenvalue, say $\lambda_1^* = 0$, and the Jacobian matrix $J(E_1, b_4^*)$ becomes

$$J(E_1, b_4^*) = J_1^* = \left(a_{ij}^*\right)_{3\times 3}, \tag{52}$$

where $a_{ij}^* = a_{ij}; \forall i, j = 1, 2, 3$, with $a_{33}^* = 0$. Now let, $U^{[1]} = (u_1^{[1]}, u_2^{[1]}, u_3^{[1]})^T$ be the eigenvector corresponding to the eigenvalue $\lambda_1^* = 0$. Thus $J_1^* U^{[1]} = \mathbf{0}$ gives $U^{[1]} = (\Lambda_1 u_3^{[1]}, \Lambda_2 u_3^{[1]}, u_3^{[1]})^T$, where $\Lambda_1 = (a_{12}a_{23} - a_{22}a_{13})/(a_{11}a_{22} - a_{12}a_{21})$ and $\Lambda_2 = (a_{21}a_{13} - a_{11}a_{31})/(a_{11}a_{22} - a_{12}a_{21})$ are negative according to the sign of the Jacobian elements and $u_3^{[1]}$ represents any nonzero real numbers. Also, let $W^{[1]} = (w_1^{[1]}, w_2^{[1]}, w_3^{[1]})^T$ represent the eigenvector corresponding to eigenvalue $\lambda_1^* = 0$ of J_1^{*T}. Hence $J_1^{*T} W^{[1]} = \mathbf{0}$ gives that $W^{[1]} = (0, 0, w_3^{[1]})^T$, where $w_3^{[1]}$ stands for any nonzero real numbers. Now, since

$$\frac{df}{db_4} = f_{b_4}(Y, b_4) = \left(\frac{df_1}{db_4}, \frac{df_2}{db_4}, \frac{df_3}{db_4}\right)^T = (0, 0, -y_3)^T \tag{53}$$

thus $f_{b_4}(E_1, b_4^*) = (0,0,0)^T$, which gives $(W^{[1]})^T f_{b_4}(E_1, b_4^*) = 0$. So, according to Sotomayor's theorem for local bifurcation, system (3) has no saddle node bifurcation at $b_4 = b_4^*$. Also, since

$$Df_{b_4}(E_1, b_4) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{54}$$

then, we can have

$$\begin{aligned}&\left(W^{[1]}\right)^T \left(Df_{b_4}(E_1, b_4^*) U^{[1]}\right) \\ &\quad = \left(0, 0, w_3^{[1]}\right)\left(0, 0, -u_3^{[1]}\right)^T = -u_3^{[1]} w_3^{[1]} \neq 0.\end{aligned} \tag{55}$$

Moreover, substituting E_1, b_4^*, and $U^{[1]}$ in (44) gives

$$\begin{aligned}&D^2 f(E_1, b_4^*)\left(U^{[1]}, U^{[1]}\right) = 2\left(u_3^{[1]}\right)^2 \\ &\quad \cdot \left(-a_2\Lambda_1^2 - a_4\Lambda_1, -b_2\Lambda_2^2 - b_3\Lambda_2, \Lambda_1 + \Lambda_2 - 1\right)^T.\end{aligned} \tag{56}$$

Hence, it is obtained that

$$\begin{aligned}&\left(W^{[1]}\right)^T D^2 f(E_1, b_4^*)\left(U^{[1]}, U^{[1]}\right) \\ &\quad = 2\left(\Lambda_1 + \Lambda_2 - 1\right)\left(u_3^{[1]}\right)^2 w_3^{[1]} \neq 0.\end{aligned} \tag{57}$$

Thus, according to Sotomayor's theorem system (3) has a transcritical bifurcation at E_1 as the parameter b_4 passes through the value b_4^*; thus the proof is complete. □

Theorem 7. *Assume that condition (21) holds; then system (3) undergoes a saddle node bifurcation near the predator free equilibrium point E_1 when the parameter a_1 passes through the bifurcation value $a_1^* = (2a_2\tilde{y}_1 + a_3)(2b_2\tilde{y}_2 + 1)/b_1$.*

Proof. According to the Jacobian matrix $J(E_1)$ given by (19), system (3) at the equilibrium point E_1 with $a_1 = a_1^*$ has zero eigenvalue, say $\lambda_1^{**} = 0$, and the Jacobian matrix $J(E_1, a_1^*)$ becomes

$$J(E_1, a_1^*) = J_1^{**} = \left(a_{ij}^{**}\right)_{3\times 3}, \tag{58}$$

where $a_{ij}^{**} = a_{ij}; \forall i, j = 1, 2, 3$, with $a_{12}^{**} = a_1^*$. Now let $U^{[11]} = (u_1^{[11]}, u_2^{[11]}, u_3^{[11]})^T$ be the eigenvector corresponding to the eigenvalue $\lambda_1^{**} = 0$. Thus $J_1^{**} U^{[11]} = \mathbf{0}$ gives $U^{[11]} = (-(a_1^*/a_{11}) u_2^{[11]}, u_2^{[11]}, 0)^T$, where $u_2^{[11]}$ represents any nonzero real numbers. Also, let $W^{[11]} = (w_1^{[11]}, w_2^{[11]}, w_3^{[11]})^T$ represent the eigenvector corresponding to eigenvalue $\lambda_1^{**} = 0$ of J_1^{**T}. Hence $J_1^{**T} W^{[11]} = \mathbf{0}$ gives that $W^{[11]} = (-(a_{21}/a_{11}) w_2^{[11]}, w_2^{[11]}, (\Psi/a_{11}a_{33}) w_2^{[11]})^T$, where $\Psi = a_{13}a_{21} - a_{11}a_{23}$ is negative due to the sign of the Jacobian elements and $w_2^{[11]}$ denotes any nonzero real numbers. Now, since

$$\frac{df}{da_1} = f_{a_1}(Y, a_1) = \left(\frac{df_1}{da_1}, \frac{df_2}{da_1}, \frac{df_3}{da_1} \right)^T = (y_2, 0, 0)^T \quad (59)$$

thus $f_{a_1}(E_1, a_1^*) = (\tilde{y}_2, 0, 0)^T$; hence $(W^{[11]})^T f_{a_1}(E_1, a_1^*) = -(a_{21}/a_{11}) w_2^{[11]} \tilde{y}_2 \neq 0$. So, according to Sotomayor's theorem for local bifurcation the first condition of saddle node bifurcation is satisfied in system (3) at $a_1 = a_1^*$. Moreover, substituting E_1, a_1^*, and $U^{[11]}$ in (44) gives

$$\begin{aligned} & D^2 f(E_1, a_1^*)(U^{[11]}, U^{[11]}) \\ & = (u_2^{[11]})^2 \left(-2\frac{a_2 a_1^{*2}}{a_{11}^2}, -2b_2, 0 \right)^T. \end{aligned} \quad (60)$$

Hence, it is obtained that

$$\begin{aligned} & (W^{[11]})^T D^2 f(E_1, a_1^*)(U^{[11]}, U^{[11]}) \\ & = 2\left(\frac{a_2 a_1^{*2} a_{21}}{a_{11}^3} - b_2 \right) (u_2^{[11]})^2 w_2^{[11]} \neq 0. \end{aligned} \quad (61)$$

Thus, according to Sotomayor's theorem system (3) has a saddle node bifurcation at E_1 as the parameter a_1 passes through the value a_1^*; thus the proof is complete. □

Theorem 8. *Assume that*

$$(2a_2 y_1^* + a_4 y_3^* + a_3)(2b_2 y_2^* + b_3 y_3^* + 1) < a_1 b_1 \quad (62)$$

$$y_1^* < \frac{a_1}{a_4}. \quad (63)$$

Then system (3) undergoes a saddle node bifurcation near the interior equilibrium point E_2, as the parameter b_1 passes through the bifurcation value $b_1^* = \Gamma_1/\Gamma_2$, where Γ_1 and Γ_2 are given in the proof.

Proof. According to the determinant of the Jacobian matrix $J(E_2)$ given by D_3 in (25), condition (62) represents a necessary condition to have nonpositive determinant for $J(E_2)$. Now rewrite the form of the determinant as follows:

$$D_3 = \Gamma_1 - b_1 \Gamma_2. \quad (64)$$

Here $\Gamma_1 = b_{11}b_{23}b_{32} + b_{22}b_{31}b_{13} - b_{11}b_{22}b_{33} - b_{12}b_{23}b_{31}$ and $\Gamma_2 = b_{13}b_{32} - b_{33}b_{12}$. Obviously, Γ_1 is positive always, while Γ_2 is positive under the condition (63). Thus it is easy to verify that $D_3 = 0$ and hence $J(E_2)$ has zero eigenvalue, say $\lambda_2^* = 0$, as b_1 passes through the value $b_1^* = \Gamma_1/\Gamma_2$, which means that E_2 becomes a nonhyperbolic point. Let now the Jacobian matrix of system (3) at E_2 with $b_1 = b_1^*$ be given by

$$J(E_2, b_1^*) = J_2^* = (b_{ij}^*)_{3\times 3}, \quad (65)$$

where $b_{ij}^* = b_{ij}\ \forall i, j = 1, 2, 3$, and $b_{21}^* = b_1^*$.

Let $U^{[*]} = (u_1^{[*]}, u_2^{[*]}, u_3^{[*]})^T$ be the eigenvector corresponding to the eigenvalue $\lambda_2^* = 0$. Thus $J_2^* U^{[*]} = \mathbf{0}$ gives $U^{[*]} = (\Phi_1 u_3^{[*]}, \Phi_2 u_3^{[*]}, u_3^{[*]})^T$, where $\Phi_1 = (b_{12}b_{23} - b_{22}b_{13})/(b_{11}b_{22} - b_{12}b_1^*)$ and $\Phi_2 = (b_1^* b_{13} - b_{11}b_{23})/(b_{11}b_{22} - b_{12}b_1^*)$ are positive due to the Jacobian elements and $u_3^{[*]}$ represents any nonzero real number. Also, let $W^{[*]} = (w_1^{[*]}, w_2^{[*]}, w_3^{[*]})^T$ represent the eigenvector corresponding to eigenvalue $\lambda_2^* = 0$ of J_2^{*T}. Hence $J_2^{*T} W^{[*]} = \mathbf{0}$ gives that $W^{[*]} = (\Theta_1 w_3^{[*]}, \Theta_2 w_3^{[*]}, w_3^{[*]})^T$, where $\Theta_1 = (b_{21}b_{32} - b_{22}b_{31})/(b_{11}b_{22} - b_{12}b_1^*)$ and $\Theta_2 = (b_{12}b_{31} - b_{11}b_{32})/(b_{11}b_{22} - b_{12}b_1^*)$ are negative due to the Jacobian elements and $w_3^{[*]}$ denotes any nonzero real numbers. Now, since

$$\frac{df}{db_1} = f_{b_1}(Y, b_1) = \left(\frac{df_1}{db_1}, \frac{df_2}{db_1}, \frac{df_3}{db_1} \right)^T = (0, y_1, 0)^T \quad (66)$$

thus $f_{b_1}(E_2, b_1^*) = (0, y_1^*, 0)^T$, which gives $(W^{[*]})^T f_{b_1}(E_2, b_1^*) = \Theta_2 y_1^* w_3^{[*]} \neq 0$. Consequently the first condition of saddle node bifurcation is satisfied. Moreover, by substituting E_2, b_1^*, and $U^{[*]}$ in (44) we get that

$$\begin{aligned} & D^2 f(E_2, b_1^*)(U^{[*]}, U^{[*]}) = 2(u_3^{[*]})^2 \\ & \cdot \left(-a_2\Phi_1^2 - a_4\Phi_1, -b_2\Phi_2^2 - b_3\Phi_2, \Phi_1 + \Phi_2 - 1 \right)^T. \end{aligned} \quad (67)$$

Hence, it is obtained that

$$\begin{aligned} & (W^{[*]})^T D^2 f(E_2, b_1^*)(U^{[*]}, U^{[*]}) = 2(u_3^{[*]})^2 w_3^{[*]} \\ & \times \left[-(a_2\Phi_1^2 + a_4\Phi_1)\Theta_1 - (b_2\Phi_2^2 + b_3\Phi_2)\Theta_2 \right. \\ & \left. + (\Phi_1 + \Phi_2 - 1) \right] \neq 0. \end{aligned} \quad (68)$$

So, according to Sotomayor's theorem, system (3) has a saddle node bifurcation as b_1 passes through the value b_1^* and hence the proof is complete. □

6. Numerical Simulations

In this section, the global dynamics of system (3) is investigated numerically. The objectives first confirm our obtained analytical results and second specify the control set of parameters that control the dynamics of the system. Consequently, system (3) is solved numerically using the following biologically feasible set of hypothetical parameters with different sets

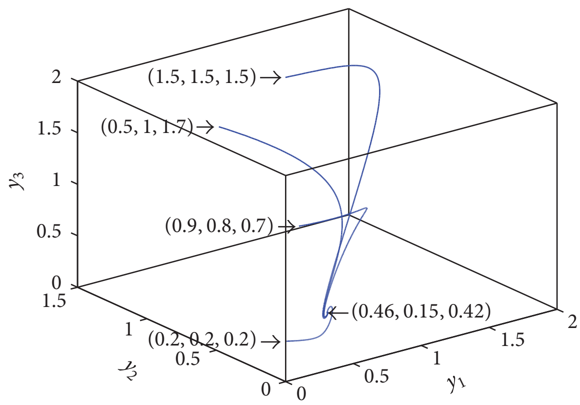

Figure 1: 3D phase plot of the system (3) for the data given by (69) starting from different initial values in which the solution approaches asymptotically to 0.46, 0.15, and 0.42.

of initial points and then the resulting trajectories are drawn in the form of phase portrait and time series figures.

$$\begin{aligned} a_1 &= 2, \\ a_2 &= 0.1, \\ a_3 &= 0.4, \\ a_4 &= 0.5, \\ b_1 &= 0.4, \\ b_2 &= 0.1, \\ b_3 &= 0.5, \\ b_4 &= 0.2. \end{aligned} \tag{69}$$

Clearly, Figure 1 shows the asymptotic approach of the solutions, which started from different initial points to a positive equilibrium point (0.46, 0.15, 0.42), for the data given by (69). This confirms our obtained result regarding the existence of globally asymptotically stable positive point of system (3) provided that certain conditions hold.

Now in order to discuss the effect of the parameters values of system (3) on the dynamical behavior of the system, the system is solved numerically for the data given in (69) with varying one parameter each time. It is observed that varying parameters values a_2, a_4, b_2, b_3, and b_4 have no qualitative effect on the dynamical behavior of system (3) and the system still approaches to a positive equilibrium point. On the other hand, when a_1 decreases in the range ($a_1 \leq 1.05$) keeping other parameters fixed as given in (69) the dynamical behavior of system (3) approaches asymptotically to the vanished equilibrium point as shown in the typical figure given by Figure 2. Similar observations have been obtained on the behavior of system (3) in case of increasing the parameter a_3 in the range ($a_3 \geq 0.8$) or decreasing the parameter b_1 in the range ($b_1 \leq 0.2$), with keeping other parameters fixed as given in (69), and then the solution of system (3) is depicted in Figures 3 and 4, respectively. Finally, for the parameters $a_2 = 0.9$ and $b_4 = 0.7$ with other parameters fixed as given in (69), the solution of system (3) approaches asymptotically to the predator free equilibrium point as shown in the Figure 5.

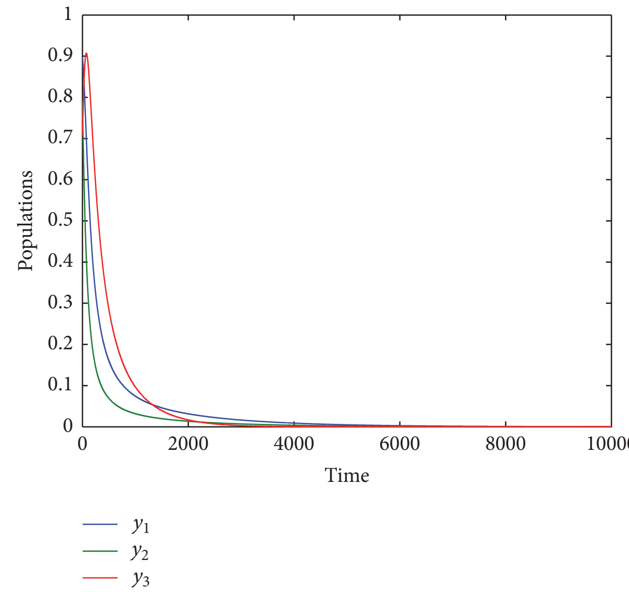

Figure 2: Time series of system (3) approaches asymptotically to the vanishing equilibrium point for $a_1 = 0.8$ with other parameters given by (69).

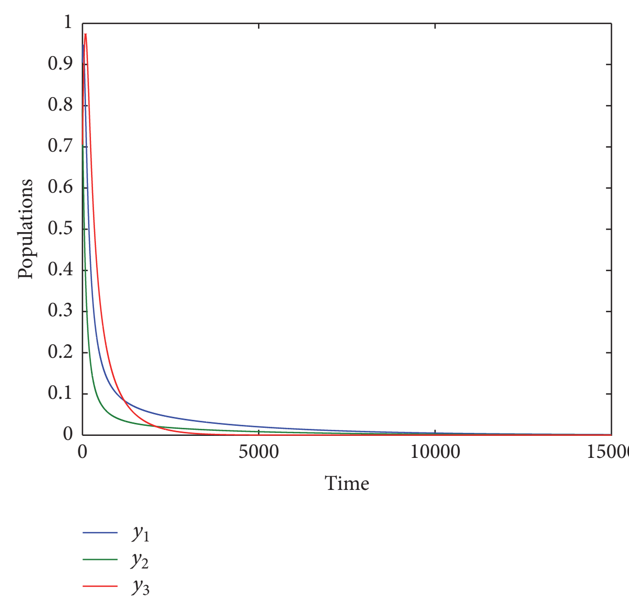

Figure 3: Time series of system (3) approaches asymptotically to the vanishing equilibrium point for $a_3 = 0.85$ with other parameters given by (69).

7. Discussion

In this paper, a model that describes the prey-predator system having a refuge and stage structure properties in the prey population has been proposed and studied analytically as well as numerically. Sufficient conditions which ensure the

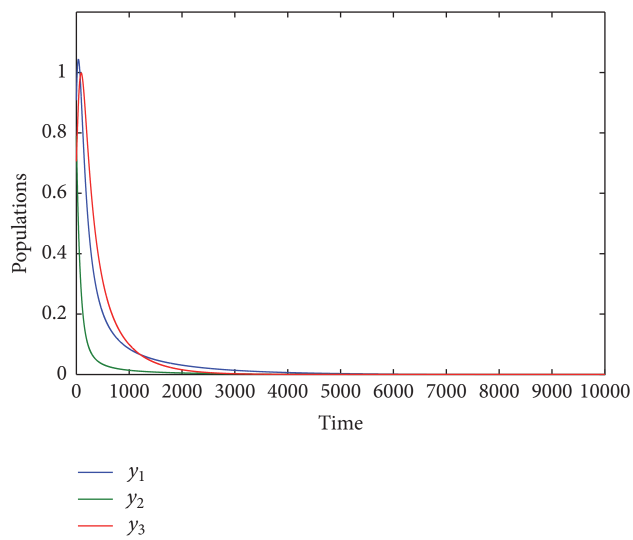

FIGURE 4: Time series of system (3) approaches asymptotically to the vanishing equilibrium point for $b_1 = 0.15$ with other parameters given by (69).

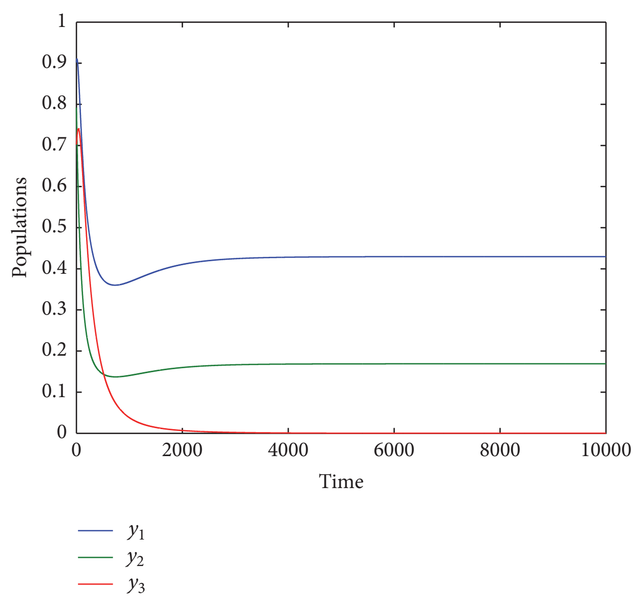

FIGURE 5: Time series of system (3) approaches asymptotically to the predator free equilibrium point $E_1 = (0.42, 0.16, 0)$ for $a_2 = 0.9$ and $b_4 = 0.7$ with other parameters given by (69).

stability of equilibria and the existence of local bifurcation are obtained. The effect of each parameter on the dynamical behavior of system (3) is studied numerically and the trajectories of the system are drowned in the typical figures. According to these figures, which represent the solution of system (3) for the data given by (69), the following conclusions are obtained.

(1) System (3) has no periodic dynamics rather than the fact that the system approaches asymptotically to one of their equilibrium points depending on the set of parameter data and the stability conditions that are satisfied.

(2) Although the position of the positive equilibrium point in the interior of R_+^3 changed as varying in the parameters values a_2, a_4, b_2, b_3, and b_4, there is no qualitative change in the dynamical behavior of system (3) and the system still approaches to a positive equilibrium point. Accordingly adding the refuge factor which is included implicitly in these parameters plays a vital role in the stabilizing of the system at the positive equilibrium point.

(3) Decreasing in the value of growth rate of immature prey or in the value of conversion rate from immature prey to mature prey keeping the rest of parameter as in (69) leads to destabilizing of the positive equilibrium point and the system approaches asymptotically to the vanishing equilibrium point, which means losing the persistence of system (3).

(4) Increasing in the value of grownup rate of immature prey keeping the rest of parameter as in (69) leads to destabilizing of the positive equilibrium point and the system approaches asymptotically to the vanishing equilibrium point too, which means losing the persistence of system (3).

(5) Finally, for the data given by (69), increasing intraspecific competition of immature prey and natural death rate of the predator leads to destabilizing of the positive equilibrium point and the solution approaches instead asymptotically to the predator free equilibrium point, which confirm our obtained analytical results represented by conditions (21)-(22).

Competing Interests

The authors declare that they have no competing interests.

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