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Research Article

Piecewise Approximate Analytical Solutions of High-Order Singular Perturbation Problems with a Discontinuous Source Term

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A reliable algorithm is presented to develop piecewise approximate analytical solutions of third- and fourth-order convection diffusion singular perturbation problems with a discontinuous source term. The algorithm is based on an asymptotic expansion approximation and Differential Transform Method (DTM). First, the original problem is transformed into a weakly coupled system of ODEs and a zero-order asymptotic expansion of the solution is constructed. Then a piecewise smooth solution of the terminal value reduced system is obtained by using DTM and imposing the continuity and smoothness conditions. The error estimate of the method is presented. The results show that the method is a reliable and convenient asymptotic semianalytical numerical method for treating high-order singular perturbation problems with a discontinuous source term.

1. Introduction

Many mathematical problems that model real-life phenomena cannot be solved completely by analytical means. Some of the most important mathematical problems arising in applied mathematics are singular perturbation problems. These problems commonly occur in many branches of applied mathematics such as transition points in quantum mechanics, edge layers in solid mechanics, boundary layers in fluid mechanics, skin layers in electrical applications, and shock layers in fluid and solid mechanics. The numerical treatment of these problems is accompanied by major computational difficulties due to the presence of sharp boundary and/or interior layers in the solution. Therefore, more efficient and simpler computational methods are required to solve these problems.

For the past two decades, many numerical methods have appeared in the literature, which cover mostly second-order singular perturbation boundary value problems (SPBVPs) [1–3]. But only few authors have developed numerical methods for higher order SPBVPs (see, e.g., [4–10]). However, most of them have concentrated on problems with smooth data. In fact some authors have developed numerical methods for problems with discontinuous data which gives rise to an interior layer in the exact solution of the problem, in addition to the boundary layer at the outflow boundary point. Most notable among these methods are piecewise-uniform mesh finite difference method [11–14] and fitted mesh finite element method [15, 16] for third- and fourth-order SPBVPs with a discontinuous source term. The aim of this paper is to employ a semianalytical method which is Differential Transform Method (DTM) as an alternative to existing methods for solving high-order SPBVPs with a discontinuous source term.

DTM is introduced by Zhou [17] in a study of electric circuits. This method is a formalized modified version of Taylor series method where the derivatives are evaluated through recurrence relations and not symbolically as the traditional Taylor series method. The method has been used effectively to obtain highly accurate solutions for large classes of linear and nonlinear problems (see, e.g., [17–21]). There is no need for discretization, perturbations, and further large computational work and round-off errors are avoided. Additionally, DTM does not generate secular terms (noise terms) and does not need analytical integrations as other semianalytical methods like HPM, HAM, ADM, or VIM and so DTM is an attractive tool for solving differential equations.

In this paper, a reliable algorithm is presented to develop piecewise approximate analytical solutions of third- and fourth-order convection diffusion singular perturbation problems with a discontinuous source term. The algorithm is based on an asymptotic expansion approximation and DTM. First, the original problem is transformed into a weakly coupled system of ODEs and a zero-order asymptotic expansion of the solution is constructed. Then a piecewise smooth solution of the terminal value reduced system is obtained by using DTM and imposing the continuity and smoothness conditions. The error estimate of the method is presented. The results show that the method is a reliable and convenient asymptotic semianalytical method for treating high-order singular perturbation problems with a discontinuous source term.

2. Differential Transform Method for ODE System

Let us describe the DTM for solving the following system of ODEs:

$$\begin{aligned}
u_1'(t) &= f_1(t,u_1,u_2,\dots,u_n),\\
u_2'(t) &= f_2(t,u_1,u_2,\dots,u_n),\\
&\vdots\\
u_n'(t) &= f_n(t,u_1,u_2,\dots,u_n),
\end{aligned} \tag{1}$$

subject to the initial conditions

$$u_i(t_0) = c_i, \quad i = 1,2,\dots,n. \tag{2}$$

Let $[t_0, T]$ be the interval over which we want to find the solution of (1)-(2). In actual applications of the DTM, the Nth-order approximate solution of (1)-(2) can be expressed by the finite series

$$u_i(t) = \sum_{k=0}^{N} U_i(k)(t-t_0)^k, \qquad t \in [t_0, T], \; i = 1,2,\dots,n, \tag{3}$$

where

$$U_i(k) = \frac{1}{k!}\left[\frac{d^k u_i(t)}{dt^k}\right]_{t=t_0}, \quad i = 1,2,\dots,n, \tag{4}$$

which implies that $\sum_{k=N+1}^{\infty} U_i(k)(t-t_0)^k$ is negligibly small. Using some fundamental properties of DTM (Table 1), the ODE system (1)-(2) can be transformed into the following recurrence relations:

$$\begin{aligned}
U_i(k+1) &= \frac{F_i(k,U_1,U_2,\dots,U_n)}{(k+1)},\\
Y_i(0) &= c_i,\\
& \qquad i = 1,2,\dots,n,
\end{aligned} \tag{5}$$

where $F_i(k,U_1,U_2,\dots,U_n)$ is the differential transform of the function $f_i(t,u_1,u_2,\dots,u_n)$, for $i = 1,2,\dots,n$. Solving the recurrence relation (5), the differential transform $U_i(k)$, $k > 0$, can be easily obtained.

3. Description of the Method

Motivated by the works of [11, 13, 16], we, in the present paper, suggest an asymptotic semianalytic method which is DTM to develop piecewise approximate analytical solutions for the following class of SPBVPs.

Third-Order SPBVP [13]. Find $y \in C^1(\overline{\Omega}) \cap C^2(\Omega) \cap C^3(\Omega^- \cup \Omega^+)$ such that

$$\begin{aligned}
-\varepsilon y'''(t) + a(t)\,y''(t) + b(t)\,y'(t) + c(t)\,y(t) &\\
= h(t), \quad t \in (\Omega^- \cup \Omega^+)&,\\
y(0) = p,&\\
y'(0) = q,&\\
y'(1) = r,&
\end{aligned} \tag{6}$$

where $a(t)$, $b(t)$, and $c(t)$ are sufficiently smooth functions on $\overline{\Omega}$ satisfying the following conditions:

$$\begin{aligned}
a(t) &< 0,\\
b(t) &\ge 0,\\
0 \ge c(t) &\ge -\gamma, \quad \gamma > 0,\\
\alpha - \theta\gamma &\ge \eta > 0,
\end{aligned} \tag{7}$$

where θ is arbitrarily close to 1, for some η.

Fourth-Order SPBVP [11]. Find $y \in C^2(\overline{\Omega}) \cap C^3(\Omega) \cap C^4(\Omega^- \cup \Omega^+)$ such that

$$\begin{aligned}
-\varepsilon y^{(iv)}(t) + a(t)\,y'''(t) + b(t)\,y''(t) - c(t)\,y(t) &\\
= -h(t), \quad t \in (\Omega^- \cup \Omega^+)&,\\
y(0) = p,&\\
y(1) = q,&\\
y''(0) = -r,&\\
y''(1) = -s,&
\end{aligned} \tag{8}$$

where $a(t)$, $b(t)$, and $c(t)$ are sufficiently smooth functions on $\overline{\Omega}$ satisfying the following conditions:

$$\begin{aligned}
a(t) &< 0,\\
b(t) &\ge 0,\\
0 \ge c(t) &\ge -\gamma, \quad \gamma > 0,\\
\alpha - \theta\gamma &\ge \eta > 0,
\end{aligned} \tag{9}$$

Table 1: Some fundamental operations of DTM.

Original function	Transformed function
$u(t) = \beta(v(t) \pm w(t))$	$U(k) = \beta V(k) \pm \beta W(k)$
$u(t) = v(t)w(t)$	$U(k) = \sum_{\ell=0}^{k} V(\ell)W(k-\ell)$
$u(t) = \dfrac{d^m v(t)}{dt^m}$	$U(k) = \dfrac{(k+m)!}{k!}V(k+m)$
$u(t) = (\beta + t)^m$	$U(k) = H[m,k]\dfrac{m!}{k!\lvert(m-k)\rvert!}(\beta+t_0)^{m-k},\ H[m,k] = \begin{cases} 1, & \text{if } m-k \geq 0 \\ 0, & \text{if } m-k < 0 \end{cases}$
$u(t) = e^{\lambda t}$	$U(k) = \dfrac{\lambda^k}{k!}e^{\lambda t_0}$
$u(t) = \sin(\omega t + \beta)$	$U(k) = \dfrac{\omega^k}{k!}\sin\left(\omega t_0 + \beta + \dfrac{k\pi}{2}\right)$
$u(t) = \cos(\omega t + \beta)$	$U(k) = \dfrac{\omega^k}{k!}\cos\left(\omega t_0 + \beta + \dfrac{k\pi}{2}\right)$

where θ is arbitrarily close to 1, for some η. For both the problems defined above, $\Omega = (0,1)$, $\Omega^- = (0,d)$, $\Omega^+ = (d,1)$, $\overline{\Omega} = (\Omega^- \cup \Omega^+)$, $\overline{\Omega} = \Omega \setminus \{d\}$, and $0 < \varepsilon \ll 1$. It is assumed that $h(t)$ is sufficiently smooth on $\overline{\Omega}$ and its derivatives have discontinuity at the point d and the jump at d is given as $[h](d) = h(d^+) - h(d^-)$.

3.1. Zero-Order Asymptotic Expansion Approximations. The SPBVP (6) can be transformed into an equivalent problem of the form

$$\begin{aligned} &y_1'(t) - y_2(t) = 0, \quad t \in \Omega \cup \{1\}, \\ &-\varepsilon y_2''(t) + a(t)y_2'(t) + b(t)y_2(t) + c(t)y_1(t) \\ &\quad = h(t), \quad t \in \overline{\Omega}, \\ &y_1(0) = p, \\ &y_2(0) = q, \\ &y_2(1) = r, \end{aligned} \tag{10}$$

where $y_1 \in C^1(\overline{\Omega})$ and $y_2 \in C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$ [14].

Similarly the SPBVP (8) can be transformed into

$$\begin{aligned} &-y_1''(t) - y_2(t) = 0, \quad t \in \Omega \cup \{1\}, \\ &-\varepsilon y_2''(t) + a(t)y_2'(t) + b(t)y_2(t) + c(t)y_1(t) \\ &\quad = h(t), \quad t \in \overline{\Omega}, \\ &y_1(0) = p, \\ &y_1(1) = q, \\ &y_2(0) = r, \\ &y_2(1) = s, \end{aligned} \tag{11}$$

where $y_1 \in C^2(\overline{\Omega}) \cap C^3(\Omega) \cap C^4(\Omega^- \cup \Omega^+)$ and $y_2 \in C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$ [11].

Remark 1. Hereafter, only the above systems (10) and (11) are considered.

Using some standard perturbation methods [11, 13, 16, 22] one can construct an asymptotic expansion for the solution of (10) and (11) as follows.

Find a continuous function $\mathbf{u} = (u_1, u_2)^T$ of the terminal value reduced system of (10) such that

$$\begin{aligned} u_1'(t) - u_2(t) &= 0, \\ a(t)u_2'(t) + b(t)u_2(t) + c(t)u_1(t) &= h(t), \\ u_1(0) &= p, \\ u_1(d^-) &= u_1(d^+), \\ u_2(d^-) &= u_2(d^+), \\ u_2(1) &= r. \end{aligned} \tag{12}$$

That is, find a smooth function u_1 on $\overline{\Omega}$ that satisfies the following equivalent reduced BVP:

$$\begin{aligned} a(t)u_1''(t) + b(t)u_1'(t) + c(t)u_1(t) &= h(t), \quad t \in \overline{\Omega}, \\ u_1(0) &= p, \\ u_1(d^-) &= u_1(d^+), \\ u_1'(d^-) &= u_1'(d^+), \\ u_1'(1) &= r. \end{aligned} \tag{13}$$

Then find

$$u_2(t) = u_1'(t). \tag{14}$$

Define $\mathbf{y}_{\text{as}} = (y_{1,\text{as}}, y_{2,\text{as}})^T$ on $\overline{\Omega}$ as

$$y_{1,\text{as}}(t) = u_1(t) + \frac{\varepsilon}{a(0)}(q - u_2(0))e^{ta(0)/\varepsilon} + \begin{cases} kt, & t \in \Omega^- \cup \{0, d\}, \\ \dfrac{k\varepsilon}{a(d)} e^{(t-d)a(d)/\varepsilon}, & t \in \Omega^+ \cup \{1\}, \end{cases}$$
$$y_{2,\text{as}}(t) = u_2(t) + (q - u_2(0))e^{ta(0)/\varepsilon} + \begin{cases} k, & t \in \Omega^- \cup \{0, d\}, \\ ke^{(t-d)a(d)/\varepsilon}, & t \in \Omega^+ \cup \{1\}, \end{cases} \tag{15}$$

where $k = -\varepsilon[u_2'(d^+) - u_2'(d^-)]/a(d)$.

Similarly one can construct an asymptotic expansion for the solution of (11). In fact, for this problem $\mathbf{u} = (u_1, u_2)^T$ is the solution of the terminal value reduced system

$$\begin{aligned} -u_1''(t) - u_2(t) &= 0, \\ a(t)u_2'(t) + b(t)u_2(t) + c(t)u_1(t) &= h(t), \\ &\quad t \in (\Omega^- \cup \Omega^+), \\ u_1(0) &= p, \\ u_1(d^-) &= u_1(d^+), \\ u_1'(d^-) &= u_1'(d^+), \\ u_2(d^-) &= u_2(d^+), \\ u_1(1) &= q, \\ u_2(1) &= s. \end{aligned} \tag{16}$$

That is, in particular, find a smooth function u_1 on $\overline{\Omega}$ that satisfies the following equivalent reduced BVP:

$$\begin{aligned} a(t)u_1'''(t) + b(t)u_1''(t) - c(t)u_1(t) &= -h(t), \\ &\quad t \in (\Omega^- \cup \Omega^+), \\ u_1(0) &= p, \\ u_1(d^-) &= u_1(d^+), \\ u_1'(d^-) &= u_1'(d^+), \\ u_1''(d^-) &= u_1''(d^+), \\ u_1(1) &= q, \\ u_1''(1) &= -s. \end{aligned} \tag{17}$$

Then find $u_2(t) = -u_1''(t)$.

Define $\mathbf{y}_{\text{as}} = (y_{1,\text{as}}, y_{2,\text{as}})^T$ on $\overline{\Omega}$ as

$$y_{1,\text{as}}(t) = u_1(t) + \begin{cases} -k_1 \dfrac{\varepsilon^2}{[a(0)]^2} e^{a(0)t/\varepsilon}, & t \in \Omega^- \cup \{0, d\}, \\ -k_2 \dfrac{\varepsilon^2}{[a(d)]^2} e^{(t-d)a(d)/\varepsilon}, & t \in \Omega^+ \cup \{1\}, \end{cases}$$
$$y_{2,\text{as}}(t) = u_2(t) + \begin{cases} k_1 e^{a(0)t/\varepsilon}, & t \in \Omega^- \cup \{0, d\}, \\ k_2 e^{(t-d)a(d)/\varepsilon}, & t \in \Omega^+ \cup \{1\}, \end{cases} \tag{18}$$

where

$$k_1 = [r - u_2(0)],$$
$$k_2 = -\frac{\varepsilon}{a(0)}[u_2'(d^+) - u_2'(d^-)] + k_1 e^{a(0)d/\varepsilon}. \tag{19}$$

Theorem 2 (see [11, 13]). *The zero-order asymptotic expansion $\mathbf{y}_{as} = (y_{1,as}, y_{2,as})^T$ defined above for the solution $\mathbf{y} = (y_1, y_2)^T$ of (10) and (11) satisfies the inequality*

$$|\mathbf{y} - \mathbf{y}_{as}| \le C\varepsilon. \tag{20}$$

Now, in order to obtain piecewise analytical solutions of (10) and (11), we only need to obtain piecewise analytical solutions of the terminal value reduced systems (12) and (16), that is, the solution of equivalent reduced BVPs (13) and (17).

3.2. Piecewise Approximate Analytical Solutions. The solution $u_1(t)$ of BVP (13) can be represented as a piecewise solution form:

$$u_1(t) = \begin{cases} u_{1L}(t), & t \in \Omega^- \\ u_{1R}(t), & t \in \Omega^+. \end{cases} \tag{21}$$

Thus the BVP (13) is transformed into

$$\begin{aligned} a(t)u_{1L}''(t) + b(t)u_{1L}'(t) + c(t)u_{1L}(t) &= h(t), \\ u_{1L}(0) &= p, \\ u_{1L}'(0) &= \alpha_1, \\ &\quad t \in \Omega^-, \\ a(t)u_{1R}''(t) + b(t)u_{1R}'(t) + c(t)u_{1R}(t) &= h(t), \\ u_{1R}(1) &= \beta_1, \\ u_{1R}'(1) &= r, \\ &\quad t \in \Omega^+, \end{aligned} \tag{22}$$

with continuity and smoothness conditions $u_{1L}(d^-) = u_{1R}(d^+)$, $u_{1L}'(d^-) = u_{1R}'(d^+)$ and α_1, β_1 are unknown constants.

Applying Nth-order DTM on (22) results in the recurrence relations

$$\begin{aligned}
&\sum_{\ell=0}^{k} A(k-\ell)(\ell+2)(\ell+1)U_{1L}(\ell+2) \\
&\quad + \sum_{\ell=0}^{k} B(k-\ell)(\ell+1)U_{1L}(\ell+1) \\
&\quad + \sum_{\ell=0}^{k} C(k-\ell)U_{1L}(\ell) = H(k), \\
&U_{1L}(0) = p, \\
&U_{1L}(1) = \alpha_1, \\
&\sum_{\ell=0}^{k} A(k-\ell)(\ell+2)(\ell+1)U_{1R}(\ell+2) \\
&\quad + \sum_{\ell=0}^{k} B(k-\ell)(\ell+1)U_{1R}(\ell+1) \\
&\quad + \sum_{\ell=0}^{k} C(k-\ell)U_{1R}(\ell) = H(k), \\
&U_{1R}(0) = \beta_1, \\
&U_{1R}(1) = r,
\end{aligned} \tag{23}$$

where $A(k)$, $B(k)$, $C(k)$, $H(k)$, $U_{1L}(k)$, and $U_{1R}(k)$ are the differential transform of $a(t)$, $b(t)$, $c(t)$, $h(t)$, $u_{1L}(t)$, and $u_{1R}(t)$, respectively, and α_1 and β_1 values are determined from the transformed continuity and smoothness conditions:

$$\begin{aligned}
&\sum_{k=0}^{N} U_{1L}(k)(d^-)^k = \sum_{k=0}^{N} U_{1R}(k)(d^+-1)^k, \\
&\sum_{k=0}^{N} (k+1)U_{1L}(k+1)(d^-)^k \\
&\quad = \sum_{k=0}^{N} (k+1)U_{1R}(k+1)(d^+-1)^k.
\end{aligned} \tag{24}$$

The recurrence relations (23) with transformed conditions (24) represent a system of algebraic equations in the coefficients of the power series solution of the reduced BVP (13) and the unknowns α_1 and β_1. Solving this algebraic system, the piecewise smooth approximate solution $\tilde{\mathbf{u}} = (\tilde{u}_1(t), \tilde{u}_2(t))^T$ of (13) is obtained and given by

$$\tilde{u}_1(t) = \begin{cases} \sum_{k=0}^{N} U_{1L}(k)t^k, & t \in \Omega^- \\ \sum_{k=0}^{N} U_{1R}(k)(t-1)^k, & t \in \Omega^+, \end{cases}$$

$$\tilde{u}_2(t) = \begin{cases} \sum_{k=0}^{N} (k+1)U_{1L}(k+1)t^k, & t \in \Omega^- \\ \sum_{k=0}^{N} (k+1)U_{1R}(k+1)(t-1)^k, & t \in \Omega^+. \end{cases} \tag{25}$$

And thus, the piecewise approximate analytical solution $\mathbf{y}_{\mathrm{ap}} = (y_{1,\mathrm{ap}}, y_{2,\mathrm{ap}})^T$ of (10) is obtained and given by

$$\begin{aligned}
y_{1,\mathrm{ap}}(t) &= \tilde{u}_1(t) + \frac{\varepsilon}{a(0)}(q - \tilde{u}_2(0))e^{ta(0)/\varepsilon} \\
&\quad + \begin{cases} k\varepsilon t, & t \in [0,d], \\ \dfrac{\kappa\varepsilon^2}{a(d)}e^{(t-d)a(d)/\varepsilon}, & t \in (d,1], \end{cases} \\
y_{2,\mathrm{ap}}(t) &= \tilde{u}_2(t) + (q - \tilde{u}_2(0))e^{ta(0)/\varepsilon} \\
&\quad + \begin{cases} k\varepsilon, & t \in [0,d], \\ k\varepsilon e^{(t-d)a(d)/\varepsilon}, & t \in (d,1], \end{cases}
\end{aligned} \tag{26}$$

where $k = [\tilde{u}_2'(d^-) - \tilde{u}_2'(d^+)]/a(d)$.

Similarly the reduced BVP (17) can be transformed into

$$\begin{aligned}
&a(t)u_{1L}'''(t) + b(t)u_{1L}''(t) - c(t)u_{1L}(t) = -h(t), \\
&\qquad u_{1L}(0) = p, \\
&\qquad u_{1L}'(0) = \alpha_1, \\
&\qquad u_{1L}''(0) = \alpha_2, \\
&\qquad\qquad t \in \Omega^-, \\
&a(t)u_{1R}'''(t) + b(t)u_{1R}''(t) - c(t)u_{1R}(t) = -h(t), \\
&\qquad u_{1R}(1) = q, \\
&\qquad u_{1R}'(1) = \beta_1, \\
&\qquad u_{1R}''(1) = -s, \\
&\qquad\qquad t \in \Omega^+,
\end{aligned} \tag{27}$$

with continuity and smoothness conditions $u_{1L}(d^-) = u_{1R}(d^+)$, $u_{1L}'(d^-) = u_{1R}'(d^+)$, $u_{1L}''(d^-) = u_{1R}''(d^+)$ and α_1, α_2, β_1 are unknown constants.

Applying Nth-order DTM on (27) results in the recurrence relations

$$\begin{aligned}
&\sum_{\ell=0}^{k} A(k-\ell)(\ell+3)(\ell+2)(\ell+1)U_{1L}(\ell+3) \\
&\quad+\sum_{\ell=0}^{k} B(k-\ell)(\ell+2)(\ell+1)U_{1L}(\ell+2) \\
&\quad-\sum_{\ell=0}^{k} C(k-\ell)U_{1L}(\ell)=-H(k), \\
&U_{1L}(0)=p, \\
&U_{1L}(1)=\alpha_1, \\
&2U_{1L}(2)=\alpha_2, \\
&\sum_{\ell=0}^{k} A(k-\ell)(\ell+3)(\ell+2)(\ell+1)U_{1R}(\ell+3) \\
&\quad+\sum_{\ell=0}^{k} B(k-\ell)(\ell+2)(\ell+1)U_{1R}(\ell+2) \\
&\quad-\sum_{\ell=0}^{k} C(k-\ell)U_{1R}(\ell)=-H(k), \\
&U_{1R}(0)=q, \\
&U_{1R}(1)=\beta_1, \\
&2U_{1R}(2)=-s,
\end{aligned} \tag{28}$$

where the unknown constants α_1, α_2, and β_1 are determined from the transformed continuity and smoothness conditions:

$$\begin{aligned}
&\sum_{k=0}^{N} U_{1L}(k)(d^-)^k=\sum_{k=0}^{N} U_{1R}(k)(d^+-1)^k, \\
&\sum_{k=0}^{N}(k+1)U_{1L}(k+1)(d^-)^k \\
&\quad=\sum_{k=0}^{N}(k+1)U_{1R}(k+1)(d^+-1)^k, \\
&\sum_{k=0}^{N}(k+2)(k+1)U_{1L}(k+2)(d^-)^k \\
&\quad=\sum_{k=0}^{N}(k+2)(k+1)U_{1R}(k+2)(d^+-1)^k.
\end{aligned} \tag{29}$$

And the piecewise approximate analytical solution $\mathbf{y}_{\text{ap}} = (y_{1,\text{ap}}, y_{2,\text{ap}})^T$ of (11) is obtained and given by

$$\begin{aligned}
y_{1,\text{ap}}(t) &= \tilde{u}_1(t) \\
&\quad+\begin{cases} k_1 \dfrac{\varepsilon^2}{[a(0)]^2} e^{a(0)t/\varepsilon}, & t\in[0,d], \\ k_2 \dfrac{\varepsilon^2}{[a(d)]^2} e^{(t-d)a(d)/\varepsilon}, & t\in(d,1], \end{cases} \\
y_{2,\text{ap}}(t) &= \tilde{u}_2(t)+\begin{cases} k_1 e^{a(0)t/\varepsilon}, & t\in[0,d], \\ k_2 e^{(t-d)a(d)/\varepsilon}, & t\in(d,1], \end{cases}
\end{aligned} \tag{30}$$

where

$$\begin{aligned}
k_1 &= [r-\tilde{u}_2(0)], \\
k_2 &= \frac{-\varepsilon}{a(0)}\left[\tilde{u}_2'(d^-)-\tilde{u}_2'(d^+)\right]+[r-\tilde{u}_2(0)]\,e^{a(0)d/\varepsilon}.
\end{aligned} \tag{31}$$

3.3. Error Estimate. The error estimate of the present method has two sources: one from the asymptotic approximation and the other from the truncated series approximation by DTM.

Theorem 3. *Let $\mathbf{y} = (y_1, y_2)^T$ be the solution of (10). Further let $\mathbf{y}_{ap} = (y_{1,ap}, y_{2,ap})^T$ be the approximate solution (26). Then*

$$\left\|\mathbf{y}-\mathbf{y}_{ap}\right\| \le C\left(\varepsilon+\frac{1}{(N+1)!}\right). \tag{32}$$

Proof. Since the DTM is a formalized modified version of the Taylor series method, then we have a bounded error given by

$$\|\mathbf{u}-\tilde{\mathbf{u}}\| \le \frac{M}{(N+1)!}, \quad M \le \left\|\mathbf{u}^{(N+1)}(\xi)\right\|, \ 0\le\xi\le 1. \tag{33}$$

From Theorem 2 and the above bounded error, we have

$$\left\|\mathbf{y}-\mathbf{y}_{\text{ap}}\right\| \le \left\|\mathbf{y}-\mathbf{y}_{\text{as}}\right\| + \left\|\mathbf{y}_{\text{as}}-\mathbf{y}_{\text{ap}}\right\| \le C_1\varepsilon+\frac{M}{(N+1)!}. \tag{34}$$

Since the singular perturbation parameter ε is extremely small, the present method works well for singular perturbation problems. □

Remark 4. A similar statement is true for the solution of (11) and the approximate solution (30).

4. Illustrating Examples

In this section we will apply the method described in the previous section to find piecewise approximate analytical solutions for three SPBVPs with a discontinuous source term.

Example 1. Consider the third-order SPBVP from [13, 16]

$$\begin{aligned}-\varepsilon y'''(t)-2y''(t)+4y'(t)-2y(t)&=h(t),\\ y(0)&=1,\\ y'(0)&=0,\\ y'(1)&=0,\end{aligned}\tag{35}$$

where

$$h(t)=\begin{cases}0.7, & 0\le t\le 0.5\\ -0.6, & 0.5<t\le 1.\end{cases}\tag{36}$$

Using the present method with 5th-order DTM, the piecewise analytical solution is given by

$$y_{1,\text{ap}}(t)=\begin{cases}1+0.575447t-0.099553t^2-0.162277t^3-0.072842t^4-0.021023t^5+0.287723\varepsilon e^{-2t/\varepsilon}\\ \quad+\dfrac{\varepsilon}{2}\left(0.966783-\dfrac{0.849226}{e}\right)t, & t\in[0,0.5],\\ 1.330163-0.515081(t-1)^2-0.343388(t-1)^3-0.128770(t-1)^4-0.034339(t-1)^5\\ \quad+0.287723\varepsilon e^{-2t/\varepsilon}-\dfrac{\varepsilon^2}{4}\left(0.966783-\dfrac{0.849226}{e}\right)e^{-2(t-0.5)/\varepsilon}, & t\in(0.5,1.0],\end{cases}$$

$$y_{2,\text{ap}}(t)=\begin{cases}0.575447-0.199107t-0.486830t^2-0.291369t^3-0.105115t^4-0.575447e^{-2t/\varepsilon}\\ \quad+\dfrac{\varepsilon}{2}\left(0.966783-\dfrac{0.849226}{e}\right), & t\in[0,0.5],\\ -1.030163t+1.030163-1.030163(t-1)^2-0.515081(t-1)^3-0.171694(t-1)^4\\ \quad-0.575447e^{-2t/\varepsilon}+\dfrac{\varepsilon}{2}\left(0.966783-\dfrac{0.849226}{e}\right)e^{-2(t-0.5)/\varepsilon}, & t\in(0.5,1.0].\end{cases}\tag{37}$$

Example 2. Consider the third-order SPBVP with variable coefficients from [13, 16]

$$\begin{aligned}&-\varepsilon y'''(t)-2e^t y''(t)+\cos\left(\frac{\pi t}{4}\right)y'(t)-(1+x)y(t)\\ &\quad=h(t),\\ &y(0)=0,\\ &y'(0)=0,\\ &y'(1)=1,\end{aligned}\tag{38}$$

where

$$h(t)=\begin{cases}2t^3, & 0\le t\le 0.5\\ 10t+1, & 0.5<t\le 1.\end{cases}\tag{39}$$

Using the present method with 5th-order DTM, the piecewise analytical solution is given by

$$\begin{aligned}&y_{1,\text{ap}}(t)\\ &=\begin{cases}1.865844t+0.466461t^2-0.233230t^3-0.072568t^4+0.000470t^5+0.932922\varepsilon e^{-2t/ep}\\ \quad-0.513641\varepsilon t, & t\in[0,0.5],\\ 0.814311+t-1.280360(t-1)^2-0.069249(t-1)^3+0.181448(t-1)^4-0.052651(t-1)^5\\ \quad+0.932922\varepsilon e^{-2t/\varepsilon}+0.155770\varepsilon^2 e^{-3.297442(t-0.5)/\varepsilon}, & t\in(0.5,1.0],\end{cases}\end{aligned}\tag{40}$$

$$y_{2,\text{ap}}(t)=\begin{cases}1.865844+0.932922t-0.699692t^2-0.290271t^3+0.002350t^4-1.865844e^{-2t/\varepsilon}\\ \quad-0.513641\varepsilon t, & t\in[0,0.5],\\ 3.560719-2.560719t-0.207747(t-1)^2+0.725794(t-1)^3-0.263254(t-1)^4\\ \quad-1.865844e^{-2t/\varepsilon}-0.513641\varepsilon e^{-3.297442(t-0.5)/\varepsilon}, & t\in(0.5,1.0].\end{cases}$$

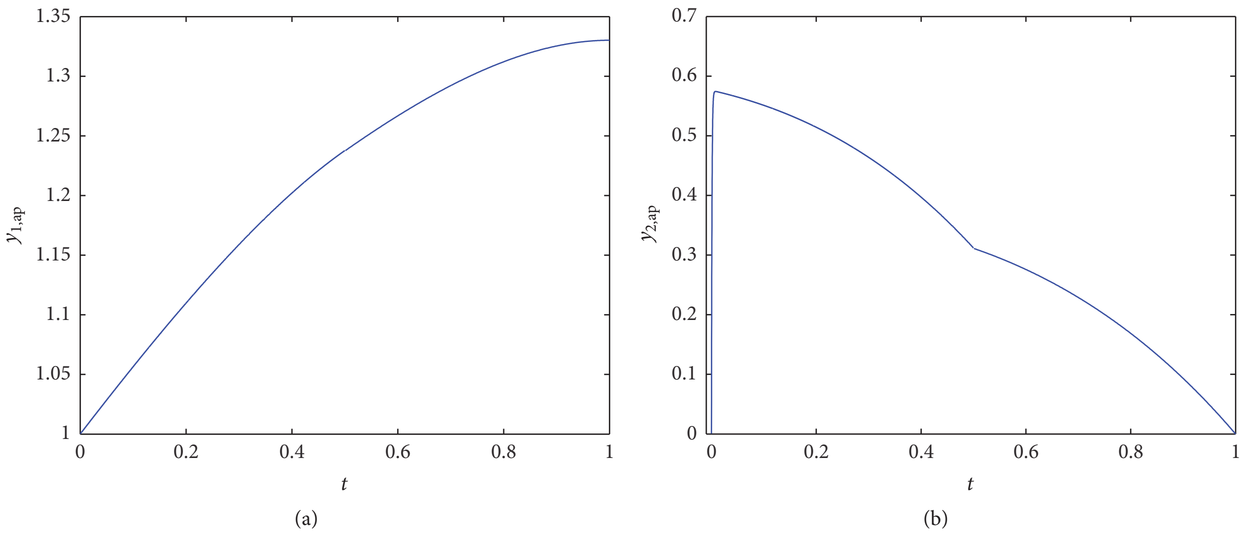

FIGURE 1: Graphs of the approximate solution $y_{1,ap}$ and its first derivative $y_{2,ap}$ for Example 1 at $\varepsilon = 2^{-9}$ and $N = 5$.

Example 3. Consider the fourth-order SPBVP from [13, 16]

$$\begin{gathered}
-\varepsilon y^{(iv)}(t) - 4y'''(t) + 4y''(t) = -h(t), \\
t \in (\Omega^- \cup \Omega^+), \\
y(0) = 1, \\
y(1) = 1, \\
y''(0) = -1, \\
y''(1) = -1,
\end{gathered} \tag{41}$$

where

$$h(t) = \begin{cases} 0.7, & 0 \le t \le 0.5 \\ -0.6, & 0.5 < t \le 1. \end{cases} \tag{42}$$

Using the present method with 5th-order DTM, the piecewise analytical solution is given by

$$\begin{gathered}
y_{1,ap}(t) = \begin{cases} 1 + 0.256751t - 0.199842t^2 - 0.037447t^3 - 0.009362t^4 - 0.0018724t^5 - 0.037520\varepsilon^2 e^{-4t/\varepsilon}, & t \in [0, 0.5], \\ 1.353655 - 0.353655t - \dfrac{1}{2}(t-1)^2 - \dfrac{23}{120}(t-1)^3 - \dfrac{23}{480}(t-1)^4 - \dfrac{23}{2400}(t-1)^5 & \\ \quad -\dfrac{1}{16}\left(2.5\varepsilon E - 11 + 0.600316e^{-2.0/\varepsilon}\right)\varepsilon^2 e^{-4(t-0.5)/\varepsilon}, & t \in (0.5, 1.0], \end{cases} \\
y_{2,ap}(t) = \begin{cases} 0.399683 + 0.224683t + 0.112342t^2 + 0.037447t^3 + 0.600316e^{-4t/\varepsilon}, & t \in [0, 0.5], \\ -\dfrac{3}{20} + \dfrac{23}{20}t + \dfrac{23}{40}(t-1)^2 + \dfrac{23}{120}(t-1)^3 + \left(2.5\varepsilon E - 11 + 0.600316e^{-2.0/\varepsilon}\right)e^{-4(t-0.5)/\varepsilon}, & t \in (0, 1.0]. \end{cases}
\end{gathered} \tag{43}$$

The numerical solution for each example is presented overall the problem domain as shown in Figures 1–3.

The corresponding maximum pointwise errors are taken to be

$$\begin{aligned}
E_\varepsilon^N &= \max_{t_i \in \overline{\Omega}_\varepsilon^N} \left| y_{ap}^N(t_i) - y_{ap}^{50}(t_i) \right|, \\
E^N &= \max E_\varepsilon^N,
\end{aligned} \tag{44}$$

where $y_{ap}^N(t_i)$ is the obtained approximate solution using Nth-order DTM over a uniform mesh $t_i = ih$, $t_i \in [0, 1]$, $h = 10^{-3}$, $i = 0, 1, 2, \ldots$ and $y_{ap}^{50}(t_i)$ is our numerical reference solution obtained using DTM with order $N = 50$.

The computed maximum pointwise errors E_ε^N and E^N for the above solved BVPs are given in Tables 2–7. The numerical results in Tables 2–7 agree with the theoretical ones present in this paper where the obtained solutions and their derivatives converge rapidly to the reference solutions with increasing the order of the DTM.

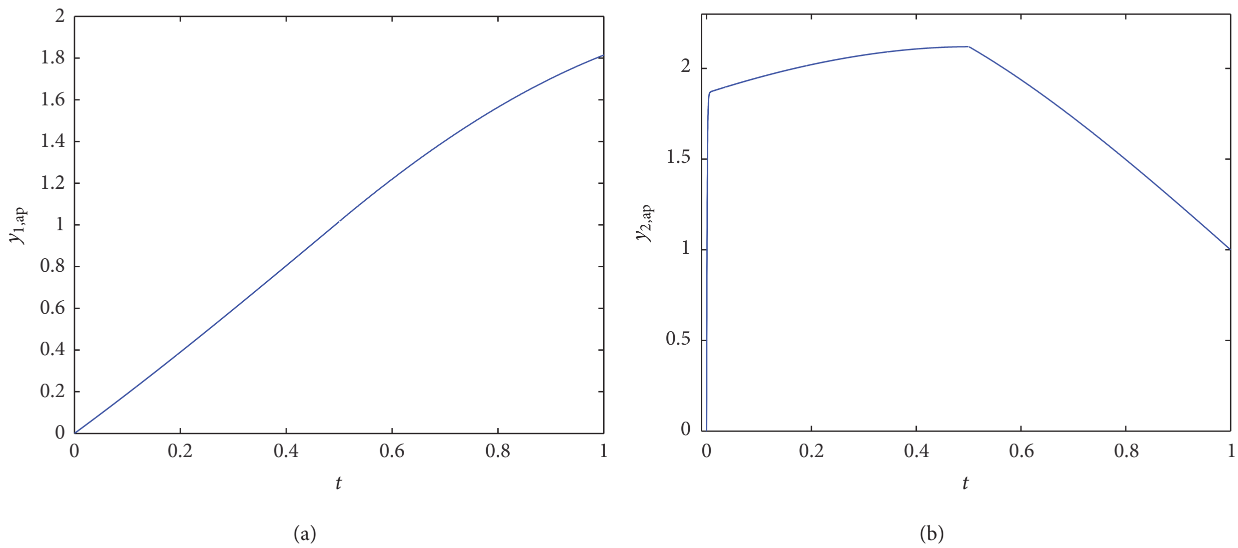

Figure 2: Graphs of the approximate solution $y_{1,ap}$ and its first derivative $y_{2,ap}$ for Example 2 at $\varepsilon = 2^{-9}$ and $N = 5$.

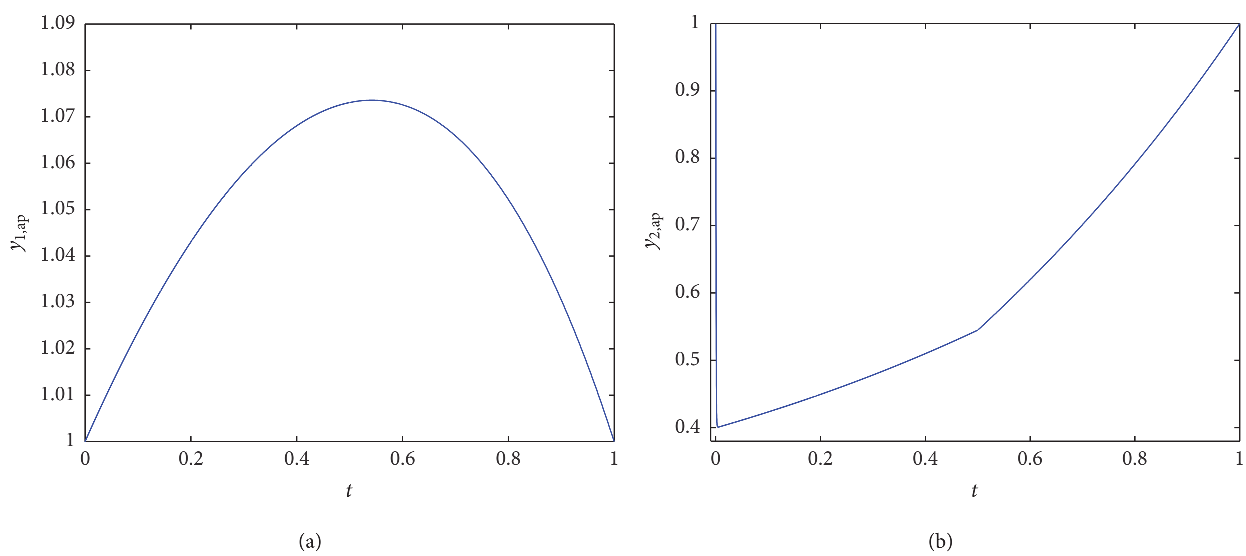

Figure 3: Graphs of the approximate solution $y_{1,ap}$ and the second derivative $y_{2,ap}$ for Example 3 at $\varepsilon = 2^{-9}$ and $N = 5$.

Table 2: Maximum pointwise errors E_ε^N and E^N for the solution $y_{1,ap}$ of Example 1.

ε	Approximation order of DTM, N			
	4	6	8	10
2^{-3}	$2.9574e-3$	$6.5563e-5$	$5.7278e-7$	$6.3740e-9$
2^{-9}	$2.9573e-3$	$3.0803e-5$	$1.6200e-7$	$4.0000e-9$
2^{-15}	$2.9573e-3$	$3.0803e-5$	$1.6200e-7$	$4.0000e-9$
2^{-21}	$2.9573e-3$	$3.0803e-5$	$1.6200e-7$	$4.0000e-9$
E^N	$2.9574e-3$	$6.5563e-5$	$5.7278e-7$	$6.3740e-9$

TABLE 3: Maximum pointwise errors E_ε^N and E^N for the first derivative solution $y_{2,\mathrm{ap}}$ of Example 1.

ε	Approximation order of DTM, N			
	4	6	8	10
2^{-3}	$8.8781e-3$	$1.6938e-4$	$1.3570e-6$	$8.6900e-9$
2^{-9}	$1.0321e-2$	$1.3075e-4$	$7.8084e-7$	$2.3233e-9$
2^{-15}	$1.0444e-2$	$1.3312e-4$	$8.0028e-7$	$2.6769e-9$
2^{-21}	$1.0444e-2$	$1.3316e-4$	$8.0067e-7$	$2.4481e-9$
E^N	$1.0444e-2$	$1.6938e-4$	$8.0067e-7$	$8.6900e-9$

TABLE 4: Maximum pointwise errors E_ε^N and E^N for the solution $y_{1,\mathrm{ap}}$ of Example 2.

ε	Approximation order of DTM, N			
	4	6	8	10
2^{-3}	$1.2547e-2$	$1.0369e-3$	$1.3608e-5$	$1.5365e-6$
2^{-9}	$1.2547e-2$	$1.0077e-3$	$1.3488e-5$	$6.8490e-7$
2^{-15}	$1.2547e-2$	$1.0077e-3$	$1.3488e-5$	$6.8100e-7$
2^{-21}	$1.2547e-2$	$1.0077e-3$	$1.3488e-5$	$6.8100e-7$
E^N	$1.2547e-2$	$1.0369e-3$	$1.3488e-5$	$1.5365e-6$

TABLE 5: Maximum pointwise errors E_ε^N and E^N for the first derivative solution $y_{2,\mathrm{ap}}$ of Example 2.

ε	Approximation order of DTM, N			
	4	6	8	10
2^{-3}	$1.8191e-2$	$2.2563e-3$	$2.9397e-5$	$3.2970e-6$
2^{-9}	$2.1534e-2$	$2.0165e-3$	$2.6724e-5$	$1.5660e-6$
2^{-15}	$2.1583e-2$	$2.0124e-3$	$2.6679e-5$	$1.5390e-6$
2^{-21}	$2.1584e-2$	$2.0123e-3$	$2.6678e-5$	$1.5390e-6$
E^N	$2.1584e-2$	$2.2563e-3$	$2.9397e-5$	$3.2970e-6$

TABLE 6: Maximum pointwise errors E_ε^N and E^N for the solution $y_{1,\mathrm{ap}}$ of Example 3.

ε	Approximation order of DTM, N			
	4	6	8	10
2^{-3}	$1.8918e-3$	$2.3001e-5$	$1.3700e-7$	$3.3000e-9$
2^{-9}	$1.8918e-3$	$2.3001e-5$	$1.3700e-7$	$3.3000e-9$
2^{-15}	$1.8918e-3$	$2.3001e-5$	$1.3700e-7$	$3.3000e-9$
2^{-21}	$1.8918e-3$	$2.3001e-5$	$1.3700e-7$	$3.3000e-9$
E^N	$1.8918e-3$	$2.3001e-5$	$1.3700e-7$	$3.3000e-9$

TABLE 7: Maximum pointwise errors E_ε^N and E^N for the second derivative solution $y_{2,\mathrm{ap}}$ of Example 3.

ε	Approximation order of DTM, N			
	4	6	8	10
2^{-3}	$2.1578e-2$	$2.8775e-4$	$1.7839e-6$	$6.5000e-9$
2^{-9}	$2.1578e-2$	$2.8775e-4$	$1.7839e-6$	$6.5000e-9$
2^{-15}	$2.1578e-2$	$2.8775e-4$	$1.7839e-6$	$6.5000e-9$
2^{-21}	$2.1578e-2$	$2.8775e-4$	$1.7839e-6$	$6.5000e-9$
E^N	$2.1578e-2$	$2.8775e-4$	$1.7839e-6$	$6.5000e-9$

5. Conclusion

We have presented a new reliable algorithm to develop piecewise approximate analytical solutions of third- and fourth-order convection diffusion SPBVPs with a discontinuous source term. The algorithm is based on constructing a zero-order asymptotic expansion of the solution and the DTM which provides the solutions in terms of convergent series with easily computable components. The original problem is transformed into a weakly coupled system of ODEs and a zero-order asymptotic expansion for the solution of the transformed system is constructed. For simplicity, the result terminal value reduced system is replaced by its equivalent reduced BVP with suitable continuity and smoothness conditions. Then a piecewise smooth solution of the reduced BVP is obtained by using DTM and imposing the continuity and smoothness conditions. The error estimate of the method is presented and shows that the method results in high-order convergence for small values of the singular perturbation parameter. We have applied the method on three SPBVPs and the piecewise analytical solution is presented for each one overall the problem domain. The numerical results confirm that the obtained solutions and their derivatives converge rapidly to the reference solutions with increasing the order of the DTM. The results show that the method is a reliable and convenient asymptotic semianalytical numerical method for treating high-order SPBVPs with a discontinuous source term. The method is based on a straightforward procedure, suitable for engineers.

Competing Interests

The author declares that he has no competing interests.

References

[1] M. K. Kadalbajoo and K. C. Patidar, "A survey of numerical techniques for solving singularly perturbed ordinary differential equations," *Applied Mathematics and Computation*, vol. 130, no. 2-3, pp. 457–510, 2002.

[2] M. Kumar, P. Singh, and H. K. Mishra, "A recent survey on computational techniques for solving singularly perturbed boundary value problems," *International Journal of Computer Mathematics*, vol. 84, no. 10, pp. 1439–1463, 2007.

[3] K. K. Sharma, P. Rai, and K. C. Patidar, "A review on singularly perturbed differential equations with turning points and interior layers," *Applied Mathematics and Computation*, vol. 219, no. 22, pp. 10575–10609, 2013.

[4] S. Valarmathi and N. Ramanujam, "An asymptotic numerical method for singularly perturbed third-order ordinary differential equations of convection-diffusion type," *Computers and Mathematics with Applications*, vol. 44, no. 5-6, pp. 693–710, 2002.

[5] J. C. Roja and A. Tamilselvan, "Numerical method for singularly perturbed third order ordinary differential equations of convection-diffusion type," *Numerical Mathematics: Theory, Methods and Applications*, vol. 7, no. 3, pp. 265–287, 2014.

[6] V. Shanthi and N. Ramanujam, "A boundary value technique for boundary value problems for singularly perturbed fourth-order ordinary differential equations," *Computers and Mathematics with Applications*, vol. 47, no. 10-11, pp. 1673–1688, 2004.

[7] M. Cui and F. Geng, "A computational method for solving third-order singularly perturbed boundary-value problems," *Applied Mathematics and Computation*, vol. 198, no. 2, pp. 896–903, 2008.

[8] M. Kumar and S. Tiwari, "An initial-value technique to solve third-order reaction-diffusion singularly perturbed boundary-value problems," *International Journal of Computer Mathematics*, vol. 89, no. 17, pp. 2345–2352, 2012.

[9] M. I. Syam and B. S. Attili, "Numerical solution of singularly perturbed fifth order two point boundary value problem," *Applied Mathematics and Computation*, vol. 170, no. 2, pp. 1085–1094, 2005.

[10] E. R. El-Zahar, "Approximate analytical solutions of singularly perturbed fourth order boundary value problems using differential transform method," *Journal of King Saud University—Science*, vol. 25, no. 3, pp. 257–265, 2013.

[11] V. Shanthi and N. Ramanujam, "Asymptotic numerical method for boundary value problems for singularly perturbed fourth-order ordinary differential equations with a weak interior layer," *Applied Mathematics and Computation*, vol. 172, no. 1, pp. 252–266, 2006.

[12] V. Shanthi and N. Ramanujam, "An asymptotic numerical method for fourth order singular perturbation problems with a discontinuous source term," *International Journal of Computer Mathematics*, vol. 85, no. 7, pp. 1147–1159, 2008.

[13] T. Valanarasu and N. Ramanujam, "An asymptotic numerical method for singularly perturbed third-order ordinary differential equations with a weak interior layer," *International Journal of Computer Mathematics*, vol. 84, no. 3, pp. 333–346, 2007.

[14] T. Valanarasu and N. Ramanujam, "Asymptotic numerical method for singularly perturbed third order ordinary differential equations with a discontinuous source term," *Novi Sad Journal of Mathematics*, vol. 37, no. 2, pp. 41–57, 2007.

[15] A. R. Babu and N. Ramanujam, "An asymptotic finite element method for singularly perturbed third and fourth order ordinary differential equations with discontinuous source term," *Applied Mathematics and Computation*, vol. 191, no. 2, pp. 372–380, 2007.

[16] A. R. Babu and N. Ramanujam, "An asymptotic finite element method for singularly perturbed higher order ordinary differential equations of convection-diffusion type with discontinuous source term," *Journal of Applied Mathematics & Informatics*, vol. 26, no. 5-6, pp. 1057–1069, 2008.

[17] J. K. Zhou, *Differential Transformation and its Applications for Electrical Circuits*, Huazhong University Press, Wuhan, China, 1986.

[18] Z. Šmarda, J. Diblík, and Y. Khan, "Extension of the differential transformation method to nonlinear differential and integro-differential equations with proportional delays," *Advances in Difference Equations*, vol. 2013, article 69, 2013.

[19] E. R. El-Zahar, "Applications of adaptive multi step differential transform method to singular perturbation problems arising in science and engineering," *Applied Mathematics & Information Sciences*, vol. 9, no. 1, pp. 223–232, 2015.

[20] H. Fatoorehchi, H. Abolghasemi, and R. Zarghami, "Analytical approximate solutions for a general nonlinear resistor-nonlinear capacitor circuit model," *Applied Mathematical Modelling*, vol. 39, no. 19, pp. 6021–6031, 2015.

[21] Z. M. Odibat, C. Bertelle, M. A. Aziz-Alaoui, and G. H. E. Duchamp, "A multi-step differential transform method and

application to non-chaotic or chaotic systems," *Computers and Mathematics with Applications*, vol. 59, no. 4, pp. 1462–1472, 2010.

[22] A. H. Nayfeh, *Introduction to Perturbation Methods*, John Wiley and Sons, New York, NY, USA, 1981.