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Research Article

Numerical Integration of a Class of Singularly Perturbed Delay Differential Equations with Small Shift

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We have presented a numerical integration method to solve a class of singularly perturbed delay differential equations with small shift. First, we have replaced the second-order singularly perturbed delay differential equation by an asymptotically equivalent first-order delay differential equation. Then, Simpson's rule and linear interpolation are employed to get the three-term recurrence relation which is solved easily by discrete invariant imbedding algorithm. The method is demonstrated by implementing it on several linear and nonlinear model examples by taking various values for the delay parameter δ and the perturbation parameter ε.

1. Introduction

The singularly perturbed delay differential equations with small shift arise very frequently in the modeling of various physical and biological phenomena, for example, micro scale heat transfer [1], hydrodynamics of liquid helium [2], second-sound theory [3], thermoelasticity [4], diffusion in polymers [5], reaction-diffusion equations [6], stability [7], control of chaotic systems [8], a variety of models for physiological processes or diseases [9] and so forth. Hence in the recent times, many researchers have been trying to develop numerical methods for solving these problems. Amiraliyev and Cimen [10] presented numerical method comprising a fitted difference scheme on a uniform mesh to solve second-order delay differential equations. Lange and Miura [11, 12] gave an asymptotic approach for a class of boundary-value problems for linear second-order differential-difference equations. Kadalbajoo and Sharma [13–15] presented numerical approaches to solve singularly perturbed differential-difference equations, which contains negative shift in the convention term (i.e., in the derivative term). Lange and Miura [16] considered the boundary value problem for a singularly perturbed nonlinear differential difference equation with shift and discussed the existence and uniqueness of their solutions. Furthermore, Kadalbajoo and Sharma [17] have discussed the numerical solution of the singularly perturbed nonlinear differential equations with small negative shifts.

In this paper, we have presented a numerical integration method for solving a class of singularly perturbed delay differential equations with small shift. First, the second-order singularly perturbed delay differential equation is replaced by an asymptotically equivalent first-order delay differential equation. Then we employed Simpson's rule and linear interpolation to get three-term recurrence relation which is solved easily by discrete invariant imbedding algorithm. The method is demonstrated by implementing it on several linear and nonlinear model examples by taking various values for the delay and perturbation parameters.

2. Description of the Method

Consider a class of singularly perturbed boundary value problems of the following form:

$$Ly \equiv \varepsilon y''(x) + a(x)y'(x-\delta) + b(x)y(x) = f(x), \quad 0 \le x \le 1, \tag{2.1}$$

with the interval and boundary conditions

$$y(0) = \alpha, \quad -\delta \le x \le 0, \tag{2.2a}$$

$$y(1) = \beta, \tag{2.2b}$$

where ε is small parameter, $0 < \varepsilon \ll 1$, and δ is also a small shifting parameter, $0 < \delta \ll 1$; $b(x)$, and $f(x)$ are bounded continuous functions in $(0,1)$, and α, β are finite constants. Further, we assume that $a(x) \ge M > 0$ throughout the interval $[0,1]$, where M is positive constant. This assumption merely implies that the boundary layer will be in the neighborhood of $x = 0$.

By using Taylor series expansion in the neighborhood of the point x, we have

$$y(x-\sqrt{\varepsilon}) = y(x) - \sqrt{\varepsilon}y'(x) + \frac{\varepsilon}{2}y''(x) \tag{2.3}$$

and consequently, (2.1) is replaced by the following first-order differential equation:

$$y'(x) = p(x)y'(x-\delta) + q(x)y(x-\sqrt{\varepsilon}) + r(x)y(x) + s(x), \tag{2.4}$$

where

$$p(x) = \frac{-a(x)}{2\sqrt{\varepsilon}}, \qquad q(x) = \frac{-1}{\sqrt{\varepsilon}}, \qquad r(x) = \frac{2-b(x)}{2\sqrt{\varepsilon}}, \qquad s(x) = \frac{f(x)}{2\sqrt{\varepsilon}}. \tag{2.5}$$

The transition from (2.1) to (2.4) is admitted, because of the condition that ε is small, $0 < \varepsilon \ll 1$. This replacement is significant from the computational point of view. Further details on the validity of this transition can be found in [18].

Now we divide the interval $[0,1]$ into N equal subintervals of mesh size $h = 1/N$ so that $x_i = ih$, $i = 0,1,2,\ldots,N$.

Integrating (2.4) with respect to x from x_i to x_{i+1} for $= 1,2,\ldots,N-1$, we get

$$\begin{aligned} y_{i+1} - y_i = {} & p_{i+1} y(x_{i+1} - \delta) - p_i y(x_i - \delta) \\ & + \int_{x_i}^{x_{i+1}} \left(-p'(x) y(x-\delta) + q(x) y\left(x - \sqrt{\varepsilon}\right) + r(x) y(x) + s(x)\right) dx, \end{aligned} \tag{2.6}$$

where $y_i = y(x_i)$, $p_i = p(x_i)$, $q_i = q(x_i)$, $r_i = r(x_i)$, $s_i = s(x_i)$.

By using Simpson's rule to evaluate the integral in (2.6), we get

$$\begin{aligned} y_{i+1} - y_i = {} & p_{i+1} y(x_{i+1} - \delta) - p_i y(x_i - \delta) \\ & - \frac{h}{6}\left(p'_i y(x_i - \delta) + 4p'_{i+1/2} y(x_{i+1/2} - \delta) + p'_{i+1} y(x_{i+1} - \delta)\right) \\ & + \frac{h}{6}\left(q_i y\left(x_i - \sqrt{\varepsilon}\right) + 4q_{i+1/2} y\left(x_{i+1/2} - \sqrt{\varepsilon}\right) + q_{i+1} y\left(x_{i+1} - \sqrt{\varepsilon}\right)\right) \\ & + \frac{h}{6}\left(r_i y_i + 4r_{i+1/2} y_{i+1/2} + r_{i+1} y_{i+1}\right) + \frac{h}{6}\left(s_i + 4s_{i+1/2} + s_{i+1}\right). \end{aligned} \tag{2.7}$$

By the means of Taylor series expansion and then by approximating $y'(x)$ by linear interpolation, we get

$$y(x_i - \delta) = y(x_i) - \delta y'(x_i) = y_i - \delta\left(\frac{y_i - y_{i-1}}{h}\right) = \left(1 - \frac{\delta}{h}\right) y_i + \frac{\delta}{h} y_{i-1}, \tag{2.8a}$$

$$y(x_{i+1} - \delta) = y(x_{i+1}) - \delta y'(x_{i+1}) = y_{i+1} - \delta\left(\frac{y_{i+1} - y_i}{h}\right) = \left(1 - \frac{\delta}{h}\right) y_{i+1} + \frac{\delta}{h} y_i, \tag{2.8b}$$

$$y\left(x_i - \sqrt{\varepsilon}\right) = y(x_i) - \sqrt{\varepsilon} y'(x_i) = y_i - \sqrt{\varepsilon}\left(\frac{y_i - y_{i-1}}{h}\right) = \left(1 - \frac{\sqrt{\varepsilon}}{h}\right) y_i + \frac{\sqrt{\varepsilon}}{h} y_{i-1}, \tag{2.8c}$$

$$y\left(x_{i+1} - \sqrt{\varepsilon}\right) = y(x_{i+1}) - \sqrt{\varepsilon} y'(x_{i+1}) = y_{i+1} - \sqrt{\varepsilon}\left(\frac{y_{i+1} - y_i}{h}\right) = \left(1 - \frac{\sqrt{\varepsilon}}{h}\right) y_{i+1} + \frac{\sqrt{\varepsilon}}{h} y_i. \tag{2.8d}$$

In similar way,

$$y(x_{i+1/2} - \delta) = y(x_{i+1/2}) - \delta y'(x_{i+1/2}) = y_{i+1/2} - \delta\left(\frac{y_{i+1} - y_i}{h}\right) = y_{i+1/2} - \frac{\delta}{h} y_{i+1} + \frac{\delta}{h} y_i. \tag{2.8e}$$

Hence, by making use of (2.8a)–(2.8e) in (2.7) we obtain

$$\begin{aligned}
y_{i+1} - y_i = {} & \left[-\frac{\delta}{h}\left(p_i + \frac{h}{6}p_i'\right) + \frac{\sqrt{\varepsilon}}{6}q_i\right] y_{i-1} \\
& + \left[\begin{array}{l} \frac{\delta}{h}\left(p_{i+1} - \frac{h}{6}p_{i+1}'\right) - \left(1 - \frac{\delta}{h}\right)\left(p_i + \frac{h}{6}p_i'\right) - \frac{4\delta}{6}p_{i+1/2}' + \frac{h}{6}\left(1 - \frac{\sqrt{\varepsilon}}{h}\right)q_i \\ + \frac{4\sqrt{\varepsilon}}{6}q_{i+1/2} + \frac{\sqrt{\varepsilon}}{6}q_{i+1} + \frac{h}{6}r_i \end{array}\right] y_i \\
& + \left[\begin{array}{l} \left(1 - \frac{\delta}{h}\right)\left(p_{i+1} - \frac{h}{6}p_{i+1}'\right) + \frac{4\delta}{6}p_{i+1/2}' + \frac{h}{6}\left(1 - \frac{\sqrt{\varepsilon}}{h}\right)q_{i+1} \\ - \frac{4\sqrt{\varepsilon}}{6}q_{i+1/2} + \frac{h}{6}r_{i+1} \end{array}\right] y_{i+1} \\
& + \frac{4h}{6}\left[-p_{i+1/2}' + q_{i+1/2} + r_{i+1/2}\right] y_{i+1/2} + \frac{h}{6}\left[s_i + 4s_{i+1/2} + s_{i+1}\right].
\end{aligned} \tag{2.9}$$

To make (2.9) a three-term recurrence relation, we can express $y_{i+1/2}$ in terms of y_{i-1}, y_i and y_{i+1} using Hermite's interpolation as follows:

$$y_{i+1/2} = \frac{1}{2}\left[y_i + y_{i+1}\right] + \frac{h}{8}\left[y_i' - y_{i+1}'\right] + O\left(h^4\right). \tag{2.10}$$

In view of (2.4) and (2.10), we get

$$\begin{aligned}
y_{i+1/2} = {} & \frac{1}{2}\left[y_i + y_{i+1}\right] + \frac{h}{8}\left[p_i y'(x_i - \delta) + q_i y\left(x_i - \sqrt{\varepsilon}\right) + r_i y_i + s_i\right] \\
& - \frac{h}{8}\left[p_{i+1} y'(x_{i+1} - \delta) + q_{i+1} y\left(x_{i+1} - \sqrt{\varepsilon}\right) + r_{i+1} y_{i+1} + s_{i+1}\right].
\end{aligned} \tag{2.11}$$

By making use of (2.8a)–(2.8e) in (2.11) and finite difference approximations, we get

$$\begin{aligned}
y_{i+1/2} = {} & \left[\frac{\delta}{8h}(p_{i+1} - p_i) + \frac{\sqrt{\varepsilon}}{8}q_i\right] y_{i-1} \\
& + \left[\frac{1}{2} + \frac{\delta}{8h}(p_i - p_{i+1}) - \frac{1}{8}\left(1 - \frac{\delta}{h}\right)(p_i - p_{i+1}) + \frac{h}{8}\left(1 - \frac{\sqrt{\varepsilon}}{h}\right)q_i + \frac{h}{8}r_i - \frac{\sqrt{\varepsilon}}{8}q_{i+1}\right] y_i \\
& + \left[\frac{1}{2} + \frac{1}{8}\left(1 - \frac{\delta}{h}\right)(p_i - p_{i+1}) - \frac{h}{8}\left(1 - \frac{\sqrt{\varepsilon}}{h}\right)q_{i+1} - \frac{h}{8}r_{i+1}\right] y_{i+1} + \frac{h}{8}\left[s_i - s_{i+1}\right].
\end{aligned} \tag{2.12}$$

Finally, making use of (2.12) in (2.9) and rearranging as three-term recurrence relation, we get

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \tag{2.13}$$

for $i = 1, 2, \ldots, N-1$, where

$$\begin{aligned}
E_i &= \frac{\delta}{h}\left(p_i + \frac{h}{6}p_i'\right) - \frac{\sqrt{\varepsilon}}{6}q_i - \frac{4h}{6}\left(-p_{i+1/2}' + q_{i+1/2} + r_{i+1/2}\right)\left(-\frac{\delta}{8h}(p_i - p_{i+1}) + \frac{\sqrt{\varepsilon}}{8}q_i\right),\\
F_i &= 1 + \frac{\delta}{h}\left(p_{i+1} - \frac{h}{6}p_{i+1}'\right) - \left(1 - \frac{\delta}{h}\right)\left(p_i + \frac{h}{6}p_i'\right) - \frac{4\delta}{6}p_{i+1/2}'\\
&\quad + \frac{h}{6}\left(1 - \frac{\sqrt{\varepsilon}}{h}\right)q_i + \frac{4\sqrt{\varepsilon}}{6}q_{i+1/2} + \frac{\sqrt{\varepsilon}}{6}q_{i+1} + \frac{h}{6}r_i + \frac{4h}{6}\left(-p_{i+1/2}' + q_{i+1/2} + r_{i+1/2}\right)\\
&\quad \times \left(\frac{1}{2} + \frac{\delta}{8h}(p_i - p_{i+1}) - \frac{1}{8}\left(1 - \frac{\delta}{h}\right)(p_i - p_{i+1}) + \frac{h}{8}\left(1 - \frac{\sqrt{\varepsilon}}{h}\right)q_i + \frac{h}{8}r_i - \frac{\sqrt{\varepsilon}}{8}q_{i+1}\right),\\
G_i &= 1 - \left(1 - \frac{\delta}{h}\right)\left(p_{i+1} - \frac{h}{6}p_{i+1}'\right) - \frac{4\delta}{6}p_{i+1/2}' - \frac{h}{6}\left(1 - \frac{\sqrt{\varepsilon}}{h}\right)q_{i+1}\\
&\quad + \frac{4\sqrt{\varepsilon}}{6}q_{i+1/2} - \frac{h}{6}r_{i+1} - \frac{4h}{6}\left(-p_{i+1/2}' + q_{i+1/2} + r_{i+1/2}\right)\\
&\quad \times \left(\frac{1}{2} + \frac{1}{8}\left(1 - \frac{\delta}{h}\right)(p_i - p_{i+1}) - \frac{h}{8}\left(1 - \frac{\sqrt{\varepsilon}}{h}\right)q_{i+1} - \frac{h}{8}r_{i+1}\right),\\
H_i &= \frac{h}{6}(s_i + 4s_{i+1/2} + s_{i+1}) + \frac{4h}{6}\left(-p_{i+1/2}' + q_{i+1/2} + r_{i+1/2}\right)\left(\frac{h}{8}(s_i - s_{i+1})\right).
\end{aligned} \tag{2.14}$$

This tridiagonal system is solved by using method of discrete invariant imbedding algorithm which is described in the next section.

3. Discrete Invariant Imbedding Algorithm

We now describe the Thomas algorithm which is also called discrete invariant imbedding [19] to solve the three-term recurrence relation:

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \quad \text{for } i = 1, 2 \ldots, N-1. \tag{3.1}$$

Let us set a difference relation of the form

$$y_i = W_i y_{i+1} + T_i \quad \text{for } i = N-1, N-2, \ldots, 2, 1, \tag{3.2}$$

where $W_i = W(x_i)$ and $T_i = T(x_i)$ are to be determined.

From (3.2), we have

$$y_{i-1} = W_{i-1} y_i + T_{i-1}. \tag{3.3}$$

Substituting (3.3) in (3.1), we have

$$y_i = \left(\frac{G_i}{F_i - E_i W_{i-1}}\right) y_{i+1} + \left(\frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}}\right). \tag{3.4}$$

By comparing (3.2) and (3.4), we get the recurrence relations

$$W_i = \left(\frac{G_i}{F_i - E_i W_{i-1}} \right), \tag{3.5}$$

$$T_i = \left(\frac{E_i T_{i-1} - H_i}{F_i - E_i W_{i-1}} \right). \tag{3.6}$$

To solve these recurrence relations for $i = 1, 2, 3, \ldots, N-1$, we need the initial conditions for W_0 and T_0. If we choose $W_0 = 0$, then we get $T_0 = \alpha$. With these initial values, we compute W_i and T_i for $i = 1, 2, 3, \ldots, N-1$ from (3.5) and (3.6) in forward process and then obtain y_i in the backward process from (3.2).

The conditions for the discrete invariant imbedding algorithm to be stable are (see [18–21])

$$E_i > 0, \qquad G_i > 0, \qquad F_i \geq E_i + G_i, \qquad |E_i| \leq |G_i|. \tag{3.7}$$

In our method, one can easily show that if the assumptions $a(x) > 0$, $b(x) < 0$ and $(\varepsilon - \delta a(x)) > 0$ hold, then the above conditions (3.7) hold, and thus the discrete invariant imbedding algorithm is stable.

4. Numerical Experiments

To demonstrate the applicability of the method, we have implemented it on two linear and two nonlinear problems with left-end boundary layers. Computational results are compared with exact solutions wherever exact solutions are available. When exact solution is not available, we have tested the effect of small delay parameter on solution of the problem for different values of δ of $o(\varepsilon)$.

4.1. Linear Problems

Example 4.1. Consider an example of singularly perturbed delay differential equation with left layer:

$$\varepsilon y''(x) + y'(x - \delta) - y(x) = 0; \quad x \in [0, 1] \text{ with } y(0) = 1, \ y(1) = 1. \tag{4.1}$$

The exact solution is given by

$$y(x) = \frac{(1 - e^{m_2}) e^{m_1 x} + (e^{m_1} - 1) e^{m_2 x}}{e^{m_1} - e^{m_2}}, \tag{4.2}$$

where $m_1 = -1 - \sqrt{1 + 4(\varepsilon - \delta)}/2(\varepsilon - \delta)$ and $m_2 = -1 + \sqrt{1 + 4(\varepsilon - \delta)}/2(\varepsilon - \delta)$.

The computational results are presented in Tables 1, 2, 3, and 4 for $\varepsilon = 0.001$ and 0.0001 for different values of δ.

Table 1: Numerical results of Example 4.1 for $\varepsilon = 0.001$, $\delta = 0.0001$, $N = 100$.

x	Numerical solution	Exact solution	Absolute error
0.00	1.0000000	1.0000000	$0.000E+00$
0.01	0.3724909	0.3719167	$5.743E-04$
0.02	0.3753635	0.3756417	$2.781E-04$
0.03	0.3791343	0.3794135	$2.792E-04$
0.04	0.3829441	0.3832233	$2.791E-04$
0.06	0.3906790	0.3909578	$2.788E-04$
0.08	0.3985702	0.3988485	$2.784E-04$
0.20	0.4493791	0.4496520	$2.730E-04$
0.50	0.6065730	0.6068032	$2.302E-04$
0.60	0.6703575	0.6705610	$2.035E-04$
0.90	0.9048500	0.9049187	$6.870E-05$
1.00	1.0000000	1.0000000	$0.000E+00$

Table 2: Numerical results of Example 4.1 for $\varepsilon = 0.001$, $\delta = 0.0008$, $N = 100$.

x	Numerical solution	Exact solution	Absolute error
0.00	1.0000000	1.0000000	$0.000E+00$
0.02	0.3785059	0.3753847	$3.121E-03$
0.03	0.3786281	0.3791566	$5.284E-04$
0.04	0.3827057	0.3829664	$2.607E-04$
0.05	0.3865343	0.3868145	$2.802E-04$
0.06	0.3904227	0.3907013	$2.786E-04$
0.08	0.3983141	0.3985924	$2.782E-04$
0.20	0.4491281	0.4494008	$2.728E-04$
0.40	0.5486277	0.5488775	$2.498E-04$
0.60	0.6701703	0.6703736	$2.033E-04$
0.90	0.9047868	0.9048555	$6.870E-05$
1.00	1.0000000	1.0000000	$0.000E+00$

Table 3: Numerical results of Example 4.1 for $\varepsilon = 0.0001$, $\delta = 0.00001$, $N = 100$.

x	Numerical solution	Exact solution	Absolute error
0.00	1.0000000	1.0000000	$0.000E+00$
0.01	0.3737204	0.3716098	$2.111E-03$
0.02	0.3754769	0.3753442	$1.327E-04$
0.03	0.3792426	0.3791161	$1.264E-04$
0.04	0.3830523	0.3829260	$1.264E-04$
0.08	0.3986781	0.3985520	$1.261E-04$
0.20	0.4494850	0.4493613	$1.237E-04$
0.40	0.5489547	0.5488413	$1.134E-04$
0.50	0.6066623	0.6065580	$1.043E-04$
0.80	0.8188018	0.8187455	$5.630E-05$
0.90	0.9048767	0.9048455	$3.120E-05$
1.00	1.0000000	1.0000000	$0.000E+00$

Table 4: Numerical results of Example 4.1 for $\varepsilon = 0.0001$, $\delta = 0.00008$, $N = 100$.

x	Numerical solution	Exact solution	Absolute error
0.00	1.0000000	1.0000000	$0.000E+00$
0.02	0.3754557	0.3753185	$1.372E-04$
0.03	0.3792183	0.3790904	$1.279E-04$
0.04	0.3830281	0.3829003	$1.279E-04$
0.06	0.3907630	0.3906352	$1.278E-04$
0.08	0.3986540	0.3985264	$1.276E-04$
0.20	0.4494613	0.4493362	$1.251E-04$
0.40	0.5489329	0.5488182	$1.146E-04$
0.60	0.6704187	0.6703254	$9.330E-05$
0.70	0.7409000	0.7408227	$7.730E-05$
0.90	0.9048707	0.9048392	$3.150E-05$
1.00	1.0000000	1.0000000	$0.000E+00$

Table 5: Numerical results of Example 4.2 for $\varepsilon = 0.001$, $N = 100$, and different values of δ.

x	Numerical solutions			
	$\delta = 0.0001$	$\delta = 0.0003$	$\delta = 0.0006$	$\delta = 0.0008$
0.00	1.0000000	1.0000000	1.0000000	1.0000000
0.02	0.2608070	0.2574476	0.2531103	0.2507717
0.04	0.2666194	0.2664989	0.2663023	0.2661506
0.05	0.2696995	0.2695796	0.2694007	0.2692834
0.06	0.2728263	0.2727058	0.2725251	0.2724043
0.08	0.2792234	0.2791020	0.2789198	0.2787981
0.20	0.3220214	0.3218944	0.3217039	0.3215766
0.40	0.4142966	0.4141648	0.4139674	0.4138352
0.60	0.5434980	0.5433738	0.5431879	0.5430634
0.80	0.7285067	0.7284169	0.7282822	0.7281920
0.90	0.8508641	0.8508093	0.8507276	0.8506728
1.00	1.0000000	1.0000000	1.0000000	1.0000000

Example 4.2. Now we consider an example of variable coefficient singularly perturbed delay differential equation with left layer:

$$\varepsilon y''(x) + e^{-0.5x} y'(x-\delta) - y(x) = 0 \quad \text{with } y(0) = 1,\ y(1) = 1. \tag{4.3}$$

For which the exact solution is not known. This example is considered to show the effect of the small shift on the boundary layer solution.

The computational results are presented in Tables 5 and 6 for $\varepsilon = 0.001$ and 0.0001 for different values of δ.

4.2. Nonlinear Problems

Nonlinear problems are linearized by the quasilinearization process. Then we have applied the present method.

Table 6: Numerical results of Example 4.2 for $\varepsilon = 0.0001$, $N = 100$, and different values of δ.

x	Numerical solutions			
	$\delta = 0.00001$	$\delta = 0.00003$	$\delta = 0.00006$	$\delta = 0.00008$
0.00	1.0000000	1.0000000	1.0000000	1.0000000
0.01	0.4289609	0.4278078	0.4260698	0.4249051
0.02	0.2612188	0.2608663	0.2603437	0.2599993
0.03	0.2637076	0.2636943	0.2636772	0.2636673
0.04	0.2667407	0.2667287	0.2667109	0.2666990
0.06	0.2729484	0.2729363	0.2729184	0.2729064
0.09	0.2826190	0.2826068	0.2825886	0.2825764
0.20	0.3221486	0.3221358	0.3221169	0.3221042
0.60	0.5436167	0.5436044	0.5435862	0.5435733
0.70	0.6275683	0.6275574	0.6275409	0.6275294
0.90	0.8509144	0.8509088	0.8509008	0.8508952
1.00	1.0000000	1.0000000	1.0000000	1.0000000

Example 4.3. Consider a singularly perturbed nonlinear delay differential equation:

$$\varepsilon y''(x) + y(x)y'(x-\delta) - y(x) = 0 \tag{4.4}$$

under the interval and boundary conditions

$$y(x) = 1, \quad -\delta \le x \le 0, \; y(1) = 1. \tag{4.5}$$

The exact solution is not known.

The computational results are presented in Tables 7 and 8 for $\varepsilon = 0.01$ for different values of δ.

Example 4.4. Consider an example of singularly perturbed nonlinear delay differential equation:

$$\varepsilon y''(x) + 2y'(x-\delta) + e^{y(x)} = 0 \tag{4.6}$$

under the interval and boundary conditions

$$y(x) = 0, \quad -\delta \le x \le 0, \; y(1) = 0. \tag{4.7}$$

The exact solution is not known.

The computational results are presented in Tables 9 and 10 for $\varepsilon = 0.01$ and 0.001 for different values of δ.

5. Discussions and Conclusions

We have presented a numerical integration method to solve singularly perturbed delay differential equations. The scheme is repeated for different choices of the delay parameter,

Table 7: Numerical results of Example 4.3 for $\varepsilon = 0.001$, $N = 100$, and different values of δ.

x	Numerical solutions			
	$\delta = 0.0001$	$\delta = 0.0003$	$\delta = 0.0006$	$\delta = 0.0008$
0.00	1.0000000	1.0000000	1.0000000	1.0000000
0.01	0.3724909	0.3596174	0.3392884	0.3250014
0.03	0.3791343	0.3790570	0.3788695	0.3786281
0.04	0.3829441	0.3828712	0.3827668	0.3827057
0.05	0.3867922	0.3867193	0.3866106	0.3865343
0.06	0.3906790	0.3906061	0.3904977	0.3904227
0.08	0.3985702	0.3984973	0.3983891	0.3983141
0.20	0.4493791	0.4493076	0.4492016	0.4491281
0.40	0.5488575	0.5487921	0.5486948	0.5486277
0.60	0.6703575	0.6703041	0.6702249	0.6701703
0.90	0.9048500	0.9048320	0.9048053	0.9047868
1.00	1.0000000	1.0000000	1.0000000	1.0000000

Table 8: Numerical results of Example 4.3 for $\varepsilon = 0.0001$, $N = 100$, and different values of δ.

x	Numerical solutions			
	$\delta = 0.00001$	$\delta = 0.00003$	$\delta = 0.00006$	$\delta = 0.00008$
0.00	1.0000000	1.0000000	1.0000000	1.0000000
0.01	0.3737204	0.3724622	0.3705641	0.3692932
0.03	0.3792426	0.3792360	0.3792249	0.3792183
0.04	0.3830523	0.3830458	0.3830347	0.3830281
0.05	0.3869004	0.3868939	0.3868828	0.3868762
0.08	0.3986781	0.3986716	0.3986605	0.3986540
0.20	0.4494850	0.4494785	0.4494676	0.4494613
0.40	0.5489547	0.5489486	0.5489386	0.5489329
0.50	0.6066623	0.6066567	0.6066476	0.6066423
0.70	0.7409147	0.7409106	0.7409040	0.7409000
0.90	0.9048767	0.9048751	0.9048723	0.9048707
1.00	1.0000000	1.0000000	1.0000000	1.0000000

Table 9: Numerical results of Example 4.4 for $\varepsilon = 0.001$, $N = 100$, and different values of δ.

x	Numerical solutions			
	$\delta = 0.0001$	$\delta = 0.0003$	$\delta = 0.0006$	$\delta = 0.0008$
0.00	0.0000000	0.0000000	0.0000000	0.0000000
0.02	−0.2107353	−0.2105360	−0.2098469	−0.2090868
0.04	−0.2053359	−0.2053088	−0.2052677	−0.2052346
0.05	−0.2026495	−0.2026230	−0.2025854	−0.2025608
0.06	−0.1999765	−0.1999504	−0.1999132	−0.1998883
0.08	−0.1946704	−0.1946451	−0.1946091	−0.1945850
0.10	−0.1894171	−0.1893927	−0.1893577	−0.1893344
0.30	−0.1396748	−0.1396577	−0.1396331	−0.1396166
0.60	−0.0737937	−0.0737854	−0.0737733	−0.0737652
0.80	−0.0350537	−0.0350500	−0.0350445	−0.0350408
0.90	−0.0170888	−0.0170870	−0.0170844	−0.0170827
1.00	0.0000000	0.0000000	0.0000000	0.0000000

Table 10: Numerical results of Example 4.4 for $\varepsilon = 0.0001$, $N = 100$, and different values of δ.

x	Numerical Solutions			
	$\delta = 0.00001$	$\delta = 0.00003$	$\delta = 0.00006$	$\delta = 0.00008$
0.00	0.0000000	0.0000000	0.0000000	0.0000000
0.02	−0.2107172	−0.2107119	−0.2106993	−0.2106895
0.03	−0.2080040	−0.2080019	−0.2079975	−0.2079955
0.04	−0.2053046	−0.2053025	−0.2052982	−0.2052962
0.05	−0.2026187	−0.2026166	−0.2026124	−0.2026104
0.08	−0.1946411	−0.1946391	−0.1946350	−0.1946331
0.10	−0.1893887	−0.1893868	−0.1893829	−0.1893810
0.30	−0.1396550	−0.1396536	−0.1396509	−0.1396495
0.60	−0.0737842	−0.0737834	−0.0737821	−0.0737814
0.80	−0.0350494	−0.0350491	−0.0350485	−0.0350481
0.90	−0.0170868	−0.0170866	−0.0170863	−0.0170862
1.00	0.0000000	0.0000000	0.0000000	0.0000000

δ, and perturbation parameter, ε. The choice of δ is not unique but can assume any number of values satisfying the condition $\delta(\varepsilon) = \tau\varepsilon$ with $\tau = O(1)$ and τ is not too large Lange and Miura [12]. To demonstrate the efficiency of the method, we have implemented it on two linear and two nonlinear model examples with the boundary layer on the left for different values of ε and δ. From the computational results, it is observed that the proposed method approximates the exact solution very well (see Tables 1–4), and the small shift, δ, affects the boundary layer solutions. That is, as δ increases, the size/thickness of the left boundary layer decreases (see Tables 5–10). This method does not depend on asymptotic expansion as well as on the matching of the coefficients. Thus, we have devised an alternative technique of solving boundary value problems for singularly perturbed delay differential equations, which is easily implemented on computer and is also practical.

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